Agglomeration in Purely Neoclassical and Symmetric Economies

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Abstract

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> "Therefore, it follows that if assumptions a1-a4 are upheld, there exists either a trivial solution, or no (price taking) competitive equilibrium. In short, the spatial impossibility theorem says that the smooth market mechanism alone cannot generate spatial agglomeration of activities." (Fujita [Fuj86], pp. 113-114).

1 Introduction

Here we examine the circumstances underlying equilibrium population agglomeration in the context of a completely standard economy, namely without externalities or imperfect competition, but with ordinary utility functions and constant returns to scale production. Whatever equilibria there are will clearly be Pareto efficient. And symmetric equilibria will be present. In such a situation, what force can possibly cause population to agglomerate, and importantly, can this force complement or substitute for the agglomerative forces more commonly used in the literature, such as the New Economic Geography or externalities?

As we shall explain, it is a bit puzzling and surprising that agglomeration can be generated in such a simple neoclassical model, starting with a completely symmetric situation. In fact, transportation cost can be zero or positive; the results are identical. In equilibrium, the regions or locations are autarkic, but the population distributions can be asymmetric. Next, we detail the strategy for our analysis.

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Our focus is on a very specific example for tractability and expository reasons. We adopt and then adapt the example of Kehoe [Keh85]. This classical example is aspatial, so it is best to imagine it to have only one region. There are four commodities and four consumers with different Cobb-Douglas utilities, but two different producers with constant returns to scale technologies. Constant returns to scale simplifies matters, since equilibrium profits must be zero. Thus, there is no need to worry about profit distribution and the zero profit conditions yield restrictions on equilibrium prices, useful for computational purposes. The key properties of this example are that it is quite simple **but features 3 equilibria**. Heterogeneous income effects play a big role both in Kehoe's example and in our work.

Next, we adapt Kehoe's model to the spatial context. There will be 2 identical regions or locations. There will be measure 1 of each of the four types of consumer. The same production technologies are available in each region. There are now 8 commodities, 4 in each region. Consumers can move between regions at no cost, as is standard in the literature.

We consider two versions of the model with differing portability of endowments. In the first version, endowments move with the consumers. An example of a mobile endowment is labor. In the second, endowments are immobile but income derived from endowments moves with the consumers. The differences between the two are in the market clearing conditions and possibly in the income derived from endowments. The latter turns out to be irrelevant in equilibrium. Notice that land is an example of an endowment that is not portable.

Our model and results are perfectly consistent with the spatial impossibility theorem as stated by Fujita and Thisse ([FT13], p. 39), even though we have a continuum of agents:

> **The Spatial Impossibility Theorem**. Assume a two-region economy with a finite number of consumers and firms. If space is homogeneous, transport is costly, and preferences are locally nonsatiated, there is no competitive equilibrium involving transportation.

The remainder of the paper proceeds as follows: In the following section, we lay out the model. Whereas section 3 presents the equilibria when endowments are mobile, section 4 presents the equilibria when they are not. We then discuss the possibility of inter-regional trade in section 6. Section 8 concludes.

2 The Model

We build our model on the production economy analyzed by Kehoe [Keh85]. His model features a single region with four commodities $i = 1, \cdots, 4$, four consumers $j = 1, \cdots, 4$, and linear technology. We add one more region to it and examine if agglomeration takes place in the absence of scale economies.

There is a unit mass of each of four types of consumers, who take up residence in either region a or b. Their relocation incurs no cost. We denote the population distribution by $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}$, where $\lambda_j \in [0,\ 1]$ is a fraction of type-j consumers who live in region a. In what follows we use a superscript to denote a row and commodity i, and a subscript to denote a column and consumer type j or region a or b.

A consumer of type j maximizes $u_j\left(x_j\right) = \prod_{i=1}^4 \left(x_j^i\right)^{\alpha_j^i}$ subject to $\pi \cdot x_j \leq \pi \cdot w_j$, where $x_j = \begin{bmatrix} x_j^1 & x_j^2 & x_j^3 & x_j^4 \end{bmatrix}^\top$ is his consumption bundle, $w_j = \begin{bmatrix} w_j^1 & w_j^2 & w_j^3 & w_j^4 \end{bmatrix}^\top$ is his endowment, and $\pi = \begin{bmatrix} \pi^1 & \pi^2 & \pi^3 & \pi^4 \end{bmatrix}^\top$ is a price vector. **As we will show below, the equilibrium price vector will be the same in both regions**. Expenditure share α and endowment w are specified as

$$\alpha = \begin{bmatrix} .52 & .86 & .5 & .06 \\ .4 & .1 & .2 & .25 \\ .04 & .02 & .2975 & .0025 \\ .04 & .02 & .0025 & .6875 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 50 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 \\ 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 400 \end{bmatrix}. \tag{1}$$

For instance, type-4 consumer's expenditure share of commodity 1 is .06, and his endowment of commodity 1 is zero.

Technology is linear and specified by technological process

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 6 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 & -4 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}. \tag{2}$$

The supply is Ay, where y is a 6×1 nonnegative vector indicating how much of each column of A a firm deploys in production.

Inter-regional trade does not occur in equilibrium, no matter transport cost. That is because equilibrium prices equate across regions. This is similar to the Factor Price Equalization Theorem. Here, utility levels play the role of product prices, and goods prices play the role of factor prices.

A region is in **intra-regional equilibrium** when each consumer maximizes his utility level subject to his budget and each commodity market clears. Namely, excess demand $(x_a - w)\lambda^\top - Ay_a = \mathbb{0}$ in region a; similarly, $(x_b - w)(\mathbb{1} - \lambda)^\top - Ay_b = \mathbb{0}$ in region b.[1] Furthermore, two regions are in **inter-regional equilibrium** if 1) every region is in intra-regional equilibrium, and 2) utility levels are the same in both regions type by type.[2] Whereas the

[1] A number in script font denotes a column or row vector (whichever is appropriate) consisting of repeated entries of a same number, e.g., $\mathbb{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$ and $\mathbb{1} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ in the preceding equations.

[2] A type might not be present in a region at equilibrium, but that won't happen in this example.

first requirement guarantees that the gains from trade are exhausted region by region, the second requirement guarantees that the utility gains from relocation are exhausted across regions.

A firm earns zero profit in equilibrium because of constant returns to scale. Thus, the intra-regional equilibrium price vector must be orthogonal to the column space of A. In addition, Walras' law enables the normalization of prices, $\sum \pi^i = 1$. Combined, these imply that the intra-regional equilibrium price vector π must be of the form $\begin{bmatrix} \pi^1 & \frac{1}{4} & \frac{7\pi^1-1}{3} & \frac{-10\pi^1}{3} + \frac{13}{12} \end{bmatrix}^\top$, $\pi^1 \in \left(\frac{1}{7},\, \frac{13}{14}\right)$ in intra-regional equilibrium. Let $\Pi^\perp$ be a set of all such price vectors. Note that four units of commodity 2 function as a numéraire in our economy. Also note that π^2, π^3 and π^4 are a linear and thus monotone function of π^1 over $\Pi^\perp$. Moreover, non-numéraire commodity prices π^3 and π^4 are strictly monotone in π^1, rendering them interchangeable when evaluating the monotonicity of a function. In what follows we say a function is monotone over $\Pi^\perp$ to mean that within the restricted domain $\Pi^\perp (\subset \mathbb{R}^4_{++})$ a function is monotone in terms of a non-numéraire price π^1, π^3 or π^4.

The value functions or utility levels of type 3 and 4 are strictly monotone over $\Pi^\perp$ (cf. figure 1. We picked π^1 for illustrative purposes). In inter-regional equilibrium, consumer j achieves the same utility level regardless of his residency thanks to free mobility. If a price differs between the regions the utility level will not equate among type 3, nor among type 4. Therefore, in inter-regional equilibrium, π must be identical in both regions. Furthermore, since consumers face the same prices wherever they live and the individual endowments are independent from λ, the individual consumption levels are the same in both regions. The next proposition summarizes this observation:

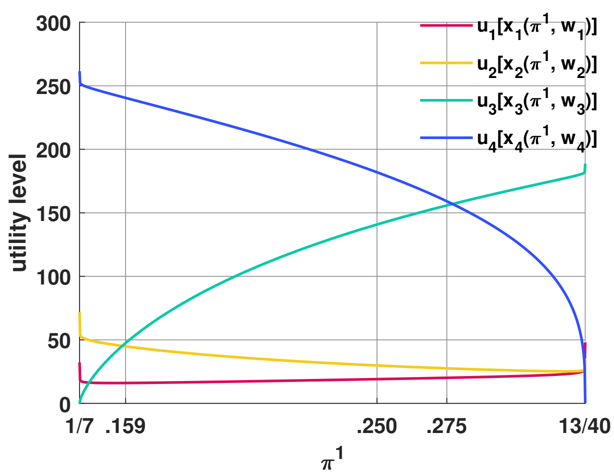


Figure 1. Value functions

Proposition 2.1 Inter-Regional Equilibrium

Suppose that at least one type of consumer has a strictly monotone value function over $\Pi^\perp$. If an inter-regional equilibrium exists, $\pi_a = \pi_b$ and $x_a = x_b$.

Proof. Suppose that a type-j consumer has a strictly monotone value function over $\Pi^\perp$, and that $\pi_a \neq \pi_b$. Then his utility level changes depending on where he is: $u_j\left[x_j\left(\pi_a,\, w_j\right)\right] \neq u_j\left[x_j\left(\pi_b,\, w_j\right)\right]$, and thus π_a and π_b do not make an inter-regional equilibrium price vector. Therefore, if an inter-regional equilibrium exists, $\pi_a = \pi_b$. Accordingly, $x\left(\pi_a,\, w\right) = x\left(\pi_b,\, w\right)$. □

Regardless, region-wide consumption $x\lambda^\top$ and $x(\mathbb{1}-\lambda)^\top$ will differ from each other because the population will not necessarily split evenly between two regions.

3 Mobile Endowments

3.1 Equilibria

We begin with the case where endowments are mobile, in other words they move with consumers. There are many inter-regional equilibria (see appendix A.2 for details). We present below three of them for example. The equilibrium specification includes the inter-regional equilibrium price vector π $(=\pi_a=\pi_b)$, population distribution λ, total population in each region $\sum\lambda_j$ and $\sum(1-\lambda_j)$, individual demand x $(=x_a=x_b)$, utility level u $(=u_a=u_b)$, and activity levels y_a and y_b along with the region-wide excess demand $(x-w)\lambda^\top - Ay_a$ in region a and $(x-w)(\mathbb{1}-\lambda)^\top - Ay_b$ in region b. We begin with the equilibria in Kehoe [Keh85] mirrored across the two regions. Whereas this is not part of our inter-regional equilibria, we place Kehoe's value of y side by side with corresponding y_a and y_b in each equilibrium. The subsequent section will draw a comparison between them. What is crucial is the value of λ, representing agglomeration.

Equilibrium #1

$$\begin{aligned}
\pi &= \begin{bmatrix} 0.15942 & 0.25000 & 0.03865 & 0.55193 \end{bmatrix}^\top \\
\lambda &= \begin{bmatrix} 0.70481 & 0.49341 & 0.82233 & 0.87084 \end{bmatrix} \\
\begin{bmatrix} \sum\lambda_j & \sum(1-\lambda_j) \end{bmatrix} &= \begin{bmatrix} 2.8914 & 1.1086 \end{bmatrix} \\
x &= \begin{bmatrix} 26.000 & 67.431 & 48.490 & 83.089 \\ 12.754 & 5.000 & 12.368 & 220.771 \\ 8.249 & 6.468 & 119.000 & 14.280 \\ 0.578 & 0.453 & 0.070 & 275.000 \end{bmatrix} \\
u &= \begin{bmatrix} 16.039 & 44.880 & 47.411 & 240.484 \end{bmatrix} \\
y_a &= \begin{bmatrix} 0 & 0 & 0 & 0 & 33.822 & 74.345 \end{bmatrix}^\top \\
y_b &= \begin{bmatrix} 0 & 0 & 0 & 0 & 8.879 & 6.853 \end{bmatrix}^\top \\
y &= \begin{bmatrix} 0 & 0 & 0 & 0 & 42.701 & 81.198 \end{bmatrix}^\top (= y_a + y_b) \\
(x-w)\lambda^\top - Ay_a &= \begin{bmatrix} -2.34E-8 & 1.34E-7 & 9.65E-8 & -6.05E-8 \end{bmatrix}^\top \approx \mathbb{0} \\
(x-w)(\mathbb{1}-\lambda)^\top - Ay_b &= \begin{bmatrix} 2.34E-8 & -1.34E-7 & -9.65E-8 & 6.05E-8 \end{bmatrix}^\top \approx \mathbb{0}
\end{aligned}$$

Equilibrium #2

$$\pi = \begin{bmatrix} 0.25000 & 0.25000 & 0.25000 & 0.25000 \end{bmatrix}^\top$$

$$\lambda = \begin{bmatrix} 0.83914 & 0.88410 & 0.25217 & 0.22671 \end{bmatrix}$$

$$\begin{bmatrix} \sum \lambda_j & \sum (1-\lambda_j) \end{bmatrix} = \begin{bmatrix} 2.2021 & 1.7979 \end{bmatrix}$$

$$x = \begin{bmatrix} 26.000 & 43.000 & 200.000 & 24.000 \\ 20.000 & 5.000 & 80.000 & 100.000 \\ 2.000 & 1.000 & 119.000 & 1.000 \\ 2.000 & 1.000 & 1.000 & 275.000 \end{bmatrix}$$

$$u = \begin{bmatrix} 19.067 & 29.832 & 140.802 & 181.909 \end{bmatrix}$$

$$y_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 14.182 & 11.342 \end{bmatrix}^\top$$

$$y_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 37.818 & 57.658 \end{bmatrix}^\top$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 52 & 69 \end{bmatrix}^\top (= y_a + y_b)$$

$$(x-w)\lambda^\top - Ay_a = \begin{bmatrix} 3.32E-8 & -1.33E-8 & -3.32E-8 & 1.33E-8 \end{bmatrix}^\top \approx \mathbb{0}$$

$$(x-w)(\mathbb{1}-\lambda)^\top - Ay_b = \begin{bmatrix} -3.32E-8 & 1.33E-8 & 3.32E-8 & -1.33E-8 \end{bmatrix}^\top \approx \mathbb{0}$$

Equilibrium #3

$$\pi = \begin{bmatrix} 0.27514 & 0.25000 & 0.30865 & 0.16621 \end{bmatrix}^\top$$

$$\lambda = \begin{bmatrix} 0.26713 & 0.92389 & 0.16127 & 0.10549 \end{bmatrix}$$

$$\begin{bmatrix} \sum \lambda_j & \sum (1-\lambda_j) \end{bmatrix} = \begin{bmatrix} 1.4578 & 2.5422 \end{bmatrix}$$

$$x = \begin{bmatrix} 26.000 & 39.072 & 224.362 & 14.499 \\ 22.011 & 5.000 & 98.768 & 66.485 \\ 1.783 & 0.810 & 119.000 & 0.539 \\ 3.311 & 1.504 & 1.857 & 275.000 \end{bmatrix}$$

$$u = \begin{bmatrix} 20.123 & 27.581 & 155.792 & 159.122 \end{bmatrix}$$

$$y_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 42.663 & 46.348 \end{bmatrix}^\top$$

$$y_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 10.517 & 18.800 \end{bmatrix}^\top$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 53.180 & 65.148 \end{bmatrix}^\top (= y_a + y_b)$$

$$(x-w)\lambda^\top - Ay_a = \begin{bmatrix} -5.51E-7 & 1.19E-7 & 4.91E-7 & -1.79E-7 \end{bmatrix}^\top \approx \mathbb{0}$$

$$(x-w)(\mathbb{1}-\lambda)^\top - Ay_b = \begin{bmatrix} 5.51E-7 & -1.19E-7 & -4.91E-7 & 1.79E-7 \end{bmatrix}^\top \approx \mathbb{0}$$

Not only the size but also the composition of types differ between the regions in equilibrium. For instance, in Equilibrium #2, most consumers of type 1 and 2 are found in region 1, and most consumers of type 3 and 4 are found in region 2. Thus, each region consists of a different mix of types even in the same equilibrium. We explain the reason behind the spatial sorting in appendix A.2.

3.2 Comparison between Single- and Two-Region Economies

All the inter-regional equilibria we found in section 3.1 are closely related to the three equilibria in Kehoe [Keh85] in several ways. Let us call our two-region economy E^{2R} and Kehoe's single-region economy E^{1R}. Before establishing an association between the two, we first point out that it is only the economy-wide **demand** that is influenced by λ. Production simply scales up or down as needed in an attempt to fill the demand both large and small regardless of the price. The demand is **non-linear** over $\Pi^{\perp}$. In contrast, the supply **does not depend on** π in the sense that so long as $\pi \in \Pi^{\perp}$ the firms always earn zero profit whatever y_a and y_b they choose. If Ay_a and Ay_b happen to square with $(x-w)\lambda^{\top}$ and $(x-w)(\mathbb{1}-\lambda)^{\top}$, then that is an inter-regional equilibrium; or else there is no inter-regional equilibrium at the $\pi \in \Pi^{\perp}$ and λ under consideration.

In E^{1R} the aggregate net demand for commodity i is a simple sum of the individual net demand, $(x^i - w^i)\mathbb{1}$. It appears as a vertical sum of each type's net demand (quantity is on the vertical axis), the black line in figure 2. By contrast, in E^{2R} the economy-wide net demand for commodity i becomes a weighted sum of the individual net demand, $(x^i - w^i)\lambda^{\top}$ and $(x^i - w^i)(\mathbb{1}-\lambda)^{\top}$. It appears as a vertical sum of each type's net demand in figure 2 with an uneven weight of λ in region a and $\mathbb{1}-\lambda$ in region b. E^{1R} can be thought of as a special case of E^{2R} in this sense, where $\lambda = \mathbb{0}$ or $\mathbb{1}$, with the requirement $u_a = u_b$ removed.

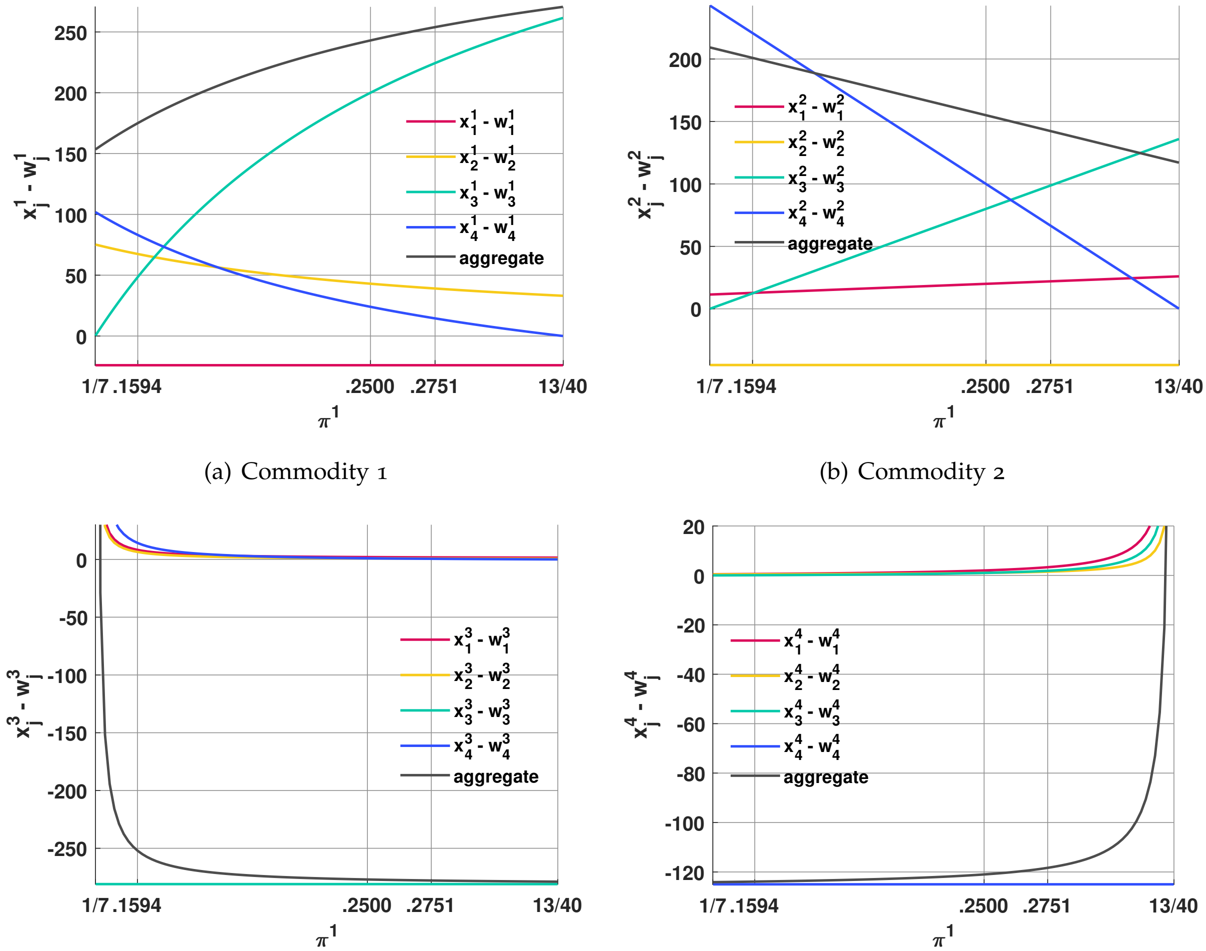
With this added degree of freedom, one may be tempted to speculate that E^{2R} takes a different equilibrium price than E^{1R}, and that there are more than three equilibrium prices possible. For instance, whereas the aggregate net demand for commodity 1 (the black line in figure 2(a)) is monotone increasing in π^1, with the right mix of λ the economy-wide net demand for commodity 1 may no longer be increasing or monotone. However, this turns out not to be the case:

PROPOSITION 3.1 EQUILIBRIUM PRICES IN SINGLE- AND TWO-REGION ECONOMIES

Suppose that at least one type of consumer has a strictly monotone value function over $\Pi^{\perp}$. *The set of inter-regional equilibrium price vectors* Π^{2R} *in* E^{2R} *is a subset of its counterpart* Π^{1R} *in* E^{1R}.

Proof. Suppose that $\pi^{2R} \in \Pi^{2R}$ but $\notin \Pi^{1R}$. Recall from proposition 2.1 that $\pi_a = \pi_b$ in inter-regional equilibrium. Since $\pi^{2R} (= \pi_a = \pi_b)$ clears all four markets in each region of E^{2R}, there exists such $y_a,\ y_b \geq \mathbb{0}$ that

$$\begin{aligned} \left[x\left(\pi^{2R}\right) - w\right]\lambda^{\top} &= Ay_a, \quad \text{and} \\ \left[x\left(\pi^{2R}\right) - w\right](\mathbb{1}-\lambda)^{\top} &= Ay_b. \end{aligned} \tag{3}$$



(a) Commodity 1 (b) Commodity 2

(c) Commodity 3 (d) Commodity 4

Figure 2. Individual net demand for each commodity by type in color, and aggregate net demand in black.

Aggregate them to obtain the countrywide market clearance in E^{2R}:

$$\left[x\left(\pi^{2R}\right)-w\right]\mathbb{1}=A(y_a+y_b). \tag{4}$$

On the other hand, since $\pi^{2R}\notin\Pi^{1R}$, there is no $y\geq\mathbb{0}$ such that

$$\left[x\left(\pi^{2R}\right)-w\right]\mathbb{1}=Ay \tag{5}$$

in E^{1R}. Since the left-hand sides of (4) and (5) are identical, $A(y_a+y_b)=Ay$. Then

$$y_a+y_b=y. \tag{6}$$

Whereas equilibrium y_a+y_b exists, equilibrium y does not, contradicting each other. Therefore, if π^{2R} clears the regional markets in E^{2R}, it also clears the markets in E^{1R}. Hence $\Pi^{2R}\subseteq\Pi^{1R}$. □

Remark. This proposition pertains only to the prices. There are only three equilibrium **prices** in E^{2R}, $\pi^1 = .2500$, $.1594$ and $.2751$. However, E^{2R} does come with many inter-regional equilibria in terms of **distribution** λ.

The proposition capitalizes on the fact that the individual demand does not depend on λ. The individual net demand $x - w$ will be the same in inter-regional equilibrium no matter where the person lives, how many regions there are, or how the population is divided, because the individual endowment w does not depend on λ. Namely, a type-j consumer does not change his demand based on how many consumers of type j (or of any type for that matter) there are in his region. Rather, λ only changes the equilibrium quantity by way of changing the region-wide (**not** individual) endowment $w\lambda^\top$ and $w(\mathbb{1} - \lambda)^\top$. The individual net demands are $x(\pi_a, w) - w$ in region a and $x(\pi_b, w) - w$ in region b. Since $\pi_a = \pi_b$ from proposition 2.1, they are the same. In conjunction with the fact that there is a unit mass of each type, when we add them together to obtain the countrywide demand in E^{2R} it will come to the same as the aggregate demand in E^{1R}. On the flip side, the sum of activity levels $y_a + y_b$ in E^{2R} comes to y in E^{1R} as well (cf. corollary 3.1 to follow).

Depending on the weight λ, one of the regions can and does feature an **intra**-regional equilibrium whose price vector falls outside Π^{1R} thanks to the added degree of freedom mentioned above. However, these equilibria will not make an **inter**-regional equilibrium because the inter-regional equilibrium price vector has to be a member of Π^{1R} as proved above. We present one such example in figure 3.[3] Observe that region a features one intra-regional equilibrium price vector in Π^{1R} and two intra-regional equilibrium price vectors outside Π^{1R}. The latter two will not make the list for inter-regional equilibria because there is no corresponding intra-regional price vectors found in region b. Thus, as we discussed earlier in this section, the expansion of a set of λ from $\{\mathbb{0}, \mathbb{1}\}$ of E^{1R} to $[0, 1]^4$ of E^{2R} does unleash lots of price vectors outside Π^{1R}, but these are only intra-regional equilibrium price vectors. In inter-regional equilibrium, the price vector still has to be selected from Π^{1R}. See appendix A.1 for more on net demand.

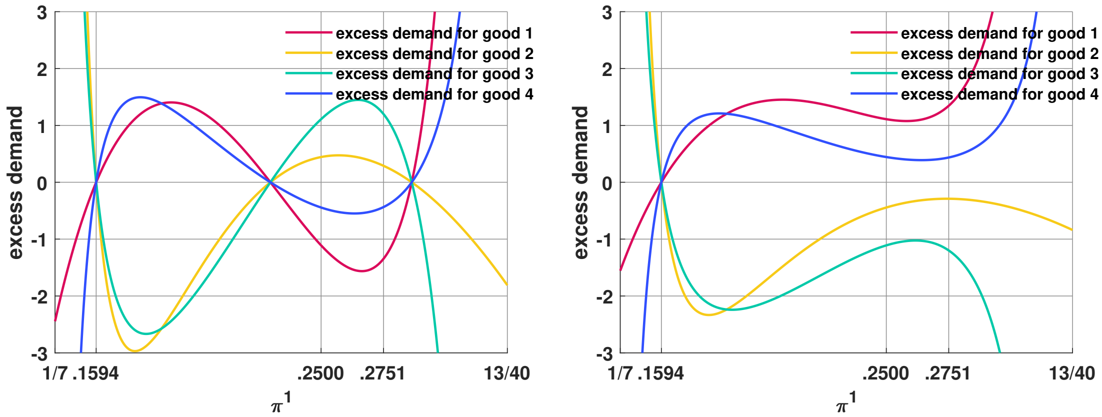
Note also that the zero-profit condition further implies $\Pi^{2R} \subseteq \Pi^{1R} \subseteq \Pi^\perp$.

We derive two equivalencies from proposition 3.1:

Corollary 3.1 Supply and Demand in Single- and Two-Region Economies

The countrywide demand in E^{2R} is identical to its corresponding aggregate demand in E^{1R} in equilibrium. Furthermore, the countrywide activity level $y_a + y_b$ in E^{2R} is equal to its corresponding y in E^{1R} in equilibrium.

[3] As mentioned earlier in this section, y_a and y_b are not a function of $\pi \in \Pi^\perp$. We compute excess demand $(x - w)\lambda^\top - Ay_a$ with the midpoint between y_a that clears the even-numbered markets and y_a that clears the odd-numbered markets in figure 3 for illustrative purposes. These y_a's are equal to each other only in intra-regional equilibrium, turning excess demand zero. The same goes for region b as well.



(a) Excess demand in region a. (b) Excess demand in region b.

Figure 3. Regional excess demand when $\lambda = \begin{bmatrix} .9723 & .1262 & .5688 & .6412 \end{bmatrix}$. A region is in intra-regional equilibrium when all four excess demands are zero. Two regions are furthermore in intra-regional equilibrium when they share the same intra-regional equilibrium price vector. In this case, region a has three intra-regional equilibria and region b has one; E^{2R} as a whole has one inter-regional equilibrium.

Proof. Immediate from (4) and (5). □

Remark. This explains why $y_a + y_b = y$ in all three equilibria we listed in section 3.1. Indeed in any inter-regional equilibrium, $y_b = y - y_a$ and this is why the right two columns in figure 7 in appendix A.2 to follow are flipped images of each other: Whereas there are many inter-regional equilibrium y_a and y_b depending on λ, y remains the same because there is only one y each for three price vectors in Π^{1R} in E^{1R}. Put differently, there is a wide range of equilibrium y_a and y_b because the possible range of λ is $[0,\ 1]^4$ in E^{2R}, but the sum of y_a and y_b has to be equal to one of only three y's because the possible range of λ in E^{1R} is $\{\mathbb{0},\ \mathbb{1}\} (\subset [0,\ 1]^4)$.

In short, the demand has to keep to $u_a = u_b$, and the supply has to keep to $y_a + y_b = y$ in inter-regional equilibrium.

3.3 Scalable Equilibria and Spatial Sorting

Whereas a sample of inter-regional equilibria listed in section 3.1 involves an uneven presence of each type in a region, heterogeneous preferences do not necessarily lead to regional sorting. This section will establish that λ of the form $\begin{bmatrix} c & c & c & c \end{bmatrix}$ $(c \in [0,\ 1])$ constitutes an inter-regional equilibrium. In particular, $c = .5$ indicates that spatial sorting is not a requisite of inter-regional equilibria.

Evidently, such distributions constitute an **intra**-regional equilibrium because linear technology allows firms to rescale their production by a factor of c and $1 - c$ in respective regions to meet the smaller, but proportionally down scaled net aggregate demand. In contrast, it is not all too obvious whether they further constitute an **inter**-regional equilibrium. We verify that each region will achieve the same utility level regardless of the value of c selected.

Consider any equilibrium in E^{1R}. Material balance implies $\left(x^{1R}-w\right)\mathbb{1}=Ay^{1R}$. Multiply both sides by c to obtain $\left(x^{1R}-w\right)(c\mathbb{1})=A(cy^{1R})$. Given the equilibrium price in E^{1R}, the optimal bundle x_a in E^{2R} coincides with x^{1R} in E^{1R} because the individual demand is independent of c. In addition let $y_a = cy^{1R}$. Then the equation can be rewritten as $(x_a-w)(c\mathbb{1})=Ay_a$, which is none other than the material balance in region a itself. Therefore, region a reaches an intra-regional equilibrium, as does region b. Furthermore, there is no inter-regional migration of consumers of any type. Since individual demand is independent of c, each type achieves the same utility level in either region under π^{1R} selected. On the supply end, $y_a+y_b=cy+(1-c)y=y$, in keeping with corollary 3.1. Therefore, any c constitutes an inter-regional equilibrium, with any equilibrium price inherited from E^{1R}. Put differently, equilibria in E^{1R} are scalable: any equilibrium in E^{1R} can be implemented as an inter-regional equilibrium in E^{2R} with arbitrary $c\in[0,\ 1]$. In this case, both regions are simply a miniature copy of E^{1R} with the identical composition of types. Consequently, spatial sorting can but does not have to take place in inter-regional equilibrium.

This observation draws on two features of the model. On the one hand, supply is linear. If Ay^{1R} is in the production set, so are $Ay_a = cAy^{1R}$ and $Ay_b = (1-c)Ay^{1R}$. On the other hand, π_a and π_b aside, there is no channel through which individual demand matrices x_a and x_b respond to c. Thus, net aggregate demands in E^{2R} are simply a scalar multiple of $(x^{1R}-w)\mathbb{1}$ in E^{1R}. Since the equilibrium price in E^{1R} clears markets in E^{2R} as shown above, the resultant allocation constitutes an inter-regional equilibrium. All combined, an equilibrium in E^{1R} can be scaled down by an arbitrary factor without changing the level of utility, which in turn guarantees an existence of (infinitely many) corresponding inter-regional equilibria in E^{2R}.

4 Immobile Endowments

We have so far assumed that transport cost is prohibitively high. The only time the endowment makes a cross-border transfer is when it relocates with its owner, who himself is perfectly mobile. Labor is a good example. This assumption effectively rules out inter-regional commuting. Therefore, unless $\lambda=\begin{bmatrix}.5 & .5 & .5 & .5\end{bmatrix}$, the distribution of endowments is not uniform.[4]

In fact, inter-regional equilibria exist whether w is mobile or not. This section considers the case where the endowment cannot move at all. Land is a good example. We call the previous economy with **mo**bile endowments E^{MO} and the economy with **im**mobile

[4] As shown in section 3.3, $\lambda=\begin{bmatrix}.5 & .5 & .5 & .5\end{bmatrix}$ **is** one of the inter-regional equilibrium values.

w currently in question E^{IM}. Let us further denote the distribution of endowments by $\mu \in [0,\ 1]^4$. Up until now $\mu = \lambda$. Suppose instead that μ is independent from λ and exogenously given. Income from the endowments moves with consumers

There are many equilibria in E^{IM} (see figure 4). We present one of them below:

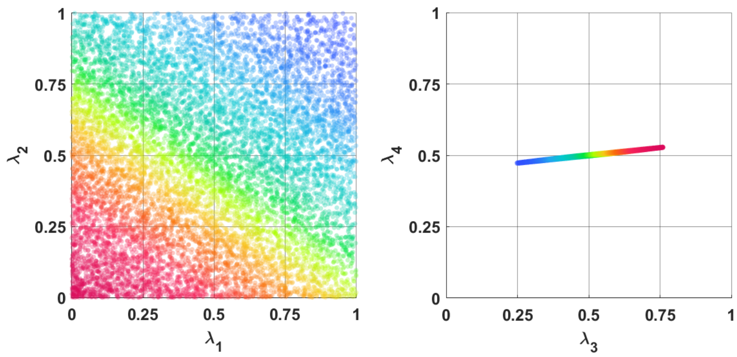
$$
\begin{aligned}
\pi &= \begin{bmatrix} 0.15942 & 0.25000 & 0.03865 & 0.55193 \end{bmatrix}^\top \\
\lambda &= \begin{bmatrix} 0.2699 & 0.2397 & 0.6296 & 0.5140 \end{bmatrix} \\
\begin{bmatrix} \sum \lambda_j & \sum (1-\lambda_j) \end{bmatrix} &= \begin{bmatrix} 1.6532 & 2.3468 \end{bmatrix} \\
\mu &= \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \\
x &= \begin{bmatrix} 26.000 & 67.431 & 48.490 & 83.089 \\ 12.754 & 5.000 & 12.368 & 220.771 \\ 8.249 & 6.468 & 119.000 & 14.280 \\ 0.578 & 0.453 & 0.070 & 275.000 \end{bmatrix} \\
u &= \begin{bmatrix} 16.039 & 44.880 & 47.411 & 240.484 \end{bmatrix} \\
y_a &= \begin{bmatrix} 0 & 0 & 0 & 0 & 18.5378 & 39.8117 \end{bmatrix}^\top \\
y_b &= \begin{bmatrix} 0 & 0 & 0 & 0 & 24.1635 & 41.3863 \end{bmatrix}^\top \\
y &= \begin{bmatrix} 0 & 0 & 0 & 0 & 42.701 & 81.198 \end{bmatrix}^\top (= y_a + y_b) \\
x\lambda^\top - w\mu^\top - Ay_a &= \begin{bmatrix} 0.4772E-8 & -0.7825E-8 & -0.8252E-8 & 0.4345E-8 \end{bmatrix}^\top \approx \mathbb{0} \\
x(\mathbb{1}-\lambda)^\top - w(\mathbb{1}-\mu)^\top - Ay_b &= \begin{bmatrix} -0.4772E-8 & 0.7825E-8 & 0.8252E-8 & -0.4345E-8 \end{bmatrix}^\top \approx \mathbb{0}.
\end{aligned}
$$

Therefore, agglomeration is not necessarily caused by the way w is geographically distributed. Note that π, x, u, and $y_a + y_b$ in any equilibrium in E^{IM} are identical to the equilibria listed in section 3.1 by construction.

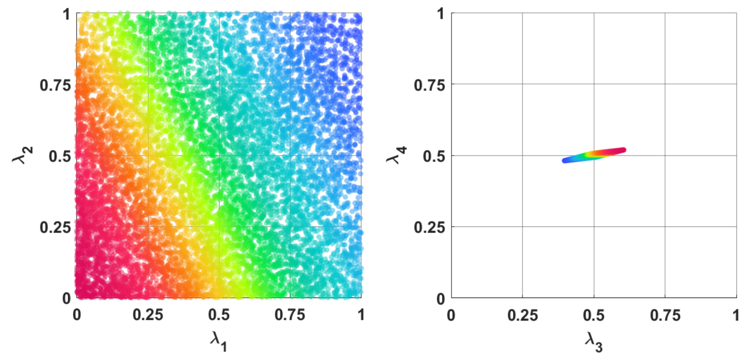
As with c in section 3.3, the individual consumption remains the same whatever μ is. Even if a consumer of type j resides in region a but his endowment is located in region b, his income $\pi_b \cdot w_j$ credited in region b can be cashed in in region a through a bank. He does not incur any transport cost because there is no transport of physical goods involved with this transfer. Therefore, his income remains intact. Moreover, since the location of w_j does not alter his income, proposition 2.1 still applies so that $\pi_b = \pi_a (= \pi)$. His income is simply $\pi \cdot w_j$ whether his endowment location coincides with his residence or not. Therefore, the individual consumption $x(\pi,\ w)$ is identical as well whether μ is equal to λ or given otherwise.

On the regional level, the markets clear when

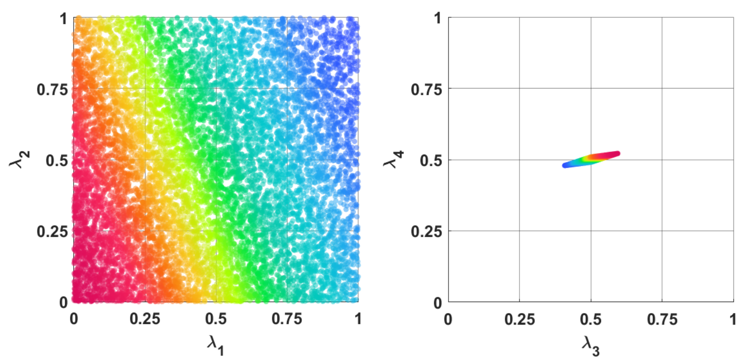
$$
\begin{aligned}
x(\pi,\ w)\lambda^\top - w\mu^\top &= Ay_a, \quad \text{and} \\
x(\pi,\ w)(\mathbb{1}-\lambda)^\top - w(\mathbb{1}-\mu)^\top &= Ay_b.
\end{aligned} \tag{7}
$$



(a) $\pi^1 = .159422$.



(b) $\pi^1 = .250000$.



(c) $\pi^1 = .275136$.

Figure 4.

On the countrywide level, aggregate (7) to derive

$$[x(\pi,\ w) - w]\mathbb{1} = A(y_a + y_b). \tag{8}$$

Then if π clears the regional markets, it also clears its corresponding markets in E^{1R}. Let Π^{IM} be the set of inter-regional equilibrium price vectors in E^{IM}.

PROPOSITION 4.1 EQUILIBRIUM PRICES WHEN ENDOWMENTS ARE IMMOBILE

Suppose that at least one type of consumer has a strictly monotone value function over $\Pi^{\perp}$. Π^{IM} *is a subset of its counterpart* Π^{1R} *in* E^{1R}.

Proof. Same as proposition 3.1. The only difference is between equations (3) and (7). What ensues is identical. □

In summary, E^{IM} compares to E^{MO} as follows:

- region-wide **net** demand: different (does involve μ)
- countrywide net demand: same (μ and $\mathbb{1} - \mu$ cancel each other out).

Then we can infer from the above that

- region-wide supply: different (should involve μ)
- countrywide supply: same (should be independent from μ)

in inter-regional equilibrium. Therefore, for a given equilibrium price vector $\pi \in \Pi^{MO} \cap \Pi^{IM}$ in E^{2R} (if exists), $y_a^{MO} + y_b^{MO} = y_a^{IM} + y_b^{IM} = y^{1R}$, but not necessarily $y_a^{MO} = y_a^{IM}$ or $y_b^{MO} = y_b^{IM}$.

Comparing (3) to (7), $Ay_a^{IM} = Ay_a^{MO} + w(\lambda - \mu)^\top$. This implies that an inter-regional equilibrium in E^{IM} may or may not have a corresponding inter-regional equilibrium in E^{MO}. It does as long as firms in region a are capable of producing the difference $w(\lambda - \mu)^\top$, and similarly, the ones in region b are capable of producing the difference $w(-\lambda + \mu)^\top$ to be made up for, while keeping to $y_a + y_b = y$ as a whole.

Whether its counterpart exists in E^{MO} or not, inter-regional equilibria in E^{IM} themselves exist. One of them presented above features $\mu = \begin{bmatrix} .5 & .5 & .5 & .5 \end{bmatrix}$ so that there is no spatial inhomogeneity at play. Therefore, agglomeration in the current model is not necessarily due to an uneven spatial distribution of endowments.

We can trivially and retroactively set μ equal to one of λ's found in section 3.1 to ensure an inter-regional equilibrium even when w is immobile. With that, in the following sections, we assume that w is mobile with its owner for simplicity and for the ease of notation. We do so with the understanding that there are at least as many equilibria in E^{IM} as in E^{MO} by simply setting $\mu = \lambda$.

5 Mixed Mobility of Endowments

We have discussed the economy where endowments are mobile (section 3) and immobile (section 4). A more realistic setup is where the first two endowments are mobile and the remainder are not. Given technological process A in (2), the last two commodities can only be an input such as land. Land is not mobile.

In this section, we set $\mu = \begin{bmatrix} \lambda_1 & \lambda_2 & .5 & .5 \end{bmatrix}$ to this effect. The first two endowments accompany their owner, type 1 and 2; the last two endowments are evenly allocated to each region regardless of where type 3 and 4 migrate to. What can be produced instead of endowed with (i.e., endowment 1 and 2) can cross the border; what cannot be produced and only be endowed with (i.e., endowment 3 and 4) cannot cross the border.

As in sections 3 and 4, equilibria exist. We present one of them below:

$$
\begin{aligned}
\pi &= \begin{bmatrix} 0.25000 & 0.25000 & 0.25000 & 0.25000 \end{bmatrix}^\top \\
\lambda &= \begin{bmatrix} 0.9350 & 0.3152 & 0.5140 & 0.4860 \end{bmatrix} \\
\begin{bmatrix} \sum \lambda_j & \sum (1-\lambda_j) \end{bmatrix} &= \begin{bmatrix} 2.2502 & 1.7498 \end{bmatrix} \\
\mu &= \begin{bmatrix} 0.9350 & 0.3152 & 0.5 & 0.5 \end{bmatrix} \\
x &= \begin{bmatrix} 26.000 & 43.000 & 200.000 & 24.000 \\ 20.000 & 5.000 & 80.000 & 100.000 \\ 2.000 & 1.000 & 119.000 & 1.000 \\ 2.000 & 1.000 & 1.000 & 275.000 \end{bmatrix} \\
u &= \begin{bmatrix} 19.067 & 29.832 & 140.802 & 181.909 \end{bmatrix} \\
y_a &= \begin{bmatrix} 0 & 0 & 0 & 0 & 24.1740 & 39.4699 \end{bmatrix}^\top \\
y_b &= \begin{bmatrix} 0 & 0 & 0 & 0 & 27.8260 & 29.5301 \end{bmatrix}^\top \\
y &= \begin{bmatrix} 0 & 0 & 0 & 0 & 52.0000 & 69.0000 \end{bmatrix}^\top (= y_a + y_b) \\
x\lambda^\top - w\mu^\top - Ay_a &= \begin{bmatrix} -1.0023E-09 & -2.0064E-09 & -4.1382E-10 & 5.9035E-10 \end{bmatrix}^\top \approx \mathbb{0} \\
x(\mathbb{1}-\lambda)^\top - w(\mathbb{1}-\mu)^\top - Ay_b &= \begin{bmatrix} 1.0023E-09 & 2.0064E-09 & 4.1382E-10 & -5.9035E-10 \end{bmatrix}^\top \approx \mathbb{0}.
\end{aligned}
$$

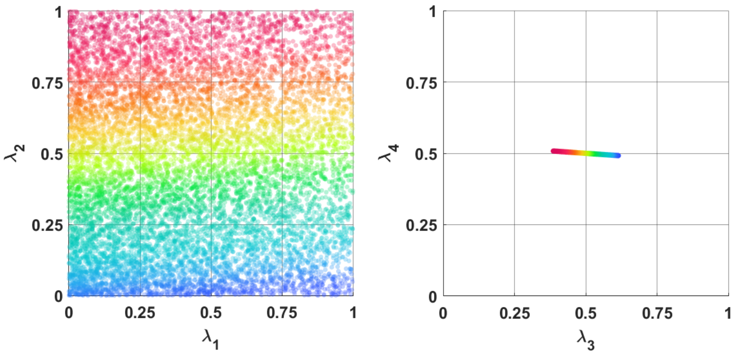
See figure 5 for a collection of equilibria in this economy.

6 Transportation Cost and Inter-Regional Trade

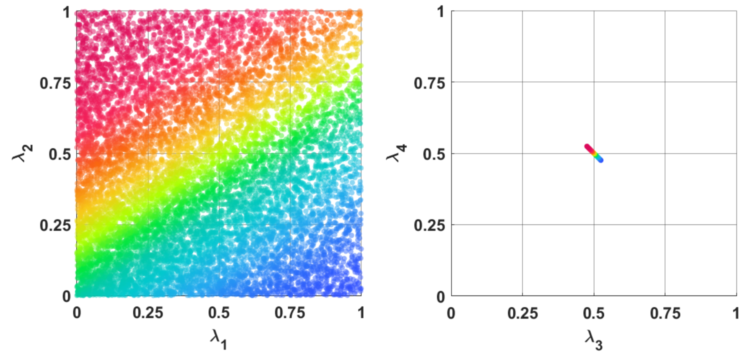
Let $t \geq 1$ be the units of commodity required to be shipped from one region to receive one unit of it in the other region. In the previous section, we assumed that $t \to \infty$ so that there is no point in engaging in inter-regional trades. Trading beyond the regional boundaries is effectively equivalent to deploying first four columns of A, i.e., disposal of commodities for free. Let us now consider two other cases: $t > 1$, and $t = 1$.

When $t > 1$, the inter-regional equilibria remain the same as above. No one engages in inter-regional trades in this case either. Such trades only incur transport costs with no gain in return. Technology is not heterogeneous by region to warrant comparative advantages, nor does it exhibit increasing returns to scale to warrant exclusive production in a particular region. Consequently, imported goods are always priced higher than locally produced goods and thus no one buys them.

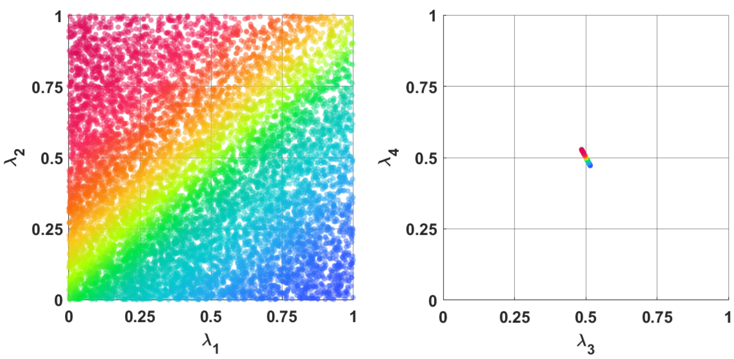
By the same token, no one ships his endowment outside his residence even if it is mobile. If he did, it would only reduce his income and consumption level for nothing in return. Unlike transfer of earnings through a bank, transfer of physical endowments does incur costs.



(a) $\pi^1 = .159422$.



(b) $\pi^1 = .250000$.



(c) $\pi^1 = .275136$.

Figure 5.

When $t = 1$, E^{2R} reduces to E^{1R} in effect, with a token presence of λ. The location of production or consumption is of no consequence due to perfect mobility. As such, any λ constitutes an inter-regional equilibrium. Moreover, endowment mobility as previously discussed in section 4 makes no difference, as there is no way to tell regions apart, which become purely ornamental in the absence of transport cost.

7 Agglomeration and Consumer Types

7.1 Economy with Two Types

We have demonstrated that agglomeration occurs in a four-commodity, four-type setting. Let us denote the number of commodities by I and the number of types by J. This section examines the role I and J play in forming agglomeration. We will start with a downscaled version with $I = 2$, $J = 2$ and $\text{rank}(A) = 1$. In particular, we isolate two commodities and two types from the preceding economy, and replace A with a 2×1 vector $\hat{A}$ with one positive and one negative entry. As in section 2, firms earn zero profit so that $\pi_a^\top \hat{A} = \pi_b^\top \hat{A} = 0$.[5] Consequently, $\pi_a = \pi_b$. Since π does not differ by region, neither does $x - w =: z$, a 2×2 individual net demand matrix, nor does $u_j(x_j)$ for any j. The argument so far does not involve any λ. As such, inter-regional utility equalization does not impose any restrictions on λ.

Material balance implies $z\lambda^\top = \hat{A}y_a$ and $z(\mathbb{1} - \lambda^\top) = \hat{A}y_b$ in respective regions. Combined,

$$\begin{bmatrix} z & -\hat{A} & \mathbb{0} \\ z & \mathbb{0} & \hat{A} \end{bmatrix} \begin{bmatrix} \lambda^\top \\ Y \end{bmatrix} = \begin{bmatrix} \mathbb{0} \\ z\mathbb{1} \end{bmatrix}, \tag{9}$$

where $Y := [y_a \quad y_b]^\top$ (note that z is independent from λ). If the first matrix was invertible, distribution λ would be unique. However, it is not full rank. Thus, there are infinitely many solutions to (9). Intuitively, constant returns to scale enable firms to counter any distribution λ by simply rescaling their production level to meet the regional demand $z\lambda^\top$ and $z(1 - \lambda^\top)$ with no footprints on the price.

The individual budget constraint is simply $\pi \cdot z = 0$. Consumers do not receive any dividends from firms because they do not make any profit. Therefore, the budget constraints do not impose any restrictions on λ either.

All in all, this economy supports any $\lambda \in [0,\ 1]^2$ in equilibrium. Linear production plays two contrasting roles in this. On the one hand, it adds rigidity to the economy: it single-handedly dictates what the equilibrium price is. It does not require any involvement of demand (and by extension, λ) because $\hat{A}$ alone determines the unique direction of the price vector. On the other hand, it adds flexibility to the economy: because it is linear, firms can easily scale up or down their production to meet the regional demand that varies with λ, without affecting the price, which itself is of linear technology's own making as explained above.

The original example with $I = J = 4$ compares to the current example with $I = J = 2$ as follows: In section 2, zero-profit condition only narrows the candidate prices down to **infinitely many** vectors in $\Pi^\perp$. Here, in contrast, such π is **unique** (up to a scalar multiple). As described above, once $\hat{A}$ is given, z is uniquely determined, as does $\pi (= \begin{bmatrix} \frac{1}{7} & \frac{6}{7} \end{bmatrix}^\top)$. Moreover, since $\hat{A}$ does not differ by region, neither does z.

By contrast, section 2 cannot be written as a linear system because $\hat{A}$ alone cannot narrow π down to a single vector and thus z involves π in it rather than being treated as a constant as in (9). This in turn helps reduce the degree of freedom to pin down λ. Distribution $\begin{bmatrix} \Sigma\lambda_j & \Sigma(1 - \lambda_j) \end{bmatrix}$ thus obtained features agglomeration.[6]

[5] Consumers are free to trade directly with each other at a rate of exchange different from such π, but one of the parties involved in such exchange will be better off trading with a firm at π anyway.

[6] Except few instances referred to in section 3.3.

7.2 Economy with Many Types

Now let us consider a general case with $I(\geq 3)$, $J \geq I$, and $R(\geq 2)$ regions. Preferences are represented by $u_j\left(x_j\right) = \sum_{i=1}^{I} \alpha_j^i \log x_j^i$, with $\sum_{i=1}^{I} \alpha_j^i = 1$ for any j. As before, denote the set of orthogonal price vector by $\Pi^{\perp} := \left\{\pi \in \mathbb{R}_{++}^{I} : \pi^{\top} A = \mathbb{0}\right\}$. Since A is not location variant, neither is $\Pi^{\perp}$. If the equilibrium price vector differs by region, that is due exclusively to the difference in the composition of consumer types in each region. Firms do not have ability to break symmetry, nor do they benefit from a thicker market, because their production is simply scalable. Thus, they cannot and do not initiate or promote agglomeration. Rather, they will unassumingly meet the net aggregate demand whatever λ turns out to be. In essence, they exist simply 1) to set the rate of exchange, that is, to narrow the set of price vectors from $\mathbb{R}_{++}^{I}$ down to $\Pi^{\perp}$ without consumers' involvement; and 2) to convert commodity i into i' as needed to align every type's marginal rate of substitution with one of $\pi \in \Pi^{\perp}$ selected.

First, we consider the case where inter-regional transport of commodity is prohibitively expensive, namely $t \to \infty$, and endowments are not mobile. Absent inter-regional trade, each market clears region by region so that there are IR market-clearing conditions. In addition, free mobility of consumers implies $J(R-1)$ utility-equalization conditions, unless $\lambda_{j,\,r} = 0$ in some region. That is, $v_j\left(\pi_r,\, w_j\right) := \sum_{i=1}^{I} \alpha_j^i \log\left(\dfrac{\alpha_j^i \pi_r \cdot w_j}{\pi_r^i}\right) = v_j\left(\pi_s,\, w_j\right)$ for each j in any region r and s. Since α is not location variant, this condition can be written simply as

$$\log\left(\pi_r \cdot w_j\right) - \sum_{i} \alpha_j^i \log \pi_r^i = \bar{v}, \tag{10}$$

where $\bar{v}$ is a countrywide constant.

An increase in π^1 unfolds $2I$ distinct effects that combined determine the change in $v_j(\cdot)$. The first I effects manifest through the first term in (10). The consumer's income $\pi \cdot w_j$ changes with π^1. His income from w_j^1 increases, whereas his income from w_j^i $(i \geq 2)$ may or may not increase, depending on whether π^i increases or decreases with π^1 inside $\Pi^{\perp}$. In total, there are I income effects to track.

The remaining I effects manifest through the second term in (10). The consumer will shift his consumption of commodity 1 to other commodities, say $i = 2$, to deflect the price increase. Nevertheless, x_j^1 may increase in the end. Prices $\pi^2, \cdots, \pi^I$ change as well to keep π inside $\Pi^{\perp}$. If π^2 is increasing in π^1, he may shift his consumption from commodity 2 back to 1. In total, there are I substitution effects to track.

Take Kehoe's model for example. In this case, income effects are simpler than the case in general because type j's only source of income is endowment $i = j$. Thus, each type experiences one income effect rather than $I(= 4)$ of them. Consequently, their income is simply monotone in π^1 as shown in figure 6(b). Along with substitution effects, in total, each type registers five effects rather than $2I(= 8)$ effects.

Furthermore, $v_3(\cdot)$ and $v_4(\cdot)$ are strictly monotone in π^1 (cf. figure 1). Since π^3 is increasing in π^1, type 3's income is increasing in π^1. Thus, its utility level will increase with π^1 for income effect. In particular, its income effect on commodity 1 exceeds its substitution effect (cf. figure 2(a)). For type 4, $v_4(\cdot)$ is decreasing in π^1 because its only source of income is w_4^4, whose price is decreasing in π^1. In contrast, a change in π^1 does not produce income effect for type 2 because she is endowed with a numéraire: Unlike other types, her income is always $\pi^2 w_2^2$ regardless of the value π^1 takes. Nevertheless, her utility level changes with π^1 for various substitution effects combined.

In general, the overall change in $v_j(\cdot)$ is indeterminate without imposing further assumptions. Appendix A.3 presents one specific environment where every region shares the same price for example.

Material balance in region r implies

$$x(\pi_r,\ w)\,\lambda_r^\top = w\mu_r^\top + Ay_r, \tag{11}$$

where $x(\pi_r,\ w)$ is a Marshallian demand matrix whose i-j entry is $\dfrac{\alpha_j^i \pi_r \cdot w_j}{\pi_r^i}$, $\lambda_r := \begin{bmatrix}\lambda_{1,\ r} & \cdots & \lambda_{j,\ r} & \cdots & \lambda_{J,\ r}\end{bmatrix}$ is a fraction of each type who resides in region r, w is a matrix whose i-j entry denotes the amount of commodity i that type j is endowed with, and $\mu_r := \begin{bmatrix}\mu_{1,\ r} & \cdots & \mu_{j,\ r} & \cdots & \mu_{J,\ r}\end{bmatrix}$ is a fraction of type j's endowments allocated to region r. If $\mu_{j,\ r} = \frac{1}{R}$ for any j and r, then endowments are evenly allocated. As in section 3.3, if E^{1R} has an equilibrium, there is at least one class of equilibria in R-region economy of the scalable form $\lambda_r = \bar{\lambda}_r \mathbb{1}$ with $\sum \bar{\lambda}_r = 1$. This **trivially** includes agglomeration if $\bar{\lambda}_r \neq \frac{1}{R}$ in at least two regions out of R regions.

Outside this, non-scalable equilibria may exist depending on whether A and μ can meet the net regional demand in (11) in every region. If there is $\lambda_r (\neq \bar{\lambda}_r \mathbb{1})$ that satisfies $x\lambda_r^\top = x\bar{\lambda}_r \mathbb{1}$ in every region, then there is an equilibrium that features an uneven presence of each type. Take some region r for instance. If $\lambda_{1,\ r}$ is lower than $\bar{\lambda}_r$ by $\Delta\lambda_{1,\ r}$, so long as $\lambda_{2,\ r}$ is above $\bar{\lambda}_r$ to make up for lost $x_{1,\ r}\Delta\lambda_{1,\ r}$ with $x_{2,\ r}\Delta\lambda_{2,\ r}$, this will be an intra-regional equilibrium. The difference $\Delta\lambda_{1,\ r}$ and $\Delta\lambda_{2,\ r}$ will be transferred to $\lambda_{1,\ s}$ and $\lambda_{2,\ s}$ in another region s.[7] If the same compensation above can take place in region s as well, then this will furthermore be an inter-regional equilibrium.

In general, $x_{2,\ r}$ is not equal to $x_{1,\ r}$. Therefore $\Delta\lambda_{2,\ r}$ may not be equal to $-\Delta\lambda_{1,\ r}$. For instance, if $x_{1,\ r}$ is small for most of the commodities and $x_{2,\ r}$ is large, $\Delta\lambda_{2,\ r}$ may be smaller than $-\Delta\lambda_{1,\ r}$ while still keeping to the same $x\lambda_r^\top$ in aggregate. Consequently, λ may feature, **not trivially**, an unequal proportion of each type. The variance stems exclusively from the heterogeneity in preferences among different types rather than regional difference in productivity (which is nonexistent in our model). Convex preferences require the presence of a similar portion of each commodity. This is aided by having the

[7] Or split and resettled in multiple regions depending on where $\bar{\lambda}_{r'}$ stands in regions $r' \neq r$.

right combination of types.[8]

As in section 3, rendering w mobile makes little difference. Mobile w only augments A in effect: demand x is served through A and w, with the second source of supply now endogenized. Since supply is not a motivating force behind agglomeration (it only narrows down the set of equilibrium prices), the storyline above remains intact with or without mobile endowments.

Now consider the case where inter-regional trade is not prohibitively expensive. If $\pi \in \Pi^{\perp}$ can be written as a function of one of the prices (cf. appendix A.3) and $v_j(\cdot)$ is strictly monotone for at least one type, no trade will occur among regions where the said type is present because each region will have an identical price vector. Otherwise, it may differ from region to region.

8 Conclusions

Agglomeration is conventionally thought of as a production-driven phenomenon. Scale economies favor a concentration of inputs within a close proximity.

Barring scale economies, can there still be agglomeration? To examine whether agglomeration can be driven by consumption rather than production, we worked on a general equilibrium model with constant returns to scale proposed by Kehoe [Keh85]. We established that agglomeration does not necessitate the presence of scale economies. Heterogeneity among consumers creates asymmetry in population distribution.

We are not intent on overriding the existing knowledge about production-oriented agglomeration. Rather, we cast light on the role consumption plays in generating agglomeration, which combined should illustrate a more realistic mechanism behind agglomeration.

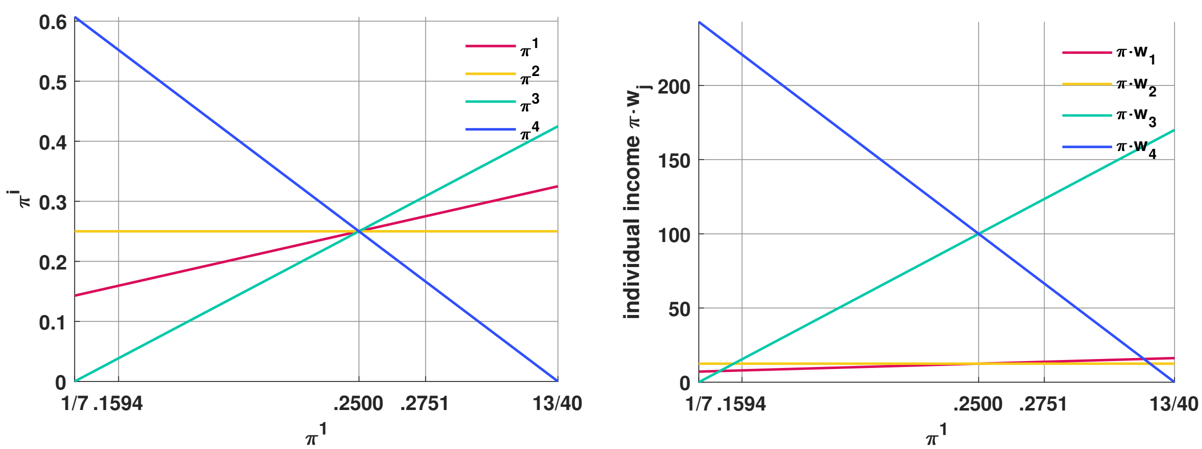
As did Kehoe [Keh85], we worked on a specific class of preferences in the interest of tractability. We defer to future research to reproduce our results in a general setting.

[8] This would be further aided by having the right combination of endowments that complement the commodities in short supply if endowments were mobile.

A Appendix

A.1 Net Demand

To further understand net demand in figure 2 let us consider consumer 3 in detail for example. He is endowed with $w_3^3 = 400$ units of commodity 3, whose price is increasing in π^1 (cf. figure 6(a)). As such, his income increases with π^1 (cf. figure 6(b)). His net demand for commodity 1, $x_3^1 - w_3^1$ is traced by the green line in figure 2(a). His income effect on commodity 1 exceeds the substitution effect. His net demand for commodity 1 **grows** with π^1 as a result. On the contrary, his net demand for commodity 4, as appearing in figure 2(d), **diminishes** with π^4 (which itself is decreasing in π^1). In this case, his income effect on commodity 4 falls behind the substitution effect. Indeed his expenditure share of commodity 4 is only $\alpha_3^4 = .0025$ and thus the effect of income growth is easily trumped by realignment of his consumption towards commodity 1 and 2. On the other hand, his net demand for commodity 3, as depicted by the flat green line in figure 2(c), remains **invariable**. His only source of income is commodity 3 and thus the effect of a change in π^3 on his demand for commodity 3 is exactly offset by the associated change in his income. The same argument goes for commodity 1 for consumer 1, commodity 2 for consumer 2, and commodity 4 for consumer 4. E^{1R} features more than one equilibrium in part because of the conflicting gradients in figures 2(a) and 2(b).



(a) A set of π that is strictly positive and orthogonal to the column space of A. Color corresponds to **commodity**.

(b) Income by type. Color corresponds to **type**.

Figure 6.

A.2 List of Inter-Regional Equilibria

A complete list of inter-regional equilibria found is available here, three of which we have already described in section 3.1. An inter-regional equilibrium takes one of three prices: $\pi^1 = .250000,\ .159422$ or $.275136 \in \Pi^{1R}$. Figure 7 sorts the equilibria according to these prices and plots population distribution λ, and activity level y_a and y_b under each price.

In general, a similar proportion of type 3 and 4 sort into the same region. This is due in part to a large excess supply of commodity 3 that type-3 consumers have, and similarly to a large excess supply of commodity 4 that type-4 consumers have, which heavily drags down the excess demand for these commodities. Without co-presence of two types, either commodity 3 or 4 will have a large excess in supply. They both can be used as an input to produce commodity 1 or 2.[9] However, commodity 3 **cannot** be used to produce commodity 4, and neither can commodity 4 be used to produce commodity 3. Thus, a region can even out the supply of commodity 1, 2 and 3; or 1, 2 and 4; but not all four of them at once through production activities. Since preferences are convex, imbalance between commodity 3 and 4 (and by extension, type 3 and 4) does not last and will be rectified through inter-regional migration. Therefore, each region tends to have a roughly equal portion of type 3 and 4. Indeed, y^5 and y^6 are positively correlated in order to produce **both** commodity 1 and 2 with the aim of providing a full range of commodities in proportion.

As for type 1 and 2, their distribution depends on the equilibrium price. When $\pi^1 = .159422$, they tend to avoid each other, but co-locate with type 3 and 4. When $\pi^1 = .250000$, they tend to co-locate with each other, but avoid type 3 and 4. When $\pi^1 = .275136$, they tend to co-locate with type 3 and 4, but they neither co-locate with nor avoid each other. As such, their pattern of distribution is not as definitive as type 3 and 4 as above. This is in part because type 1 and 2 are endowed with commodities that **can** be produced. Their lack of presence can easily be made up for by other types through production of commodity 1 and 2. In addition, their excess demand $x_1^1 - w_1^1$ and $x_2^2 - w_2^2$ are smaller in magnitude than $x_3^3 - w_3^3$ and $x_4^4 - w_4^4$ (cf. figure 2). Thus their presence, or the lack thereof, does not have as significant a bearing as type 3 and 4. This renders their distribution more fluid and sensitive to the price than type 3 and 4's.

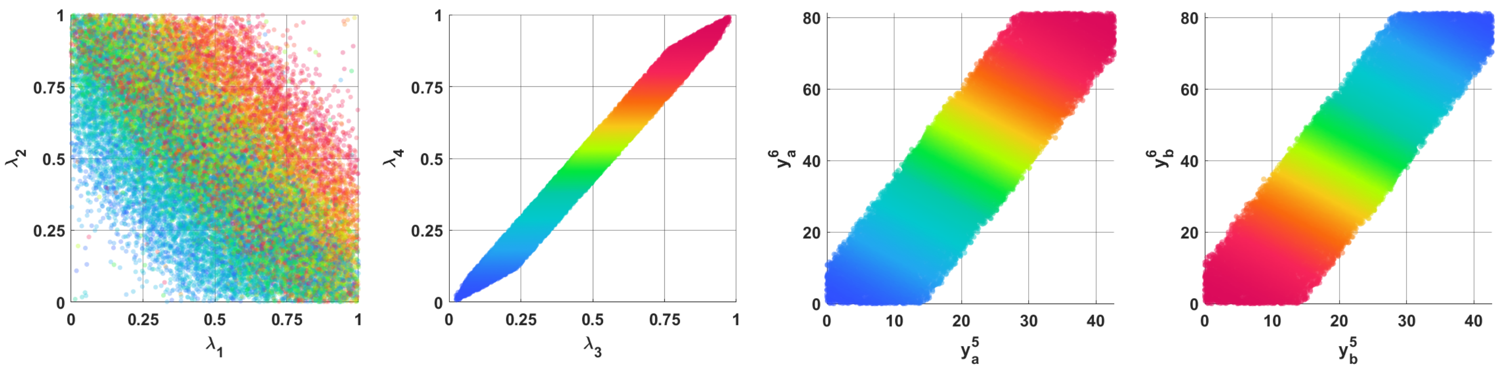
A.3 Conditions for a Universal Price

Let us consider a specific example of economy discussed in section 7.2, where every region shares a single price. Assume that $\operatorname{rank}(A) = I - 2$. This enables us to write $\pi \in \Pi^{\perp}$ as a function of one of the non-numéraire prices, say π^1, as in Kehoe's example. Let us further assume that $v_1\left(\pi^1,\ w_1\right)$ is strictly monotone in π^1. In what follows we consider an interior solution $\lambda_{1,\ r} > 0$ for any r, i.e., type 1 is present in all regions.

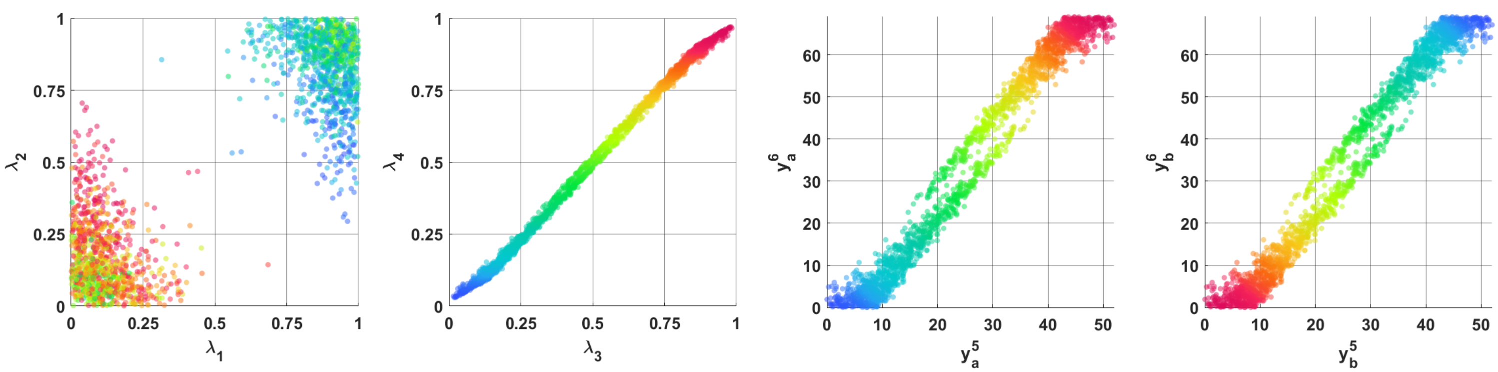
In equilibrium, $v_1(\pi_r^1,\ w_1) = \bar{v}$. Since $v_1(\cdot)$ is strictly monotone in π_r^1, $\pi_r^1 = v_1^{-1}(\bar{v},\ w_1)$. The right-hand side is independent of region. Therefore, π^1 has to be the same in any region, or else $v_1(\cdot)$ would differ from region to region. Note that we only require one

[9] Note that activity level y_a and y_b do pick up where type-3 and -4 consumers are in figure 7.

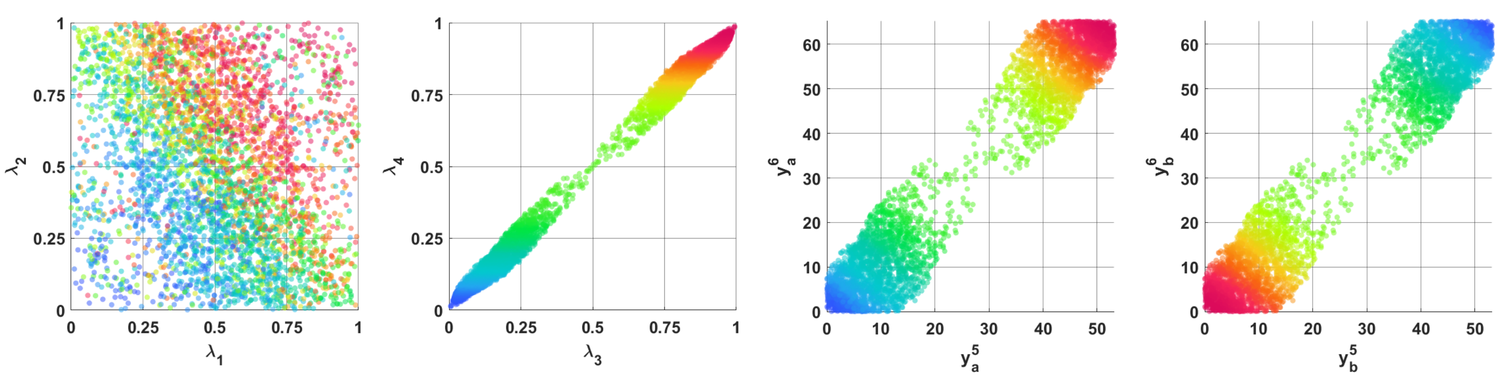
type to have a strictly monotone value function and be present in every region in order to have a location-invariant price: Type 2 may have $v_2(\pi_r^1,\ w_2) = v_2(\pi_s^1,\ w_2)$ with some $\pi_r^1 \neq \pi_s^1$, but this will induce migration among type 1 and thus will not be supported in equilibrium. Type 2 may also be absent in some regions as well without any substantive influence on the equilibrium price. Furthermore, the equilibrium price will be the same as its single-region counterpart E^{1R} as established in proposition 3.1. We will suppress r from π_r and x_r unless necessary because the price and the corresponding demand will be uniform across regions.



(a) $\pi^1 = .159422$.



(b) $\pi^1 = .250000$.



(c) $\pi^1 = .275136$.

Figure 7. Equilibrium distribution λ and activity level y under each price. The figure farthest to the left plots vector $\begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}$ and the figure next to it plots remaining entries $\begin{bmatrix}\lambda_3 & \lambda_4\end{bmatrix}$. Each equilibrium is colored according to its λ_4 value in order to show correspondence among four figures. The remaining two figures to the right plot a vector of activity levels $\begin{bmatrix}y_a^5 & y_a^6\end{bmatrix}$ in region a and $\begin{bmatrix}y_b^5 & y_b^6\end{bmatrix}$ in b. For instance, in figure 7(b), a low λ_4 (in blue) is associated with a low λ_3, and a high λ_1 and λ_2. In this case, type 1 and 2 sort into region a, and type 3 and 4 sort into region b. Production levels are low in region a and high in region b; region a deploys the fifth column of A more than the sixth, and vice versa in region b.

References

[FT13] Masahisa Fujita and Jacques-François Thisse. *Economics of Agglomeration*. Cambridge University Press, 2013.

[Fuj86] Masahisa Fujita. Urban land use theory. In Jean Jaskold Gabszewicz, Jacques-François Thisse, Masahisa Fujita, and Urs Schweizer, editors, *Location theory*, pages 73–149. Harwood Academic Publishers, Chur, Switzerland, 1986.

[Keh85] Timothy J Kehoe. Multiplicity of equilibria and comparative statics. *The Quarterly Journal of Economics*, 100(1):119–147, 1985.

↻ Last updated on April 13, 2023 at 18:20 (ET).