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"Controlling Chaotic Fluctuations through Monetary Policy"

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Controlling Chaotic Fluctuations through Monetary Policy*

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Abstract

This paper applies the chaos control method (the OGY method) proposed by Ott et al. (1990, *Physical Review Letters*) to policy making in macroeconomics. This paper demonstrates that the monetary equilibrium paths in a discrete-time, two-dimensional overlapping generations model exhibit chaotic fluctuations depending on the money supply rate and the elasticity of substitution between capital and labor under the assumption of the constant elasticity of substitution (CES) production function. We also show that the chaotic fluctuations can be stabilized by controlling the money supply rate by using the OGY method.

Keywords: Macroeconomy; Chaos Control; OGY method; Monetary Policy; OLG model; Chaos

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1 Introduction

In history, many economies have repeatedly experienced economic overheating, followed by recessions. Many studies tried to construct a dynamic general equilibrium model that can explain such phenomena. Introducing outside money into overlapping generations models, Gale (1973), Farmer (1986), and Yokoo (2000) examined the possibility of endogenous fluctuations. In particular, Gale (1973) showed that in an economy without outside money, the steady state equilibria are classified into two cases: one is the classical case in which the steady state is efficient, and the other the Samuelson case, in which the steady state is inefficient. Gale (1973) also indicated that, once outside money is introduced into the model, a positive outside money steady state exists in the Samuelson case and a negative outside money steady state exists in the classical case, and that in the latter economy, endogenous fluctuations of the economy are possible. Similar results are obtained by Farmer (1986) and Yokoo (2000). Along this line of research, we adopt a simple overlapping generations model with outside money and investigate the dynamic properties of the model.

Moreover, in the field of macroeconomics, the complex behavior of macroeconomic variables caused by economic nonlinearities has been studied over the years. For example, Benhabib and Day (1982) and Grandmont (1985) used overlapping generations models to show that perfectly competitive equilibrium can exhibit complex endogenous fluctuations. Following their work, many other studies, including Nishimura and Yano (1995), Farmer (1986), Yokoo (2000), and Matsuyama et al. (2016, 2018), have also shown that complex behavior in macroeconomic models is caused by several factors.

There are also various empirical studies on chaotic phenomena in macroeconomics. In fact, these studies show mixed results; that is, there are both supportive and negative results for the existence of chaotic behaviors. This is due to the fact that the sample size of data available for macroeconomics is small. Recently, Barnett et al. (2022a) used a different approach to confirm the possibility of chaos in macroeconomics. They adopted the new Keynesian model of Benhabib et al. (2001a,b) as a benchmark, and numerically examined whether the parameter values set by Benhabib et al. (2001a,b) satisfy Shilnikov's criterion for the existence of chaos. As a result, Barnett et al. (2022a) numerically confirmed the existence of Shilnikov chaos. Furthermore, Barnett et al. (2022b) set parameters to values obtained from the US economy and confirm that the Shilnikov chaos condition is also satisfied. These results indicate the importance of analyses on chaotic fluctuations in the actual economy.

Concerning macroeconomic policy implications, Barnett et al. (2022a) showed that in their model, an economy can be captured by a "liquidity trap" like state that emerges from chaotic fluctuations, and that the fluctuations cannot be influenced by a central bank's policy relating the interest rate to the other parts of the macroeconomic dynamics. Moreover, Barnett et al. (2022b) showed that the chaos control method proposed by Ott et al. (1990) (hereafter the OGY method) is effective against the liquidity trap brought about by chaotic fluctuations.[1]

When a chaotic attractor exists and a saddle periodic point is contained in it, the trajectories generally do not converge to the periodic point, but they enter its neighborhood recurrently due to the transitivity of chaos. Ott et al. (1990) showed that when a trajectory enters the neighborhood of its target periodic saddle point, it can be kept near that periodic point by perturbing its parameters and attempting to place the trajectory on the stable manifold of that periodic saddle point. Thus, the OGY method makes it possible to stabilize the unstable periodic orbit (see subsection 4.2 for details).

In this paper, following Farmer (1986) and Yokoo (2000), we specify the production technology as a constant elasticity of substitution (CES) type and confirm that the monetary equilibrium paths exhibit chaotic fluctuations depending on the money supply rate and the elasticity of substitution between capital and labor. We also show that chaotic equilibrium can be stabilized by applying the OGY method to the supply rule of outside money. There is a closely related paper by Bella and Mattana (2020). They also showed the existence of chaotic monetary fluctuations and the effectiveness of the OGY method in stabilizing the macroeconomy. However, there are differences

[1]As applications of the OGY method to economics, Kaas (1998) showed that the government can stabilize an unstable Walrasian equilibrium by altering income tax rates or the government's expenditure. Yokoo (2010) applied the OGY method to the population problem.

between their and our studies. The monetary equilibrium in their model depends on imperfections in the financial market in the form of credit constraints on banks, and has a more complex structure than our model. The chaotic monetary equilibrium in their analysis has not been theoretically proved to be observable. In contrast, our model structure is identical to the standard overlapping generations model (Farmer (1986) and Yokoo (2000)), and we can theoretically confirm the existence of observable monetary chaos.

2 Settings of the model

2.1 Firm and household behavior

Departing from the standard OLG model with productive capital of Diamond (1965), we add an asset (outside money) into the model. Time is discrete, extending from 0 to infinity. The representative competitive firm at the beginning of time period t exploits labor L_t and capital K_t to produce a single good which can be consumed or invested. The labor population grows at a constant rate, n, that is $L_{t+1} = (1+n)L_t$ for any t. The firm's behavior can be summarized by the following standard first-order conditions:

$$\begin{aligned} r_t &= f'(k_t) - \delta, \\ w_t &= f(k_t) - k_t f'(k_t) \equiv w(k_t), \end{aligned}$$

where $k_t = K_t/L_t$ denotes the capital–labor ratio, $f(k_t)$ the production function in the intensive (or per-capita) form, r_t the real rate of return on capital, w_t the real wage rate, $\delta \in [0,1]$ the depreciation rate of capital, and the subscript t the period of time. The production function in intensive form is specified as of the following CES (constant elasticity of substitution) type:

$$f(k) = A\left[1 - \alpha + \alpha k^{-\beta}\right]^{-\frac{1}{\beta}} = \frac{Ak}{[\alpha + (1-\alpha)k^{\beta}]^{\frac{1}{\beta}}}, \tag{1}$$

where $A > 0$, $\alpha \in (0,1)$, $\beta > -1\,(\beta \neq 0)$. The elasticity of substitution between capital and labor is given by $(1+\beta)^{-1}$. In what follows, we only consider the case where the elasticity of substitution is smaller than 1, that is, $\beta > 0$. In other words, our production technology is closer to the Leontief technology rather than the linear technology among the CES class.

Next, take a look at the household behavior. As usual, the representative agent of the household is assumed to live for two periods. She inelastically supplies one unit of labor to the labor market and earns wage income only during her young age period. We assume that the utility function of the agent born in period t is of the log-linear form:

$$u(c_t^y, c_{t+1}^o) = (1-s)\log c_t + s \log c_{t+1}^o, \tag{2}$$

where $s \in (0,1)$ is a constant, which turns out to represent the saving rate, and c_t^y denotes consumption in youth, and c_{t+1}^o denotes that in old age. The agent born in period t tries to maximize her utility given by (2) subject to the constraints:

$$w_t + x_t = c_t^y + \varsigma_t \quad \text{and} \quad (1 + r_{t+1})\varsigma_t = c_{t+1}^o,$$

where ς_t represents savings and x_t denotes the monetary transfer in real terms. By utility maximization, her optimal savings are expressed as:

$$\varsigma_t = s(w_t + x_t). \tag{3}$$

2.2 Outside money

The asset market equilibrium is then represented by

$$K_{t+1} = \varsigma_t L_t - B_t, \tag{4}$$

where

$$B_t = \frac{M_t}{P_t} \tag{5}$$

is the real value of outside money, with M_t being the nominal value of the outside money and P_t being its price. In our economy, as in Gale (1973), there is a central clearing house. Young people can accumulate (decumulate) capital by receiving goods from (handing over goods to) the clearing house, but they pay (receive) the goods when they become old. In this setting, outside money can take negative values; that is, the young sell outside money to the clearing house and buy it back from the clearing house when old. (The initial old generation has debts historically and pays the debts in terms of goods to the clearing house, and the initial young generation receives the goods by selling outside money to the clearing house). This situation is analyzed by many researchers such as Gale (1973), Grandmont (1982), and Benhabib and Laroque (1988).[2] Alternatively, following Farmer (1986), we can interpret outside money as the government's net debt. In this interpretation, negative outside money represents the situation where the private sector's asset holdings are less than the total asset holdings of the economy and the net worth held by the government is positive. IMF (2018) reported a Public Sector Balance Sheet constructed by taking a broader view of government assets and liabilities rather than standard government debt, and Figure 1.1 in the report shows that the net worth of governments in 20 of the 31 countries is positive. Thus, it would not be unrealistic to consider the case of negative outside money.

Suppose now that the policy authority such as the central bank has a policy of keeping the growth rate of outside money constant. That is, we assume that:

$$M_t = (1+\mu)M_{t-1},$$

where μ is the growth rate of outside money that can be controlled by the policy authority. The revenue from money growth (seigniorage) is distributed to the young generation as a lump sum. The aggregate money transfer is then given by

$$x_t L_t \equiv \frac{M_t - M_{t-1}}{P_t} = \mu \frac{M_{t-1}}{P_t} = \mu \frac{P_{t-1}}{P_t}\frac{M_{t-1}}{P_{t-1}}. \tag{6}$$

On the other hand, it follows from the no-arbitrage condition between productive capital and outside money that:

$$r_{t+1} = \frac{1/P_{t+1} - 1/P_t}{1/P_t} = \frac{P_t}{P_{t+1}} - 1. \tag{7}$$

Combining (6) and (7) yields:

$$x_t = \mu \left(\frac{1+r_t}{1+n}\right) b_{t-1}, \tag{8}$$

where we have used $B_t = M_t/P_t$, $b_t = B_t/L_t$, and $L_{t+1} = (1+n)L_t$.

[2]Bewley (1992) stated, "The quantity of outside money may also be negative. Negative outside money may be interpreted as demand deposits with the banking system held by the government."

2.3 The dynamic equation

By (4), we obtain

$$k_{t+1} \equiv \frac{K_{t+1}}{L_{t+1}} = \sigma_t \frac{L_t}{L_{t+1}} - \frac{B_t}{L_t}\frac{L_t}{L_{t+1}} = \frac{\sigma_t - b_t}{1+n}. \tag{9}$$

Putting (1), (3), and (8) into (9) gives:

$$k_{t+1} = \frac{1}{1+n}\left[s\left(w(k_t) + \mu\left(\frac{1-\delta+f'(k_t)}{1+n}\right)b_{t-1}\right) - b_t\right]. \tag{10}$$

As $b_t = M_t/(P_t L_t)$ for any t, we have

$$\frac{b_{t+1}}{b_t} = \frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t}\frac{L_t}{L_{t+1}} = \frac{(1+r_{t+1})(1+\mu)}{1+n},$$

which can be rewritten as

$$b_{t+1} = \frac{(1-\delta+f'(k_{t+1}))(1+\mu)}{1+n}b_t. \tag{11}$$

Shifting t by one period in (11) and substituting it into (10), we obtain:

$$k_{t+1} = \frac{1}{1+n}\left[sw(k_t) - \frac{1+(1-s)\mu}{1+\mu}b_t\right]. \tag{12}$$

Rewriting (12) gives

$$\frac{1+(1-s)\mu}{1+\mu}b_t = sw(k_t) - (1+n)k_{t+1} \equiv H(k_t, k_{t+1}). \tag{13}$$

Putting (13) into (11), we finally obtain the following second-order difference equation:

$$(1+\mu)(1-\delta+f'(k_{t+1}))H(k_t, k_{t+1}) = (1+n)H(k_{t+1}, k_{t+2}). \tag{14}$$

The OLG model given by (14) reduces to that studied by Yokoo (2000) when we let $n = \mu = 0$ and $\delta = 1$.

3 Dynamics

3.1 Steady states

For the second-order difference equation in terms of capital per capita given by (14), there are two types of potential steady states. The first type is characterized by k satisfying

$$H(k, k) = 0.$$

Evidently, $k = 0$ is always a steady state, whereas the existence of a positive steady state of this type is not necessarily ensured if the production function is given by (1). In any case, it follows from (13) that $b_t = 0$ at a steady state of this type, which means that outside money is never valued there.

The other type of possible steady state, which is more important than the first type, satisfies $H(k, k) \neq 0$. As result, it must hold that:

$$(1 + \mu)(1 - \delta + f'(k)) = 1 + n.$$

Because $f''(k) < 0$ for $k > 0$, such a steady state is unique if it exists. In this steady state, outside money is valued.

3.2 Chaotic dynamics

We want to concentrate on monetary policy, which is related to μ, so we specify non-essential parameters as follows:

$$\delta = 1 \quad \text{and} \quad n = 0,$$

which simply says that capital fully depreciates in one period and the population is constant over time.

Letting

$$x_t = k_t \quad \text{and} \quad y_t = k_{t+1}$$

turns (14) into:

$$X_{t+1} = G(X_t, \mu), \tag{15}$$

where

$$X_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad \text{and} \quad G(X_t, \mu) = \begin{pmatrix} y_t \\ sw(y_t) + (1+\mu)f'(y_t)\left(y_t - sw\left(x_t\right)\right) \end{pmatrix}.$$

Furthermore, we assume that:

$$A = \frac{1}{\alpha}. \tag{16}$$

Although economic justification is difficult with respect to the specification of this parameter, it facilitates the analysis of the model. Indeed, if $\mu = \mu_0 = 0$, then the point $\bar{X} = (1, 1)^T$, where T denotes the transpose, is always the golden-rule steady state of (15), i.e., $G(\bar{X}, \mu_0) = \bar{X}$, under the condition (16) plus $n = 0$ and $\delta = 1$, irrespective of the other parameters.

It is known from Yokoo (2000) that when β in (1) is sufficiently large and s is sufficiently close to 0, this steady state of (15) is a hyperbolic saddle point to which a transverse homoclinic orbit exists (hence the existence of Smale's horseshoe). Furthermore, it is demonstrated that a chaotic attractor is generated by a homoclinic bifurcation for (15). This ensures, by Mora-Viana's Theorem (Mora and Viana (1993), also Palis and Takens (1993) for technical details), the observability of chaotic motions in the long run for a large set of parameter values. See Yokoo (2000) for details. These theoretical results support the existence of the chaotic attractors seen in the numerical calculations, as shown in Figure 1.

Insert Figure 1 around here.

Figure 1 plots a trajectory generated by (15) for a set of empirically plausible parameter values:

$$A = 2, \quad \alpha = 0.5, \quad \beta = 8, \quad s = 0.1, \quad \text{and} \quad \mu = 0. \tag{17}$$

Some comments are in order. The parameter values except for β are commonly assumed in the literature. We mention the validity of $\beta = 8$. From the theoretical and practical viewpoints, the value of the elasticity of substitution between capital and labor is assumed to be around 1. Indeed, in the SIGMA model of the Federal Reserve Board, the value is set to 0.9. Gechert et al. (2022) recently carried out a meta-analysis based on 3186 observations of the elasticity of substitution from 121 previous works and observe that the mean of the elasticity of substitution is 0.3. Therefore, the assumption that $\beta = 8$, which means the elasticity of substitution $1/(1+\beta) = 0.11$, is not necessarily unrealistic. One can observe an attractor that seems to be a so-called Hénon-like strange attractor, in Figure 1. Actually, the transient motions are omitted in this Figure. The golden-rule steady state, which is a hyperbolic saddle, also seems to be contained in the attractor.

4 Controlling methods: a policy comparison

4.1 Fixed stabilizing monetary policy

Suppose now that a monetary policy authority such as a central bank has strong independence from the government and public opinion and is free to determine the growth rate of money. If the economy has persistent fluctuations and does not tend to converge to a steady state as shown in Figure 1, to what value should the central bank steer the growth rate of money?

To get a rough idea of the growth rate of money and the corresponding changes in its dynamic properties, for the set of parameter values given by (17), let us assume that the other parameters except μ are constant, and let the parameter μ be moved within a "realistic" range, and plot the corresponding bifurcation diagram in Figure 2.

Insert Figure 2 around here.

According to Figure 2, for the economy to have a natural tendency to converge to a steady state, the growth rate of money must be fixed at around -30 percent ($\mu \approx -0.3$) at least! However, this is totally impractical. In exchange for economic stability, we must accept the tradeoff of the virtual disappearance of money.

4.2 Fine-tuning monetary policy: the OGY method

In the previous section, we have seen that there is a tradeoff between the natural stabilization of the economy and the maintenance of the value of money. In this section, we examine how to resolve this dilemma. In other words, by exploiting the nature of chaos, we would like to stabilize the economy and maintain the value of the outside money at the same time. For this purpose, let us assume the following situation as described in Figure 1.

- There exists a golden-rule steady state, which is a hyperbolic saddle point. That is, there exist a (local) stable manifold and an unstable manifold of that point;

- The golden-rule steady state is contained in a chaotic attractor;
- The growth rate of money is manipulable by the authority such as the central bank, but its variability is limited to a very narrow range.

The OGY method is known as the current classical theory of chaos control. Let us briefly explain the theory based on the model used in this paper. From the situation in Figure 1, the golden-rule steady state, which is a saddle point, is contained in a chaotic attractor, but the dynamics on the attractor is transitive; in other words, it has a dense orbit on its invariant set, so that for any neighborhood of the steady state, the economic state will enter it at some point in time. At that time, the position of the steady state and its stable manifold are changed by moving the control parameter (in this case, the growth rate of money) so that the economic state at the next time point is on (near) the stable manifold of the steady state. In reality, due to errors, the state will deviate from the stable manifold, but by performing this operation in succession, the economic state can be brought closer to the target golden-rule steady state. The stable manifold of the steady state itself cannot be calculated exactly, but by keeping the neighborhood of the target sufficiently small, the error due to the first-order approximation can be reduced.

Let us now consider controlling the system (15) using the OGY method. The presentation here is based on Lai and Grebogi (1993). The money growth rate that is the basis for control is set to zero, i.e., $\mu = \bar{\mu} = 0$. Let μ_t be the money growth rate at time t. Remember that the golden-rule steady state $\bar{X}$ is given by

$$\bar{X} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Linearizing (15) around $(X_t, \mu_t) = (\bar{X}, \bar{\mu})$ yields

$$X_{t+1} - \bar{X} = J(X_t - \bar{X}) + B(\mu_t - \bar{\mu}), \tag{18}$$

where

$$J = D_X G(\bar{X}, \bar{\mu}) \quad \text{and} \quad B = D_\mu G(\bar{X}, \bar{\mu})$$

are a 2×2 Jacobian matrix and a 2×1 vector evaluated at $(\bar{X}, \bar{\mu})$, respectively. Some computation[3] shows that

$$J = \begin{pmatrix} 0 & 1 \\ -s\gamma & 1 + \gamma(s - \alpha)/\alpha \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 - s(1 - \alpha)/\alpha \end{pmatrix},$$

where

$$\gamma = (1 - \alpha)(1 + \beta).$$

Consider the eigenvalues of J, that is the solutions of

$$|J - \lambda I| = 0,$$

where I is the identity matrix of 2×2. One can check that for a sufficiently large $\beta > 1$ and for a sufficiently small $s > 0$, J has two real eigenvalues λ^s and λ^u, such that

$$\lambda^u < -1 < \lambda^s < 0.$$

This implies that the golden-rule steady state is a (hyperbolic) saddle point for such parameter values. Let us call λ^s the stable eigenvalue and λ^u the unstable eigenvalue of J. The corresponding stable/unstable eigenvectors $v^{s,u}$ are calculated as:

$$v^{s,u} = \begin{pmatrix} 1 \\ \lambda^{s,u} \end{pmatrix}.$$

Because the local stable manifold of $\bar{X}$ is tangent to the stable eigenspace at $\bar{X}$, we want to put X_{t+1} onto the stable eigenspace of $\bar{X}$ with X_t as given, using the linearized system (18). To do this, we need to find the contravariant vectors $e^{s,u}$ of $v^{s,u}$, respectively. These vectors are calculated as follows. First, let P be a matrix defined as

$$P = (v^s \quad v^u) = \begin{pmatrix} 1 & 1 \\ \lambda^s & \lambda^u \end{pmatrix}.$$

Then its inverse is given by

$$P^{-1} = \frac{1}{\lambda^u - \lambda^s} \begin{pmatrix} \lambda^u & -1 \\ -\lambda^s & 1 \end{pmatrix} \equiv (e^s \quad e^u)^T.$$

Note that

$$e^s \cdot v^s = e^u \cdot v^u = 1 \quad \text{and} \quad e^s \cdot v^u = e^u \cdot v^s = 0.$$

[3]Notice that $f(1) = 1/\alpha$, $f'(1) = 1$, $w(1) = 1/\alpha - 1$, and $w'(1) = -f''(1) = \gamma = (1 - \alpha)(1 + \beta)$.

Because $e^u \cdot v^s = 0$, we can apply the orthogonal condition to (18) to obtain

$$(X_{t+1} - \bar{X}) \cdot e^u = 0.$$

Thus, some calculations show that when the trajectory of the economy enters the vicinity of the golden-rule steady state in time t, the value of the money growth rate should be determined according to the following rule:

$$\begin{aligned} \mu_t &= \bar{\mu} - \frac{J(X_t - \bar{X}) \cdot e^u}{B \cdot e^u} \\ &= \frac{s\alpha\gamma(x_t - 1) + (\alpha\lambda^s - \alpha + \alpha\gamma - s\gamma)(y_t - 1)}{\alpha - s(1-\alpha)}, \end{aligned} \tag{19}$$

where

$$\lambda^s = \frac{1}{2}\left[1 - \gamma + \frac{s\gamma}{\alpha} + \sqrt{\left(1 - \gamma + \frac{s\gamma}{\alpha}\right)^2 - 4s\gamma}\right].$$

5 Controlling chaotic fluctuations

In this section, we are going to exploit the OGY method developed in the previous section to control chaotic or periodic fluctuations.

5.1 A case with no external shocks

We will first consider the case where there are no external shocks to the economy. In this case, the economy is completely described by (15). The parameters are given by (17) so that the economy exhibits chaotic dynamics with the target steady state contained in the chaotic attractor as depicted in Figure 1. Once the trajectory of (15) enters a given neighborhood of the steady state of the target, the OGY control procedure, is activated. That is, if

$$||X_t - \bar{X}|| \equiv \sqrt{(x_t - 1)^2 + (y_t - 1)^2} < \varepsilon$$

for some $\varepsilon > 0$, then the money growth rate μ_t is determined by the rule (19).

Insert Figures 3 and 4 around here.

Figure 3 shows a chaotic time series corresponding to Figure 1 without control. On the other hand, Figure 4 shows the time series when control is started at T_s $(t = 500)$ and stopped at T_e $(t = 2,500)$. It can be seen that the trajectory is completely controlled to the target steady state a short time after the start of control, and chaotic behavior appears again when the control is released.

Now let us see how the money growth rate is adjusted during the control. See Figure 5 for this. Because the values of μ_t are stuck at 0 for most of the time, it is only displayed for 50 iterations from the start of the control. The simulation shows that the control procedure is activated after only a few iterations from the start of control and that the money growth rate is immediately reduced to a value close to zero as the system is quickly stabilized.

Insert Figure 5 around here.

5.2 A case with external shocks: when the target is far away from the attractor

When simulating mathematical models that produce chaotic dynamics, it is common to observe periodic windows between chaotic behaviors in the bifurcation diagram. This is also the case for our model. Let us change β from $\beta = 8$ to $\beta = 7$, holding the other parameters unchanged. Figure 6 shows an attractor on the plane generated by (15) and Figure 7 plots the bifurcation diagram with respect to μ.

Insert Figures 6 and 7 around here.

In this example, orbit stabilization by the OGY method fails because the attractor is too far away from the given neighborhood of the target steady state. See Figure 8. We then consider a situation in which the economy is subjected to an external shock. We demonstrate below that such shocks, or stochastic perturbations, might contribute to stabilization.

Insert Figure 8 around here.

To do this, we add some stochastic perturbations to (15). To be more specific, we consider:

$$X_{t+1} = G(X_t, \mu) + \sigma \begin{pmatrix} 0 \\ \epsilon_t \end{pmatrix}, \quad \sigma \geq 0, \tag{20}$$

where ϵ_t is a random variable that follows an independent and identical distribution. For simplicity, a uniform distribution on the interval $[-1, 1]$ is assumed.

Figures 9 and 10 show the trajectory plots generated by (20) for $\sigma = 0.01$ and $\sigma = 0.05$, respectively.

Insert Figures 9 and 10 around here.

Compare the bifurcation diagram, Figure 11, corresponding to Figure 9 with that with no external shocks, Figure 6. Most of the "windows" in the bifurcations diagram seem to have disappeared due to rather small external shocks.

Insert Figure 11 around here.

Interestingly, the influence of stochastic perturbations reveals a hidden chaotic structure, which, as a result, makes it possible to capture the trajectory in the neighborhood of the target. In fact, the OGY method succeeds in stabilizing the system in such situations. See Figures 12 and 13. The corresponding money growth rate movements are shown in Figures 14-16. Figure 16 is an enlargement of Figure 15. Unlike the case with no external shocks, fine tuning of monetary policy in response to external shocks is complicated. The amplitude of the money growth rate, however, is relatively small in this example, although it depends, of course, on the magnitude of the noise.

Insert Figures 12 and 13 around here.

Insert Figures 14, 15, and 16 around here.

6 Concluding remarks

This paper applied the OGY method proposed by Ott et al. (1990) to policy making in macroeconomics. This paper demonstrated that the monetary equilibrium paths exhibit two-dimensional chaotic fluctuations depending on the money supply rate and the elasticity of substitution between capital and labor under the assumption of a constant elasticity of substitution (CES) production function. We showed that chaotic outside monetary equilibrium paths can be controlled to the outside money steady state by the OGY method. We also found numerically that moderate stochastic shocks can contribute to such stabilization.

This paper stabilizes the economy by controlling the capital–labor ratio. Note that because there is a smooth one-to-one relationship between the interest rate and the capital-labor ratio, the stabilization policy here can also be interpreted as controlling the interest rate.

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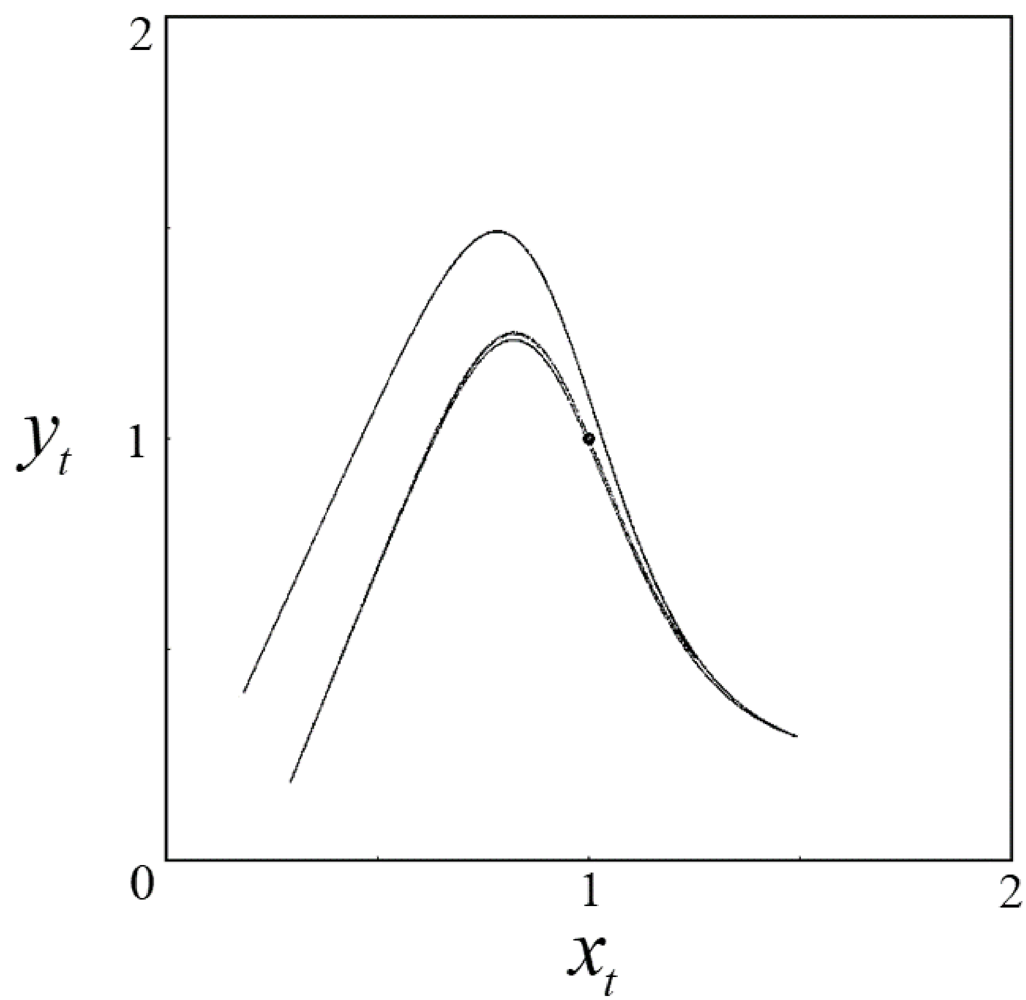

Figure 1: Chaotic attractor on the plane. The golden rule steady state $(1, 1)$ is highlighted. The transients are omitted. $A = 2, \alpha = 1/2, \beta = 8, s = 0.1$, and $\mu = 0$.

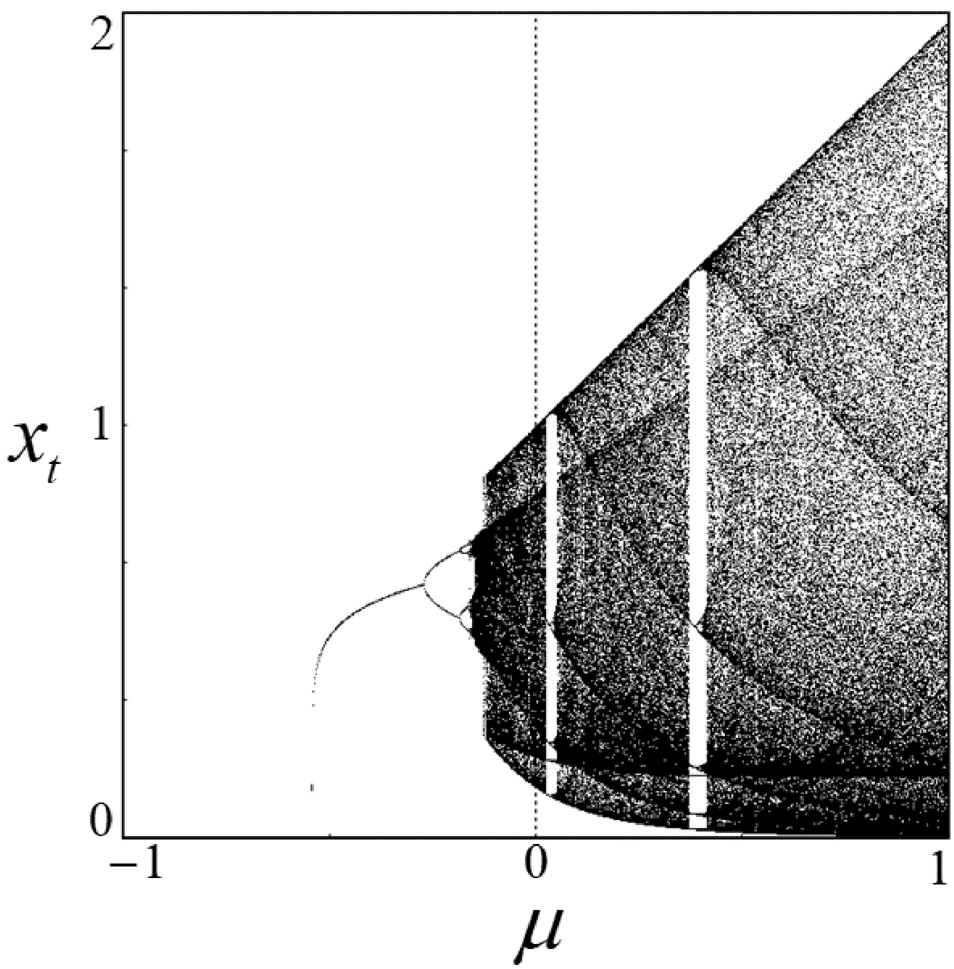

Figure 2: Bifurcation diagram. The parameter μ, the growth rate of money, moves from -1 to 1 $A = 2, \alpha = 1/2, \beta = 8$, and, $s = 0.1$. Money loses value if μ falls below a certain value. For some range of the value of μ, which is negative, the golden rule steady state is an attractor, whereas chaotic fluctuations are observed for a much larger set of μ values.

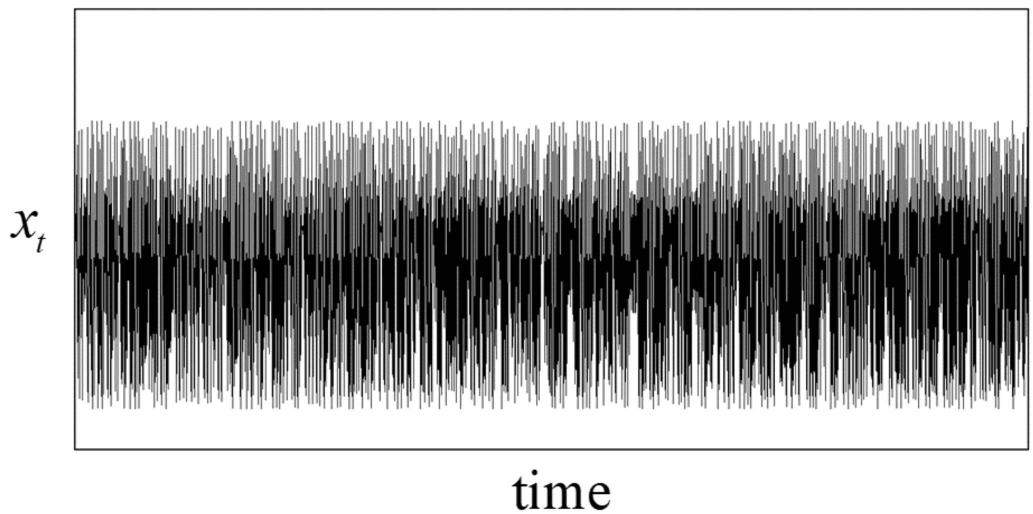

Figure 3: Uncontrolled chaotic time series. $A = 2, \alpha = 1/2, \beta = 8, s = 0.1$, and $\mu = 0$.

Figure 4: Controlled chaotic time series. It takes some time for the chaotic trajectory to enter the vicinity of the target. As soon as the control is aborted, chaotic behavior reappears. $A = 2, \alpha = 1/2, \beta = 8, s = 0.1$, and $\varepsilon = 0.1$. $t \in [0, 3 \times 10^3]$. Control starts at $T_s = 500$ and stops at $T_e = 2,500$.

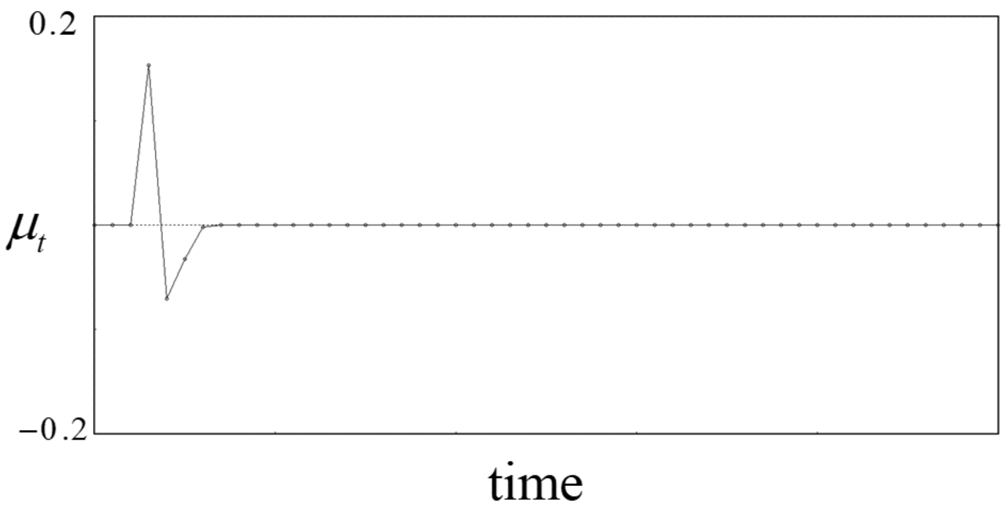

Figure 5: The values of μ_t stick to zero by 10 steps from the start of control. $A = 2, \alpha = 1/2, \beta = 8, s = 0.1$, and $\varepsilon = 0.1$. $t \in [500, 550]$.

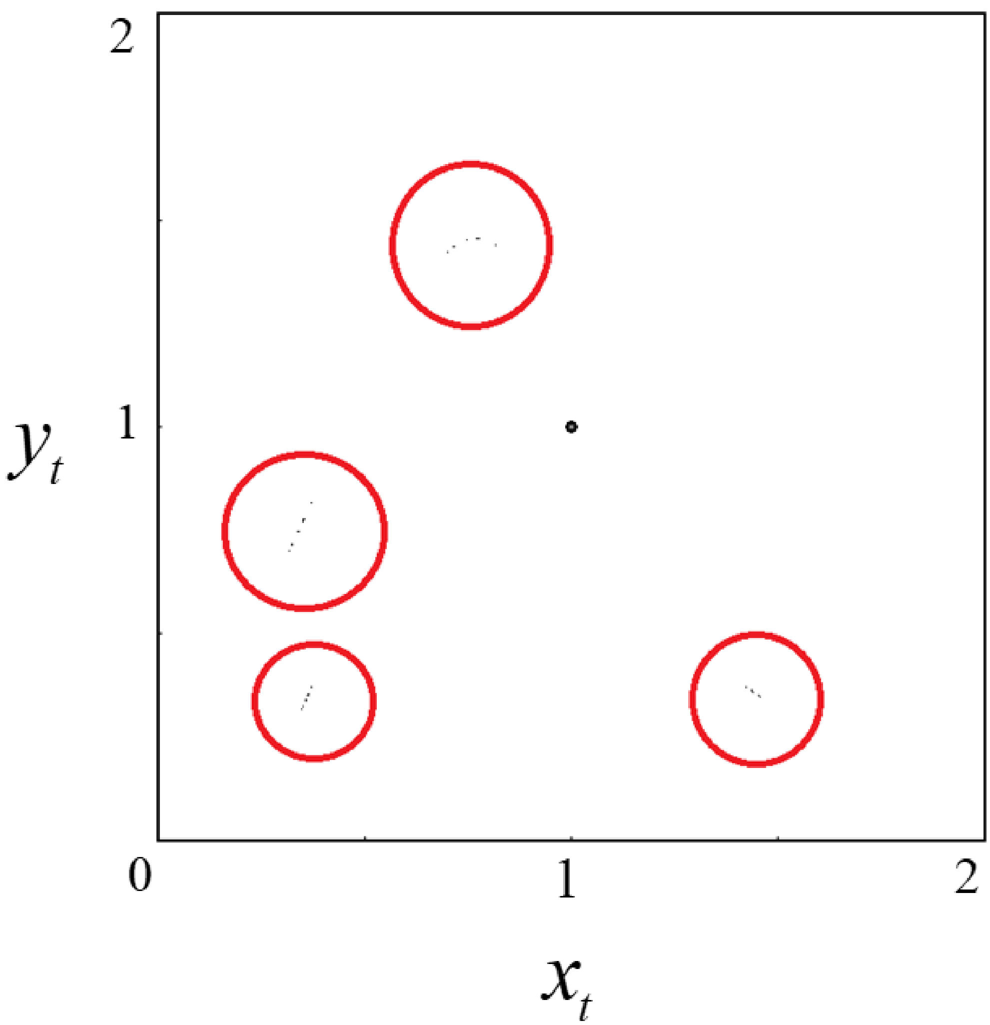

Figure 6: The target steady state is far from the attractor. The attractor is contained in the red circles. The transients are omitted. $A = 2, \alpha = 1/2, \beta = 7, s = 0.1$, and $\mu = 0$.

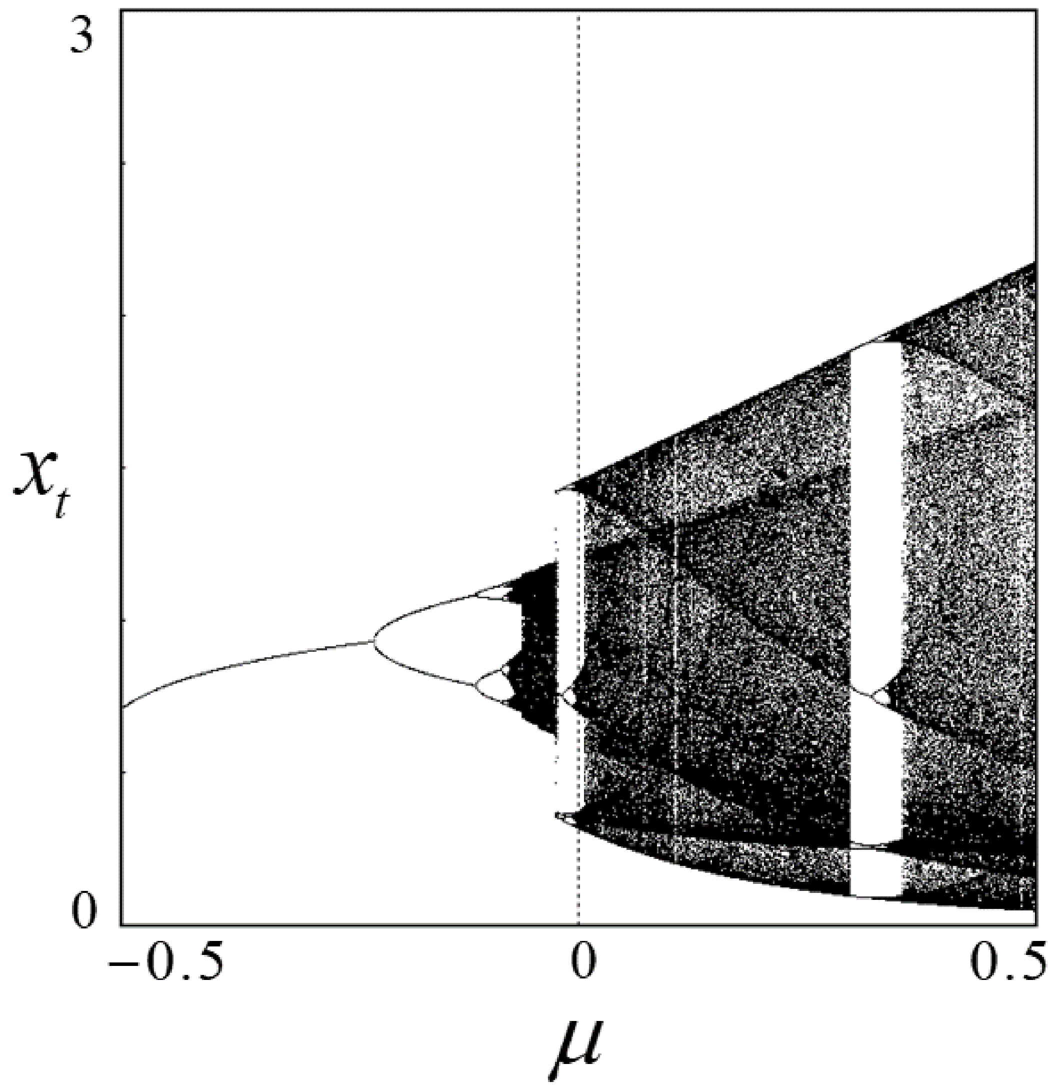

Figure 7: Bifurcation diagram with respect to μ. $A = 2, \alpha = 1/2, \beta = 7$, and $s = 0.1$.

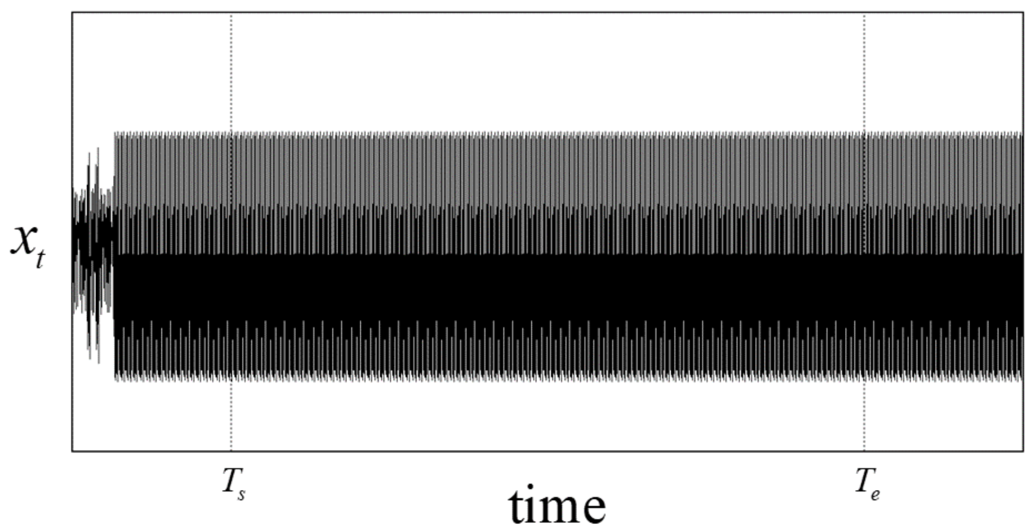

Figure 8: Time series. The OGY method fails to stabilize the system. $A = 2, \alpha = 1/2, \beta = 7, s = 0.1$, and $\varepsilon = 0.1$.

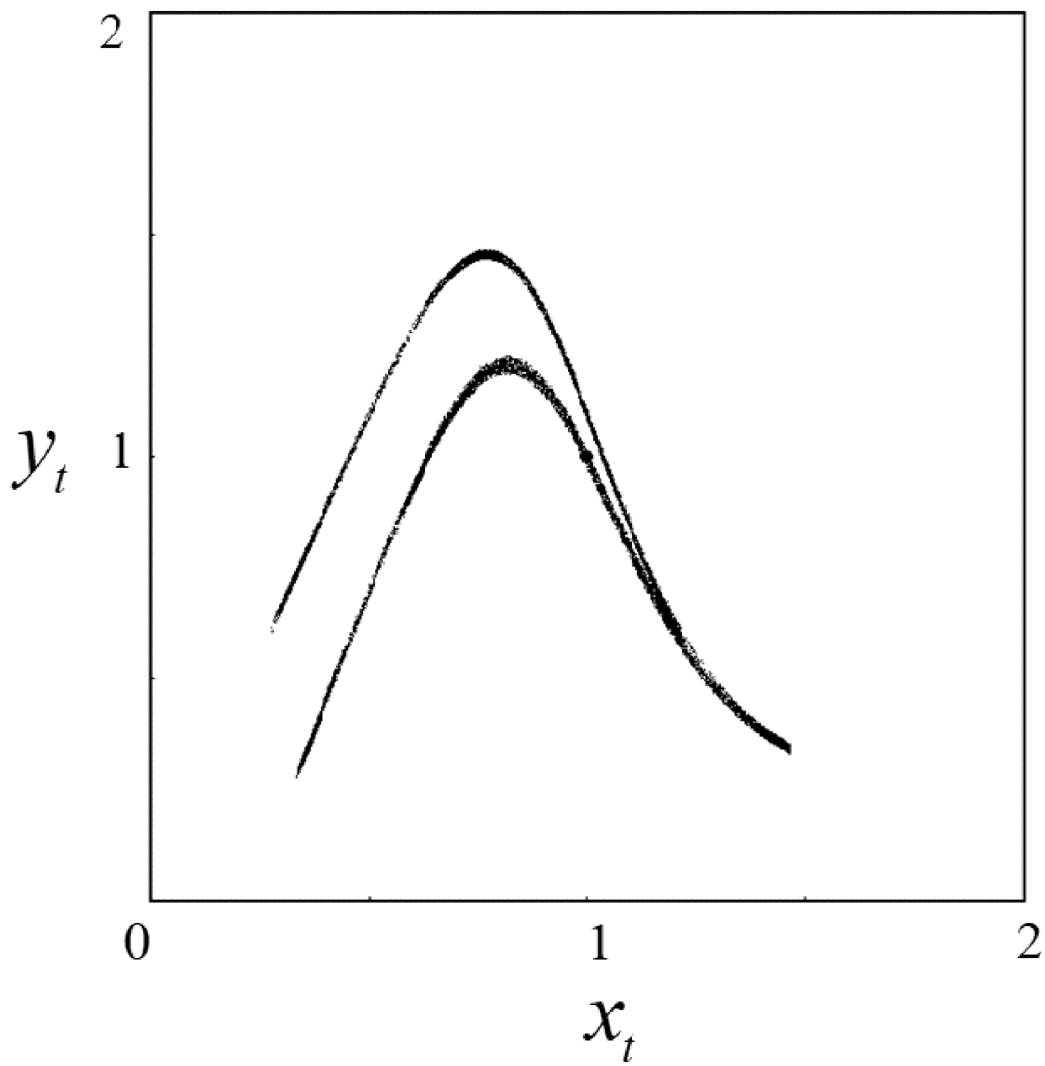

Figure 9: Noisy chaotic "attractor". The influence of stochastic perturbations reveals the hidden chaotic structure. $\sigma = 0.01$.

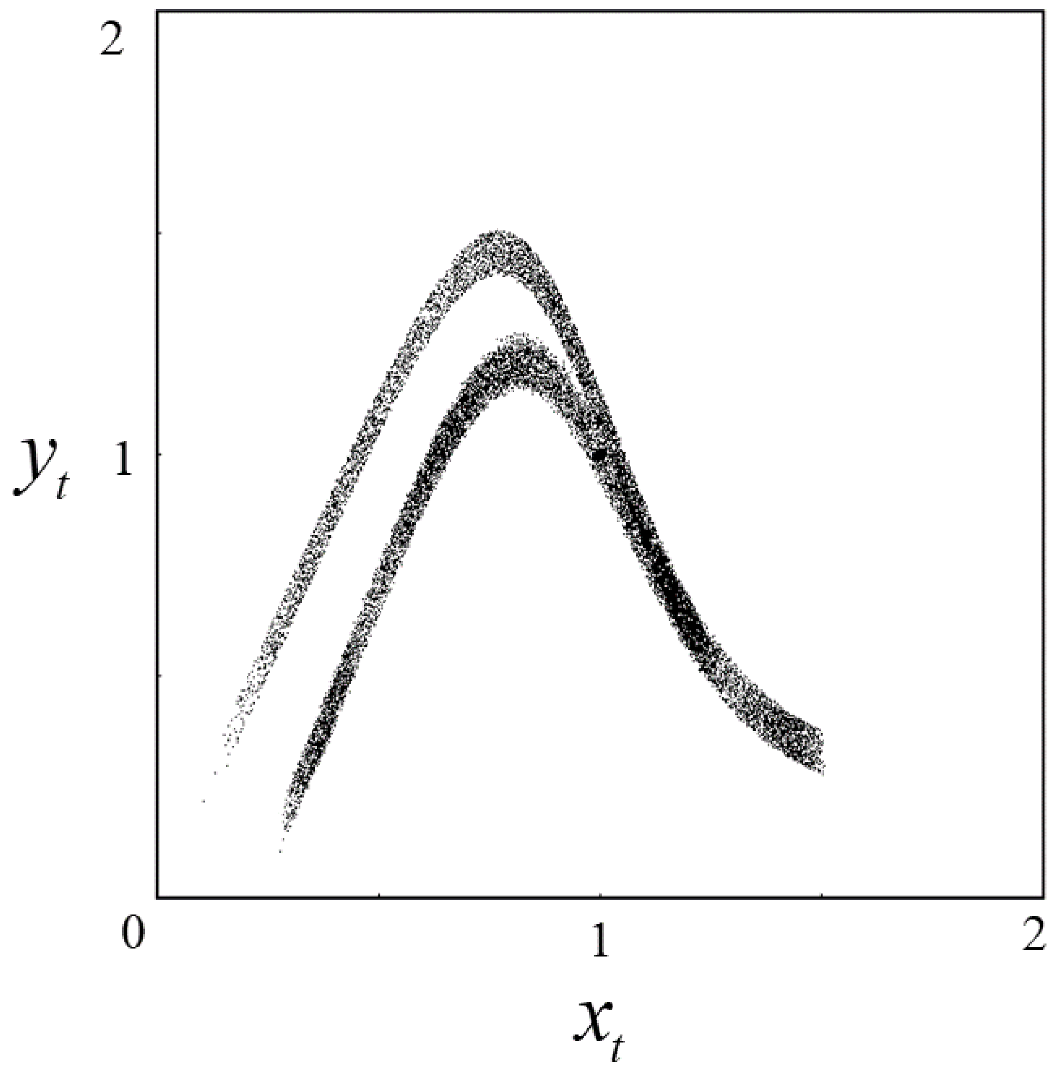

Figure 10: Noisy chaotic "attractor". The influence of stochastic perturbations reveals the hidden chaotic structure. $\sigma = 0.05$.

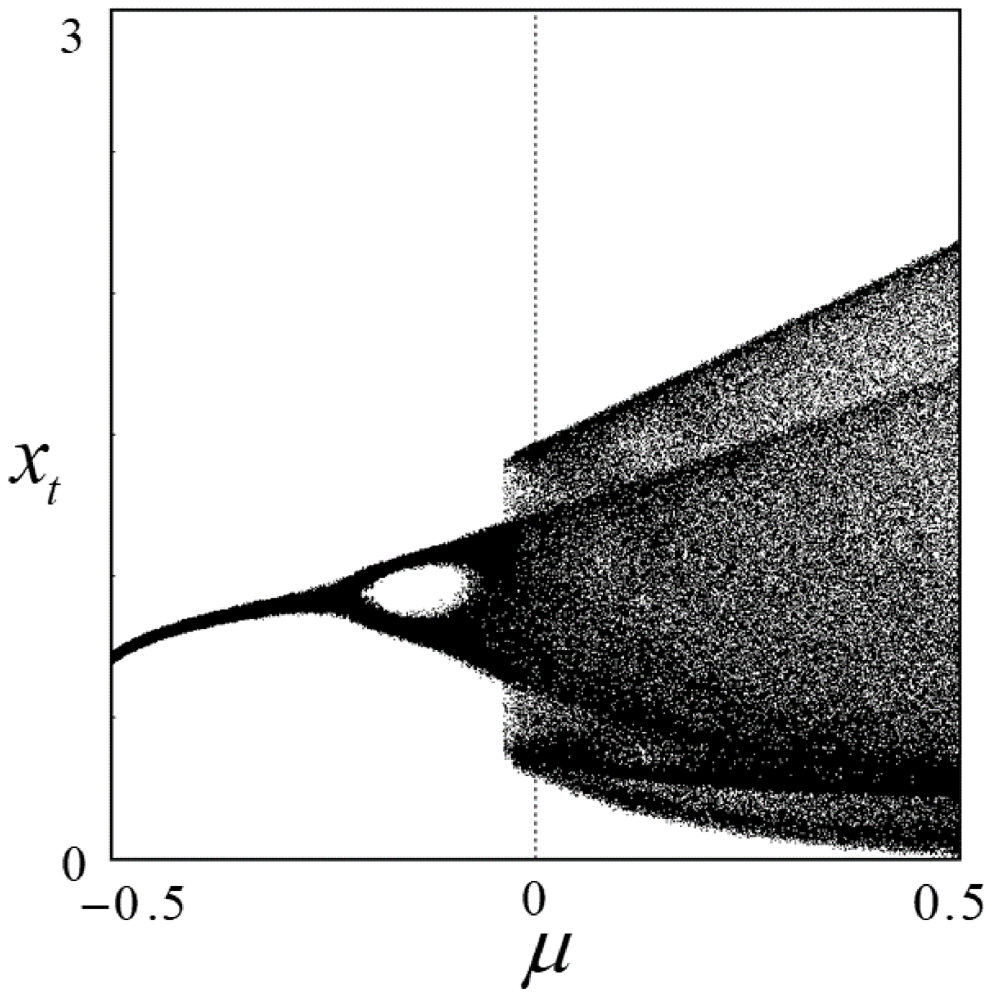

Figure 11: Compared with Figure 7, most of the windows have disappeared due to small external shocks. $\sigma = 0.01$.

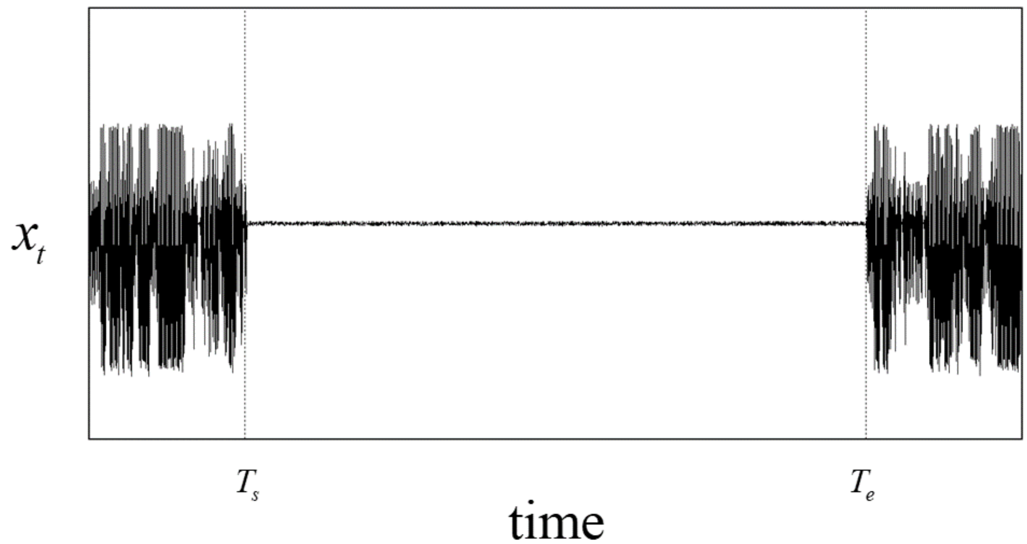

Figure 12: Time series. The OGY method succeeds in stabilizing the system subjected to external shocks. $\sigma = 0.01$.

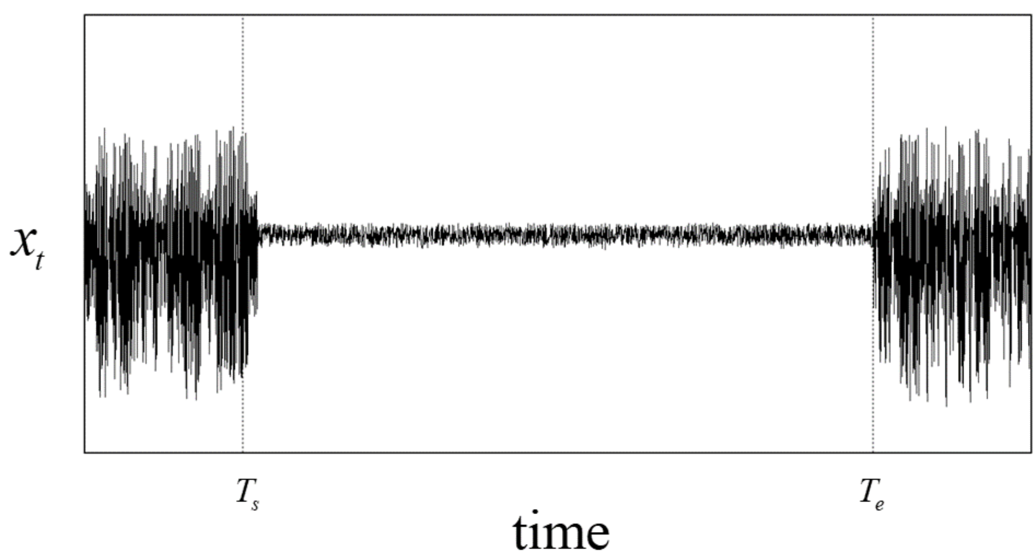

Figure 13: Time series. The OGY method succeeds in stabilizing the system subjected to external shocks. $\sigma = 0.05$.

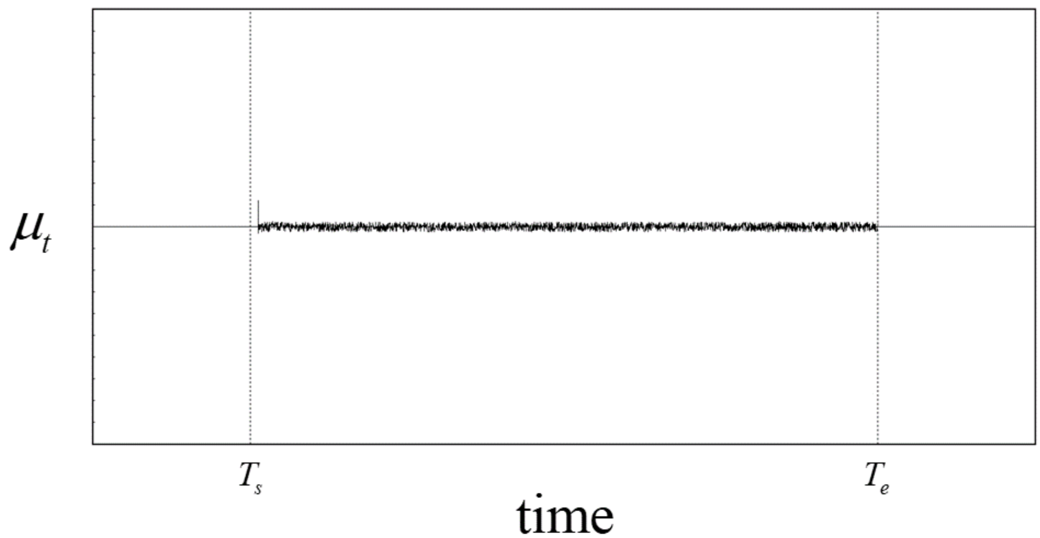

Figure 14: Monetary policy dynamics: time series of μ_t. $t \in [0, 3 \times 10^3]$. $\sigma = 0.01$.

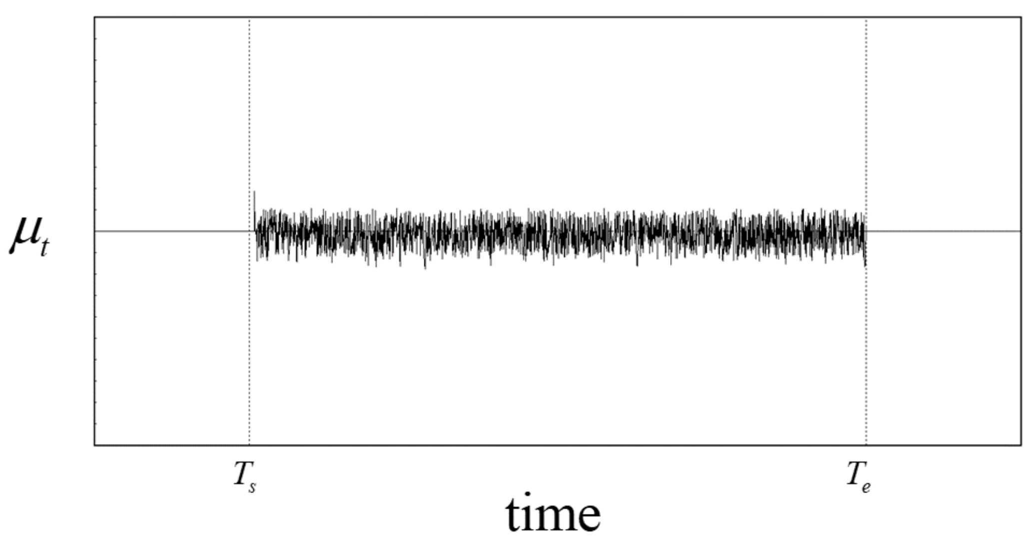

Figure 15: Monetary policy dynamics: time series of μ_t. $t \in [0, 3 \times 10^3]$. $\sigma = 0.05$.

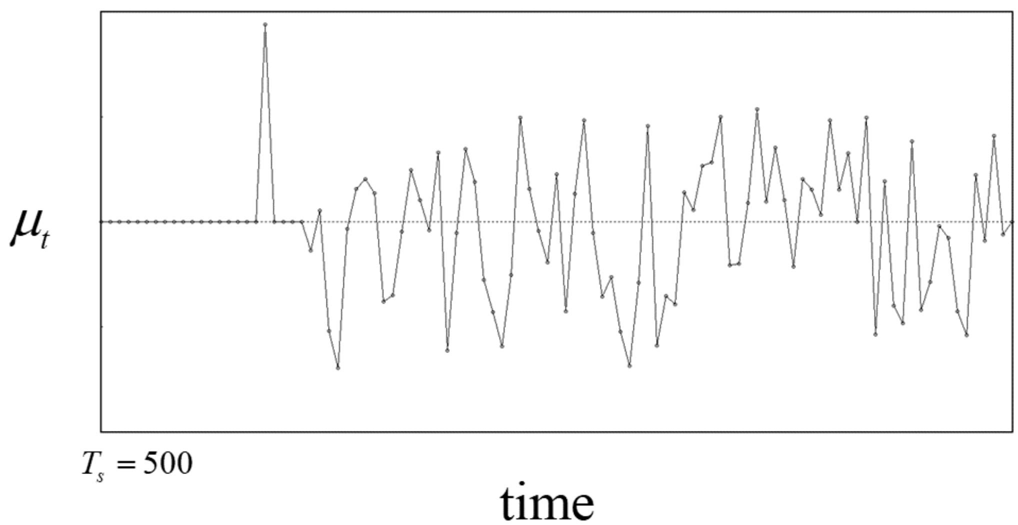

Figure 16: Closeup of Figure 15. $t \in [500, 600]$. $\sigma = 0.05$.