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a Chaotic Middle-Income Trap"

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Technology Choice, Externalities in Production, and a Chaotic Middle-Income Trap*

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Abstract

We incorporate the external effects of capital in production and endogenous technology choice into the standard overlapping generations model. We demonstrate that our model can exhibit a poverty trap, a middle-income trap, and perpetual growth paths. We also show that, under some economic conditions, an economy exhibits all three of these phenomena, depending on its initial capital level, and that the economy caught in the middle-income trap can exhibit chaotic fluctuations in the long run. In obtaining these results in the standard overlapping generations model, the combination of technology choice and externalities in production plays a crucial role.

Keywords: External effect; Technology choice; Overlapping generations model, Middle-income trap; Chaos

*This is a revised version of our preprint, Kyoto Institute of Economic Research (KIER) Discussion Paper No.1075, which is referred to in References (Asano et al. (2022c)).

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1 Introduction

In the real world, continued fluctuations in macroeconomic variables, such as GDP (gross domestic product), investment, and consumption have been observed. These phenomena are called business cycles, and are one of the main research topics in macroeconomics. By extending a standard textbook model (Diamond, 1965) in two directions, external effects and technology choice (as discussed below in detail), this paper explains, in a unified manner, various patterns of business cycles, including complex dynamics that have been considered in the literature.

In the relevant literature, theories attempting to explain business cycle phenomena can be broadly divided into two categories: exogenous and endogenous business cycle theories. Exogenous business cycle theory attributes the fundamental source of economic fluctuations to stochastic shocks. Exogenous business cycle theory, especially the dynamic stochastic general equilibrium approach, has played a dominant role in business cycle research for decades. However, after the global financial crisis, interest in endogenous business cycle theory has been renewed (e.g., Beaudry et al., 2020, Deng et al., 2021, and Schmitt-Grohé and Uribe, 2021).[1] Almost simultaneously, theoretical research on the complexity of business cycle fluctuations has been gaining momentum (e.g., Matsuyama 2013 and Matsuyama et al. 2016). To further investigate the roles of nonlinearity, this paper adopts the endogenous business cycle approach.

In this line of research, the role of technology choice in endogenous business cycles has attracted much attention. From the end of the 1990s, it has been analyzed by several authors, including Aghion et al. (1999), Iwaisako (2002), and Matsuyama (2007). These studies showed the possibility of various patterns of dynamics in their models. However, they basically relied on graphical analysis and did not characterize the properties of the equilibrium dynamics in detail. Mathematically rigorous characterizations have recently been made by Asano et al. (2012), Asano et al. (2022b),

[1]In endogenous business cycle theory, economic fluctuations occur spontaneously as a result of nonlinear factors within the economy, without any shocks from outside the economy. Early studies in this line include Benhabib and Day (1982), Grandmont (1985), Benhabib and Nishimura (1985), and Nishimura and Yano (1995).

Matsuyama et al. (2016, 2018), Sushko et al. (2014, 2016) and Umezuki and Yokoo (2019a). These studies assumed neoclassical, constant returns to scale technologies.[2] In reality, however, as Caballero and Lyons (1990) and many other studies have found,[3] there would be *external effects*[4] in production, especially in manufacturing. This fact contradicts the assumption of constant returns to scale. Thus, the role of the external effects in business cycles should be considered.

Externalities or external effects play a significant role in economics, particularly in theories of economic growth and urban economics.[5] Externalities that occur in the accumulation of knowledge obtained by firms or workers (knowledge externalities) are important for firms or countries to grow in the long run. The existence of these external effects allows the cases of increasing marginal productivity of capital or increasing returns to scale, which, combined with technology choice, can be a source of a rich variety of complex dynamics. The role of increasing returns to scale has long been analyzed in the field of international trade (Negishi, 1969). Since the 1990s, the role of increasing returns has attracted attention in various fields. For example, studies in the field of economic growth have shown that increasing returns to scale (or external effects) are an engine of long-run economic growth (Romer, 1986; Lucas, 1988), and they have become one of the foundations of modern economic growth theory. Furthermore, the field of urban economics has shown that increasing returns to scale underlie the phenomenon of urban agglomeration (Fujita and Thisse, 1996; Fujita et al., 1999). For example, Fujita and Thisse (1996) stated that "We can therefore safely conclude that increasing returns to scale are essential for explaining the geographical distribution of economic activities." In the current urban economics, increasing returns to scale have become one of its fundamental components.

[2] Iwaisako (2002) is an exception. He considered two possible technologies: a constant returns to scale technology and an increasing returns to scale technology. However, he relied exclusively on graphical analysis.

[3] For example, see Baxter and King (1991), Caballero and Lyons (1992), and Lindström (2000).

[4] In general, external effects or externalities mean that the action of an agent affects other agents' costs or benefits without going through the market transactions. For example, knowledge accumulation has positive externalities for society (e.g., newly obtained mathematical theorems are freely available to everyone). On the other hand, crime is an example of a negative externality with social costs.

[5] For example, see Klenow and Rodríguez-Clare (2005) and Moretti (2004).

Constructing an overlapping generations (OLG) model with external effects and two technologies (one of which is chosen endogenously), we show that our model can generate several growth patterns, including a poverty trap, a middle-income trap, and a perpetual growth path, and that under certain economic environments, an economy exhibits each of these three growth patterns depending on the initial capital level. A poverty trap is an economic development situation in which a low-income country cannot escape poverty in the long run.[6] In this study, we use the term middle-income trap to indicate a situation in which a country that has achieved a certain income in the middle-income category becomes stuck at that level.[7] A perpetual growth path is defined as one in which a country moves toward a high-income category and continues to grow in the long run. In fact, we show that if the external effect is mildly large in at least one technology, enough to generate a slight increase in the marginal productivity of capital, then the economy can exhibit chaotic business cycles.[8] Under the standard Cobb–Douglas technologies, in which externalities are absent, whenever we observe long-run fluctuations, they are almost certainly periodic, as shown in Umezuki and Yokoo (2019a). It should be emphasized that in obtaining long-run chaotic fluctuations in the Diamond model with a Cobb–Douglas technology choice, the introduction of externalities in production plays a crucial role.

It should be noted that our model is a so-called piecewise smooth model. Complex dynamics of piecewise smooth models have been analyzed in the literature. For example, Gardini et al. (2008) and Matsuyama et al. (2016) adopt the relatively new theory of border-collision bifurcation to prove that their macroeconomic models can exhibit chaotic behaviors. Fortunately, our model can be transformed into a piecewise linear model, which is a subclass of piecewise smooth models. In general, piecewise linear models provide clearer and sharper analytical results of complex

[6] For example, see Azariadis and Stachurski (2005) for a survey.

[7] As a recent theoretical work, Hu et al. (2022) show that the degree of externalities plays a significant part in technology choice and makes it possible to explain the empirically observed development patterns (a poverty trap, a middle-income trap, and a flying geese pattern of economic development) in a unified way. See also Asano et al. (2022a) and references therein for details.

[8] If both external effects of the two technologies are sufficiently small, our model can exhibit periodic fluctuations, which have been extensively studied, for example by Ishida and Yokoo (2004), Asano et al. (2012), and Umezuki and Yokoo (2019a).

dynamics than piecewise smooth models, which is adopted in our paper.

The remainder of this paper is organized as follows. Section 2 presents the settings of our model. Section 3 provides the main results, and Section 4 concludes this paper. Most of the proofs are relegated to appendices.

2 Settings of the model

This section describes the structure of our model, in terms of a household's and firm's behavior and the equilibrium dynamics.

2.1 Household's behavior

The basic setup follows the standard Diamond-type OLG model. Time is discrete and extends from 0 to infinity. Population is assumed to be constant over time and normalized to 1. Each generation lives two periods, supplying one unit of labor inelastically only when young. The household maximizes its Cobb–Douglas utility according to the following problem:

$$\max_{c_t^y, c_{t+1}^o, s_t} (1-s)\log c_t^y + s \log c_{t+1}^o, \qquad s \in [0,1]$$
$$\text{s.t.} \qquad s_t + c_t^y = w_t, \qquad c_{t+1}^o = r_{t+1} s_t.$$

Here, c_t^y denotes the consumption when young, c_{t+1}^o denotes the consumption when old, s_t denotes savings, w_t denotes the real wage rate, r_{t+1} is the real rate of return on the loan maturing at $t+1$, and the subscript t denotes time. Utility maximization implies that:

$$s_t = s w_t.$$

2.2 Firm's behavior

We introduce two additional factors into our model: externalities in production and multiple technologies. The firm is assumed to behave both as an owner and as a manager.[9] This economy has two available production technologies. We assume that

[9] Regarding another possible interpretation, we may assume that the firm chooses its production technology in a discrete manner in the first stage and then chooses optimal inputs in the second stage.

the firm, as the owner, has to choose one technology that maximizes the return on capital, whereas the firm manager attempts to maximize the firm's profit, which is driven away by competition. To capture our idea in the simplest possible settings, we employ Cobb–Douglas technologies, as follows:

$$F_i(k, K, L) = A_i k^{\eta_i} K^{\alpha_i} L^{1-\alpha_i},\ i \in \{1, 2\},\ A_i > 0,\ \alpha_i \in (0, 1),\ \text{and}\, \eta_i \geq 0,$$

where subscript i denotes the i-th technology, K denotes capital, L denotes labor, and k is the capital–labor ratio. Each manager regards k as given. This formulation follows that of Azariadis and Reichlin (1996).[10] The first argument of F_i is related to externalities. If $\eta_i > 0$, then positive externalities, such as knowledge spillovers, exist in production. If $\eta_i = 0$, then externalities are absent, and F_i is a standard Cobb–Douglas production function. To avoid unnecessary complications, we ignore the case of negative externalities, that is, $\eta_i < 0$. Given the first argument in F_i and $L = 1$, in a symmetric equilibrium, competition implies the following first-order conditions:

$$\begin{aligned} r_t &= \frac{\partial F_i(k_t, k_t, 1)}{\partial K_t} \equiv r_i(k_t) = \alpha_i A_i k_t^{\eta_i+\alpha_i-1}, \qquad (1) \\ w_t &= \frac{\partial F_i(k_t, k_t, 1)}{\partial L_t} \equiv w_i(k_t) = (1-\alpha_i) A_i k_t^{\eta_i+\alpha_i}. \end{aligned}$$

Thus, the shape of the marginal productivity of capital depends on the value of $\eta_i+\alpha_i$. Note that $r(k)$, given by (1), is an increasing function with respect to k if the external effect is sufficiently large, that is, $\eta + \alpha > 1$.

Upon entering the market, the representative firm's owner in period t, who was born in period $t-1$, chooses a technology that, given k_t, yields the highest return in

[10] Several studies measure external effects by estimating the percentage increase in a firm's output caused by a 1% increase in aggregate inputs (or aggregate output), keeping an individual firm's inputs unchanged. Caballero and Lyons (1989, 1992) estimated the external effect in the US manufacturing industry and obtained values ranging from 0.49 to 0.89 and from 0.32 to 0.49 in their 1989 and 1992 studies, respectively. Caballero and Lyons (1990) also provided estimates for European countries ranging from 0.29 to 1.40. Moreover, the values estimated by Lindström (2000) for Swedish manufacturing range from 0.16 to 0.53. By contrast, using the industry-level manufacturing data for the United Kingdom, Oulton (1996) found no evidence of either external effects or increasing returns to scale. These results show that the degree of external effects varies across countries and industries.

a discrete manner (see Appendix). Thus, the owner's maximization problem is given by

$$\max_{i \in \{1,2\}} r_i(k_t).$$

For notational simplicity, we sometimes write:

$$\beta_i = \eta_i + \alpha_i.$$

2.3 Equilibrium dynamic model

Considering the market equilibrium and optimization results in the previous subsections, we can represent our model in a general form:

$$k_{t+1} = s w_m(k_t), \tag{2}$$

$$m = \arg \max_{i \in \{1,2\}} r_i(k_t), \tag{3}$$

$$k_0 > 0 : \text{given, and } t = 0, 1, 2, \cdots . \tag{4}$$

Without loss of generality, we assume throughout the paper that:

$$\beta_2 > \beta_1. \tag{5}$$

We claim the following.

Claim 1. *If (5) is satisfied, then $r_1(k) > r_2(k)$ if and only if $0 < k < \theta$, where the threshold θ is the unique positive solution of $r_1(\theta) = r_2(\theta)$, that is:*

$$\theta = \left[\frac{\alpha_1 A_1}{\alpha_2 A_2} \right]^{1/(\beta_2 - \beta_1)}.$$

Proof. A simple calculation reveals that $r_1'(\theta) < r_2'(\theta)$ if and only if $\beta_2 > \beta_1$. □

Using this claim, we can rewrite our model given by (2)-(4) as the following mapping from $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$ into itself:

$$T : \mathbb{R}_+ \to \mathbb{R}_+, \tag{6}$$

$$k_{t+1} = T(k_t) = \begin{cases} T_1(k_t) = s(1-\alpha_1)A_1 k_t^{\beta_1} & \text{if} \quad k_t \leq \theta, \\ T_2(k_t) = s(1-\alpha_2)A_2 k_t^{\beta_2} & \text{if} \quad k_t > \theta. \end{cases}$$

For simplicity, we have assumed that if $k_t = \theta$, then technology 1 is chosen. Note that T is a piecewise continuous mapping with one discontinuity. Figure 1 shows a typical case where the r_1-curve is downward-sloping, whereas the r_2-curve is upward sloping and, accordingly, T_1 is chosen for $k_t \leq \theta$ and T_2 for $k_t > \theta$.

INSERT Figure 1 around here.

3 Analysis of the model

In this section, we demonstrate that the model given by (6) can exhibit poverty traps, middle-income traps, and perpetual growth paths. Moreover, we show that an economy caught in a middle-income trap can exhibit chaotic fluctuations in the long run.

To characterize the dynamics of the model given by (6), we consider the following three generic cases:

$$\begin{aligned} \text{Case 1} &: \quad 1 > \beta_2 > \beta_1, \\ \text{Case 2} &: \quad \beta_2 > \beta_1 > 1, \\ \text{Case 3} &: \quad \beta_2 > 1 > \beta_1. \end{aligned}$$

In Case 1, the external effects in both technologies are mild. The dynamics in this case have been extensively investigated, for example, by Ishida and Yokoo (2004), Asano et al. (2012), and Umezuki and Yokoo (2019a). Generically, these models are not capable of generating chaotic dynamics, in distinctive contrast to those in Cases 2 and 3. Case 2 is an extreme case where the externality is sufficiently strong for both of the technologies. Note that the condition in this case implies that each T_i is strictly convex. Consequently, we find that chaotic behavior is a ubiquitous feature for this case. Case 3, which is considered the most important, is the intermediate case where the external effect is weak or absent in one technology but strong in the other. According to empirical studies on the degree of external effects in production, in some industries the external effects in production are present and the degree of the effects ranges widely, whereas in other industries the null hypothesis of no externalities cannot be rejected (see footnote 8). Thus, the situation of Case 3 is consistent with empirical studies on the degree of external effects in production.

3.1 Case 1: Periodic fluctuations

As mentioned above, Case 1 reduces to the model studied by Umezuki and Yokoo (2019a). Therefore, we do not repeat this in detail here. The assumption that $1 > \beta_1 > \beta_2$ corresponds to the case where the external effects for both technologies $(i = 1, 2)$ are not strong and the marginal productivity of capital is decreasing. Thus, the main results in Umezuki and Yokoo (2019a) apply to Case 1 of our model, and are summarized in the following proposition:

Proposition 1. *Suppose that* $1 > \beta_2 > \beta_1 > 0$*. Then, the map (6) exhibits a periodic attractor of an arbitrarily large period when other appropriate parameters are chosen. Furthermore, aperiodic motions occur only for parameter values of measure zero.*

Proof. See Umezuki and Yokoo (2019a). □

This proposition indicates that whenever we observe long-run fluctuating behavior in a computer simulation of Case 1, this is almost certainly a periodic cycle, including an attracting steady state. It should be mentioned that Case 1 generates virtually only periodic fluctuations and does not generate chaotic fluctuations. This is in stark contrast to the other cases below that generate chaotic fluctuations.

3.2 Case 2: Chaotic middle-income trap coexisting with a poverty trap and perpetual growth paths

In this case, because $\beta_2 > \beta_1 > 1$, each T_i $(i = 1, 2)$ is strictly increasing and strictly convex. Note that the mapping T given by (6) has a trivial steady state at the origin, that is, $T(0) = 0$. For positive steady states other than the origin, T has two candidates for positive ones:

$$T_i(\bar{k}_i) = \bar{k}_i, \qquad i = 1,\, 2.$$

Solving these equations yields:

$$\bar{k}_i = \left[s\,(1-\alpha_i)\,A_i\right]^{1/(1-\beta_i)}, \qquad i = 1,\, 2.$$

Note that each potential positive steady state is a repeller, whereas the origin is an attractor. For later use, we restate this in the following lemma:

Lemma 1. *If $\beta_2 > \beta_1 > 1$, then the origin of (6) is an attractor. Furthermore, any positive steady state, if it exists, is a repeller.*

Proof. The first statement holds by $T'(0) = T_1'(0) = 0$, and the second statement holds by $T_i'(\bar{k}_i) = \beta_i > 1$. □

By drawing the graph of T, one can recognize that unless:

$$\lim_{k \to \theta+} T_2(k) \equiv T_2(\theta) < \theta < T_1(\theta), \tag{7}$$

the threshold has little effect on the dynamics of T. Therefore, we require (7) or, equivalently:

$$s(1-\alpha_2)A_2 \left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{\beta_2 - 1}{\beta_2 - \beta_1}} < 1 < s(1-\alpha_1)A_1 \left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{\beta_1 - 1}{\beta_2 - \beta_1}} \tag{8}$$

We first check that such a set of parameter values is not empty and see how to find such parameters:

Claim 2. *The set of parameter values that satisfy (8) is not empty.*

Proof. See Appendix. □

Further, we specify a closed *trapping interval* $M \subset \mathbb{R}_+$ such that $T(M) \subset M$ and $0 \notin M$. By strict monotonicity of T_i, if $T(M) \subset M$, then $\theta \in M$. Such an interval M would be regarded as a *middle-income trap*. Thus, if:

$$\bar{k}_1 < T_2(\theta) \quad \text{and} \quad T_1(\theta) < \bar{k}_2, \tag{9}$$

then $M = [T_2(\theta), T_1(\theta)]$ is such a trapping interval, and so is $M' = [\bar{k}_1, \bar{k}_2]$ with $M \subset M'$. We can show that such parametric restrictions are indeed possible:

Lemma 2. *The set of parameter values that satisfy (9) is not empty. In fact, let $\alpha_1 A_1 = \alpha_2 A_2$ with $\alpha_2 \in (\alpha_1, 1)$ and, let $\beta_2 > \beta_1 > 1$ with $1/\beta_1 + 1/\beta_2 > 1$. Then, the inequalities (9) hold.*

Proof. See Appendix. □

Lemma 2 states that the middle-income trap is more likely to occur when the βs are large but not too large; that is, the external effect for each technology is "moderately" large.

Note that (9) implies (8) by the convexity of T_i ($i = 1, 2$). By drawing the graph of T, we can summarize our observations into the following proposition.

Proposition 2. *Assume that $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 > \beta_1\beta_2$. Then, the economy represented by (6) simultaneously exhibits a poverty trap, a middle-income trap, and perpetual growth paths for an open set of parameter values.*

Proof. By Lemma 2, (6) has a middle-income trap for some specific parameter values. The coexistence of the poverty trap and perpetual growth paths follows directly from Lemma 1. Because any slight perturbations of all parameters preserve the inequalities in (9), the assertion is proved. □

This situation is depicted in Figure 2.

INSERT Figure 2 around here.

The above proposition is interesting from two perspectives. First, under some economic environment, that is, for some set of parameter values, a poverty trap, a middle-income trap, and perpetual growth paths emerge in the same economy. Second, which of the three economic phenomena in Proposition 2 will actually occur depends only on the initial condition, which is explained in more detail in Proposition 4.

Now, we focus on the dynamics on the trapping interval; that is, the middle-income trap case in Proposition 2. First, suppose that all the conditions in Proposition 2 are satisfied. Second, we restrict mapping T to M. Note that mapping T can be log-linearized as follows:

$$\log k_{t+1} = \begin{cases} \log s(1-\alpha_1)A_1 + \beta_1 \log k_t & \text{if} \quad \log k_t \leq \log \theta, \\ \log s(1-\alpha_2)A_2 + \beta_2 \log k_t & \text{if} \quad \log \theta < \log k_t. \end{cases}$$

Next, we define a variable change such that:

$$x_t = h(k_t) = \frac{\log(k_t/T_2(\theta))}{\log(T_1(\theta)/T_2(\theta))}. \tag{10}$$

By (10), the restriction mapping $T_{|_M} : M \to M$ can be transformed into the following topologically equivalent piecewise linear mapping from the unit interval $I = [0, 1]$ to itself:

$$\tau : I \to I, \tag{11}$$

$$x_{t+1} = \tau(x_t) = \begin{cases} \tau_1(x_t) = 1 + \beta_1(x_t - c) & \text{if} \quad 0 \le x_t \le c, \\ \tau_2(x_t) = \beta_2(x_t - c) & \text{if} \quad c < x_t \le 1, \end{cases}$$

where $c = h(\theta)$ and, for any $k \in M$, it holds that $h \circ T_{|_M}(k) = \tau \circ h(k)$. Note that c cannot take all the values between 0 and 1.

Claim 3. *If $\beta_2 > \beta_1 > 1$ and $1/\beta_1 + 1/\beta_2 > 1$, then the threshold c in (11) is located in the interval $(1 - 1/\beta_2, 1/\beta_1) \subset I$.*

Proof. From (9), we require that $\tau_1(0) \in (0, 1)$ and $\tau_2(1) \in (0, 1)$. From $\tau_1(0) \in (0, 1)$, it follows that $0 < 1 - c\beta_1 < 1$, implying that $c < 1/\beta_1$. From $\tau_2(1) \in (0, 1)$, $0 < \beta_2(1 - c) < 1$ implies that $c > 1 - 1/\beta_2$. □

Figure 3 depicts the graph of τ corresponding to Figure 2.

INSERT Figure 3 around here.

Let I be a closed interval and $f : I \to I$ be a piecewise smooth mapping. If there is an integer $n \ge 1$ such that $\inf |df^n(x)/dx| > 1$ whenever the derivative exists, then f is said to be *eventually expanding*.

If the above assumption holds for $n = 1$, f is said to be just expanding. It is known (for example, Lasota and Yorke, 1973) that an (eventually) expanding mapping on the interval can have absolutely continuous invariant measures, implying that there is observable *chaos* in the long run.

Proposition 3. *Suppose that the parameters are as in Proposition 2. Let M be the trapping interval for T and let $T_{|_M}$ be the restriction of T to M. Then, $T_{|_M} : M \to M$ is chaotic in the sense that it admits an absolutely continuous invariant measure.*

Proof. By conjugacy, it suffices to show that τ in (11) admits an absolutely continuous invariant measure. As $\inf |\tau'(x)| = \beta_1 > 1$, that is, τ is expanding, the assertion follows from Lasota and Yorke (1973). □

Note that the chaotic behavior described in Proposition 3 is robust in the sense that it persists for any perturbations of parameters, provided that they are as in Proposition 2.

Let us summarize our findings for Case 2 in the following proposition:

Proposition 4. *Suppose that the parameters for the model given by (6) are as in Proposition 2. Then, three cases typically emerge depending on the initial condition:*
(i) *Poverty trap; for $k_0 < \bar{k}_1$, the economy converges to 0.*
(ii) *Chaotic middle-income trap; for $k_0 \in (\bar{k}_1, \bar{k}_2)$, the economy becomes trapped in an interval, where it keeps fluctuating in a chaotic manner.*
(iii) *Perpetual growth; for $k_0 > \bar{k}_2$, the economy grows unboundedly.*

Note that, in the case of the chaotic middle-income trap, periodic points exist in the region. However, they are always unstable (i.e., repellers) and not observable.

The implication of Proposition 4 is as follows. In Case (i), if the economy starts from a sufficiently small initial value of capital k_0 with $k_0 < \bar{k}_1$, then the economy becomes caught in a poverty trap, that is, k_t is attracted to the origin. This is because the low return from capital, due to the increasing marginal productivity of capital, obstructs capital accumulation. In Case (iii), if the economy starts from a sufficiently large initial value of capital k_0 with $k_0 > \bar{k}_2$, then the economy exhibits perpetual growth. This is because the high marginal productivity of capital accelerates economic growth by the reverse logic to that of Case (i). In Case (ii), if the initial value of capital k_0 lies in the middle range, then the economy is caught in the middle-income trap. The intuition behind this can be explained as follows. If the economy starts from a value that is greater than $\bar{k}_1$ but smaller than θ, then technology 1 is chosen, and the marginal productivity of capital becomes large because of increasing marginal productivity, which accelerates economic growth until the threshold is crossed and the regime switches from technology 1 to technology 2. Then, the per capita capital stock is not large enough for technology 2 to maintain the economy's growth. Thus, it begins to shrink, which brings it back to a point near the initial value, and the story repeats itself. Such a mechanism creates middle-income traps. Furthermore, the expanding property of the underlying dynamic system causes chaotic motions. This is an intriguing case because not only does the economy become trapped in a middle-income trap, it also fluctuates chaotically in the trap. However, this cannot occur when the external effects of both technologies are weak (see Proposition 1).

Figure 4 depicts a typical trajectory that is eventually caught and chaotically fluctuates in a middle-income trap, as described in Proposition 4. Figure 5 plots four trajectories in the time series described in Proposition 4. Trajectory A in Figure 5 corresponds to a perpetual growth path. Trajectories B and C illustrate trajectories caught in a middle-income trap from above and below, respectively. Finally, trajectory D is a typical path where an economy is caught in a poverty trap.

INSERT Figures 4 and 5 around here.

3.3 Another situation in Case 2: Breakdown of the middle-income trap

When the trapping interval collapses as a result of some possible change in parameters, the economy is expected to escape the middle-income region in the long run by either eventually becoming caught in the poverty trap or achieving a perpetual growth path. Such cases occur, rather than (9), if:

$$T_2(\theta) < \bar{k}_1 \quad \text{and/or} \quad \bar{k}_2 < T_1(\theta).$$

In the rest of this subsection, we focus on the following case:

$$T_2(\theta) < \bar{k}_1 \quad \text{and} \quad \bar{k}_2 < T_1(\theta). \tag{12}$$

See Figure 6 for this situation and Figure 7 for enlargement.

INSERT Figure 6 and Figure 7 around here.

Lemma 3. *The set of parameter values that satisfy (12) is not empty.*

Proof. See Appendix. □

Proposition 5. *Let $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 < \beta_1\beta_2$ be given. Then, for some open set of parameter values, $T : [\bar{k}_1, \bar{k}_2] \to \mathbb{R}_+$ is topologically chaotic in the sense that there exists an invariant Cantor set $\Lambda \subset [\bar{k}_1, \bar{k}_2]$ such that $T_{|_\Lambda} : \Lambda \to \Lambda$ is topologically conjugate to the one-sided full shift on two symbols. Furthermore, for such Λ and any $k_0 \in [\bar{k}_1, \bar{k}_2] \setminus \Lambda$, either $\lim_{n\to\infty} T^n(k_0) = 0$ or $\lim_{n\to\infty} T^n(k_0) = \infty$ holds.*

Proof. From Lemma 3, we can take a set of parameter values satisfying (12). Using variable transformation:

$$x_t = v(k_t) = \frac{\log(k_t/\bar{k}_1)}{\log(\bar{k}_2/\bar{k}_1)}, \tag{13}$$

we obtain a piecewise linear mapping:

$$m : \mathbb{R} \to \mathbb{R},$$

$$x_{t+1} = m(x_t) = \begin{cases} m_1(x_t) = \beta_1 x_t & \text{if} \quad x_t \leq c, \\ m_2(x_t) = 1 + \beta_2(x_t - 1) & \text{if} \quad c < x_t, \end{cases}$$

where $v \circ T(k_t) = m \circ v(k_t)$ and $c = v(\theta) \in (0, 1)$, indicating that T is topologically equivalent to m. Now, consider points in the unit interval $I = [0, 1]$ that remain under the iteration of m. Because $1/\beta_1 + 1/\beta_2 < 1$, there are two closed subintervals $I_0 = [0, 1/\beta_1]$ and $I_1 = [1 - \beta_2, 1]$ such that $I_0 \cap I_1 = \phi$ and $I_0 \cup I_1 \subset \tau(I_i)$ $(i = 0, 1)$ (horseshoe condition). Furthermore, it holds that $\tau'(x) \geq \beta_1 > 1$ for all $x \in I_0 \cup I_1$ (hyperbolicity). See Figure 8. Thus, according to the standard argument of elementary dynamical systems theory (see, for example, Guckenheimer and Holmes 1983), there exists a closed m-invariant subset $\Lambda = \cap_{n\geq 0} T^{-n}(I_0 \cup I_1) = \{x \in I_0 \cup I_1 \,|\, T^n(x) \in I_0 \cup I_1,\ n \geq 0\} \subset I_0 \cup I_1$, as stated in the proposition. The trajectories that go to positive infinity correspond to perpetual growth paths, whereas the trajectories that go to negative infinity correspond to those caught in the poverty trap. □

INSERT Figure 8 around here.

The invariant set Λ in Proposition 5 corresponds to the one-dimensional version of Smale's horseshoe. This suggests the possibility that the economy starting in the middle range exhibits a transiently chaotic behavior before it either gets caught in the poverty trap or achieves a perpetual growth path. The destination in which the economy ends up can be highly random because the chaotic invariant set scrambles the nearby points. In their numerical study on endogenous business cycles, Asano et al. (2022a) called this the "pinball effect" in the middle-income trap. Figure 9 shows how two initial states that are different yet close to each other lead to different final states with transiently chaotic fluctuations.

INSERT Figure 9 around here.

3.4 Case 3: The occurrence of chaotic behaviors in the middle-income trap

Case 3, which is the intermediate case between Cases 1 and 2, is more important than the previous cases because we want to consider the smallest unit of an economy with a mix of technologies that have strong externalities and those that do not. Recall that in the middle-income trap, Case 1 generates virtually only periodic fluctuations, whereas Case 2 generates virtually only chaotic fluctuations. Thus, in Case 3, depending on the parameters, such mixed dynamic patterns would be expected. To provide an overview of this, let us consider the bifurcation diagram (Figure 10) corresponding to Case 3.

INSERT Figure 10 around here.

Figure 10, a bifurcation diagram with respect to the saving rate s, suggests that for smaller s, periodic fluctuations (including steady state dynamics) appear to occur, and for larger s, chaotic behavior appears to occur. However, for much larger s, divergence (perpetual growth) occurs and is not shown in the figure.

As in Case 2, the mapping given by (6) is valid for Case 3. The situation in Case 3 differs from that in Case 2 in that branch T_1 of map (6) becomes concave, whereas T_2 remains convex because $\beta_2 > 1 > \beta_1 > 0$. Consequently, several situations occur depending on the configuration of potential steady states $\bar{k}_i$ $(i = 1, 2)$ and the threshold θ.

Because we are interested in the occurrence of the middle-income trap, we focus on the situations in which a trapping interval appears. By the concavity of T_1 and the convexity of T_2, we can observe that the following inequality suffices to ensure the existence of such a trapping interval for Case 3:

$$\theta < \bar{k}_1 \quad \text{and} \quad T_1(\theta) < \bar{k}_2 \tag{14}$$

or equivalently:

$$\left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{1}{\beta_2-\beta_1}} < (s(1-\alpha_1)A_1)^{\frac{1}{1-\beta_1}} \quad \text{and} \tag{15}$$

$$s(1-\alpha_1)A_1\left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{\beta_1}{\beta_2-\beta_1}} < (s(1-\alpha_2)A_2)^{\frac{1}{1-\beta_2}}. \tag{16}$$

Lemma 4. *Let $\beta_2 > 1 > \beta_1 > 0$ be fixed. Then, the set of parameters that satisfies the inequalities given by (14) is not empty. Thus, there exists a trapping interval (or middle-income trap) $M = [T_2(\theta), T_1(\theta)]$ such that $T(M) \subset M$.*

Proof. See Appendix. □

As $T(0) = T_1(0) = 0$ and $\lim_{k\to+0} T_1'(k) = +\infty$, the origin is an unstable (i.e., repelling) steady state in Case 3. This implies that the poverty trap associated with the origin does not exist in this case.

Lemma 5. *For $\beta_2 > 1 > \beta_1 > 0$, the origin of T given by (6) is always a repelling steady state.*

Let us summarize what we have observed thus far. The following proposition states that, under some economic environment, two growth patterns, a middle-income trap and perpetual growth paths, arise, but a poverty trap does not occur.

Proposition 6. *Assume that $\beta_2 > 1 > \beta_1 > 0$. Then, there exists some open set of parameter values for which the economy represented by (6) simultaneously exhibits a middle-income trap and perpetual growth paths but no poverty trap.*

Proof. From Lemma 4, we can find parameters that satisfy the inequalities given by (14), which implies the existence of a trapping interval (middle-income trap). As all inequalities appearing in Lemma 4 are strict, any mapping T with slightly perturbed parameters also exhibits a middle-income trap. The nonexistence of poverty traps follows from Lemma 5. For perpetual growth paths, consider that $T'(\bar{k}_2) = T_2'(\bar{k}_2) > 1$ because of the convexity of T_2. □

Figure 11 graphically represents the meaning of Proposition 6. The figure also depicts how a trajectory with an initial value close to the origin falls into the middle-income trap.

INSERT Figure 11 around here.

The relationship between the final states and initial conditions presented in Proposition 6 can be roughly summarized by the following proposition:

Proposition 7. *Assume that $\beta_2 > 1 > \beta_1 > 0$ and let the parameters satisfy (14). Then, two cases typically occur depending on the initial condition:*
(i) *Persistent fluctuations in the middle-income trap; for $k_0 \in (0, \bar{k}_2)$, the economy becomes trapped in an interval where it exhibits persistent fluctuations.*
(ii) *Perpetual growth; for $k_0 > \bar{k}_2$, the economy diverges to infinity.*

The following is the intuition behind Proposition 7. In this case, two threshold values of k exist: the first one is θ, representing the switching point of technology, and the second one is $\bar{k}_2$, denoting the unstable steady state under the technology with increasing marginal productivity of capital. For $k < \theta$, the economy's behavior is essentially the same as the standard Solow-type model. For $k \in [\theta, \bar{k}_2)$, because the marginal productivity of capital is low, the economy shrinks. However, for $k > \bar{k}_2$, the marginal productivity of capital is high enough to promote capital accumulation. Further, the marginal productivity increases as capital accumulates, and therefore the economy grows perpetually.

Next, we examine in detail what happens in the middle-income trap.

We show that chaotic dynamics in the middle-income trap are possible in Case 3. To verify this, the same variable transformation is conducted as that performed for Case 2 to obtain the mapping $\tau : I \to I$, given by (11), with the only difference being $\beta_1 \in (0, 1)$ rather than $\beta_1 > 1$.

Similar to Case 2, the range of threshold c for τ in Case 3 is limited to some subinterval of $I = [0, 1]$.

Lemma 6. *Let $\beta_2 > 1 > \beta_1 > 0$ be fixed. Then, the threshold $c = h(\theta)$ of mapping (11) is in $((\beta_2 - 1)/\beta_2, 1) \subset (0, 1)$.*

Proof. Translating (14) through the conjugacy h implies that $h(T_1(\theta)) = 1 < h(\bar{k}_i)$ for $i = 1, 2$. As $h(\bar{k}_1) = (1 - c\beta_1)/(1 - \beta_1)$ and $h(\bar{k}_2) = c\beta_2/(\beta_2 - 1)$, rearranging the inequalities above yields $(\beta_2 - 1)/\beta_2 < c < 1$. □

Let us consider the mapping $\tau : I = [0, 1] \to I$ given by (11). Let $I_L = [0, c]$ (left interval) and $I_R = (c, 1]$ (right interval) with $c \in (0, 1)$. We consider some simplest possible patterns of trajectories generated by τ. Specifically, we find a trajectory that visits the left interval successively only once and the right interval successively at least n times.

Lemma 7. *Any trajectory generated by τ stays successively at most once in the left interval I_L if $c < 1/(1 + \beta_1)$.*

Proof. Requiring that $\tau_1(0) = 1 - c\beta_1 > c$, we obtain the result. □

Note that this condition implies that $\tau(I_L) \subset I_R$, which assures that any trajectory visits I_R at least once immediately after it has visited I_L. We can generalize the above result slightly.

Lemma 8. *Any trajectory generated by τ stays successively at least n times ($n \geq 1$) in the right interval I_R after it has visited I_L, if:*

$$c < \frac{(\beta_2 - 1)\beta_2^{n-1}}{\beta_1(\beta_2 - 1)\beta_2^{n-1} + \beta_2^n - 1}. \tag{17}$$

Proof. See Appendix. □

For the first step, we identify the condition under which the chaotic behavior occurs when the trajectory of τ successively visits I_L at most once and I_R *at least once.*

Proposition 8. *Let $\beta_2 > 1 > \beta_1 > 0$ and $1 < \beta_1\beta_2 < 1 + \beta_1$. Then, τ given by (11) is chaotic for any $c = h(\theta) \in ((\beta_2 - 1)/\beta_2, 1/(1+\beta_1)) \equiv J_1$ and so is $T_{|_M} : M \to M$, where M is the middle-income trap.*

Proof. See Appendix. ☐

Let

$$J_n = \left(\frac{\beta_2 - 1}{\beta_2}, \frac{(\beta_2 - 1)\beta_2^{n-1}}{\beta_1(\beta_2 - 1)\beta_2^{n-1} + \beta_2^n - 1} \right) \quad \text{for} \quad n \geq 1, \tag{18}$$

whenever it is well-defined.

Let us generalize the above result.

Proposition 9. *Let $\beta_2 > 1 > \beta_1 > 0$ and $1 < \beta_1\beta_2^n < 1 + \beta_1\beta_2^{n-1}$ for some $n \geq 1$. Then, τ given by (11) is chaotic for any $c \in J_n$ given by (18) and so is $T_{|_M} : M \to M$.*

Proof. Similar to Proposition 8, we have (17) from Lemma 8 and $(\beta_2 - 1)/\beta_2 < c < 1$ from Lemma 6. For such a c to be taken, it must hold that:

$$\frac{\beta_2 - 1}{\beta_2} < \frac{(\beta_2 - 1)\beta_2^{n-1}}{\beta_1(\beta_2 - 1)\beta_2^{n-1} + \beta_2^n - 1},$$

which is equivalent to $\beta_1\beta_2^n < 1 + \beta_1\beta_2^{n-1}$. Furthermore, as any trajectory of τ successively visits I_L at most once and I_R at least n times, it follows by assumption for any initial condition $x_0 \in (0, 1)$ that:

$$(\tau^{n+1})'(x_0) \geq \beta_1\beta_2^n > 1.$$

Thus, τ is eventually expanding. ☐

The following figures represent the above Propositions 8 and 9.

INSERT Figures 12 and 13 around here.

Proposition 7, along with Propositions 8 and 9, suggests that even when only one of the two technologies exhibits moderately strong externalities, chaotic dynamics in the middle-income trap can be observed for a large set of parameter values. This result contrasts sharply with Proposition 1, where virtually no chaotic behavior occurs. Finally, the following proposition is particularly important and readily derived by the above proposition.

Proposition 10. *Let* $2 > \beta_2 > 1 > \beta_1 > 0$*. Then there exists some integer* $n \geq 1$ *such that* J_n *is well-defined and* τ *is chaotic for any* $c \in J_n$*. Correspondingly,* $T_{|_M} : M \to M$ *is chaotic as well.*

Proof. By assumption, there is some integer $n \geq 1$ such that $\beta_1 \beta_2^{n-1} \leq 1$ and $1 < \beta_1 \beta_2^n$. Thus we have $\beta_1 \beta_2^n - \beta_1 \beta_2^{n-1} = \beta_1 \beta_2^{n-1}(\beta_2 - 1) < 1$, which implies the conclusion by Proposition 9. □

The above proposition shows that if one technology exhibits increasing marginal productivity, even to a small degree, there can be a middle-income trap in which the economy behaves in a persistently chaotic manner, depending on other parameter conditions, even if the other technology exhibits decreasing marginal productivity.

4 Concluding remarks

This paper introduced externalities in production into an OLG model with endogenous technology choice. Then, we analyzed how these externalities affected macroeconomic fluctuations. Specifically, we considered two types of production technologies that allow for the existence of external effects of capital and specified them as the Cobb–Douglas type. Umezuki and Yokoo (2019a) showed that, under the Cobb–Douglas specification, technology choice can generate periodic fluctuations of any lengths but never create chaotic fluctuations. By contrast, in the present model, we showed that chaotic behavior can be observed in the middle-income trap if at least one of the two technologies exhibits a moderate level of increasing marginal productivity because of externalities. In particular, we find that when one technology exhibits diminishing marginal productivity and the other, even slightly, exhibits increasing marginal productivity, a chaotic middle-income trap can still occur, depending on various other parameters. The last finding should be emphasized because it is consistent with the empirical results on external effects in production.

The present analysis has some limitations. In analyzing technology choice, the two production technologies are specified as being of the Cobb–Douglas type. However, this assumption may be slightly restrictive, and it would be worthwhile adopting a broader class of production technology, for example, the constant elasticity of substitution (CES) type. Asano et al. (2022b) analyzed the dynamic implications of technology choice under the setting of CES technologies. However, they did not consider external effects in production. Thus, an analysis using CES technologies with external effects will be our future task. Moreover, Umezuki and Yokoo (2019b) analyzed the case of a continuum of Cobb–Douglas-type technologies and showed that chaotic dynamics can appear for a wide set of parameters. An interesting extension would be to incorporate a continuum of technologies with production externalities into our model.

Appendix

Appendix A: Microfoundation of technology choice behavior

Along a similar line to Mastuyama (2007) and Asano et al. (2022b), we briefly explain a microfoundation for our technology choice behavior.

The goods and factor markets are competitive. This economy has J different types of production technologies. By use of type i technology we can transform m_i units of the final goods into $m_i R_i$ units of capital and put the capital into the final goods production. The production function of the final goods is $Y_{it} = F_i(k_t, K_t, L_t)$. K_t and L_t denote capital and labor at time t, respectively, and k_t is the capital-labor ratio capturing capital deepening externalities. The private marginal return of capital is

$$\frac{\partial}{\partial K_t} F_i(k_t, K_t, L_t) \equiv MPK_i.$$

We assume that capital depreciates completely in one period.

In each period, a unit of a new generation is born and lives in two periods, the young and old periods. Assuming that the utility function of each agent is of the long-linear type, we have a constant saving rate. We denote the saving rate by s. In managing their savings, young agents can choose to become a lender or to become an entrepreneur. An agent who chooses to become a lender lends savings when young and receives $r_{t+1} s w_t$ when old, where r_{t+1} stands for the real interest rate. An agent becoming an entrepreneur picks one technology out of the J types of technologies. Because entrepreneurs' wealth is given by their savings, if $m_i > s w_t$, the amount $m_i - s w_t$ must be borrowed. However, due to imperfections in the financial markets, each entrepreneur can only pledge up to a certain proportion of its revenues for repayment, i.e., $\lambda_i m_i R_i \cdot MPK_i$, where $0 \leq \lambda_i \leq 1$. Note that the value of λ_i differs among the J types of projects. More concretely, the entrepreneur's borrowing constraint is given by:

$$\lambda_i m_i R_i \cdot MPK_i \geq r_{t+1}(m_i - s w_t) \text{ for } i = 1, ..., J. \tag{19}$$

A smaller value of λ_i means a stricter credit constraint.

It should be noted that earnings from investment should not be lower than those from lending because an entrepreneur can always become a lender:

$$R_i \cdot MPK_i \cdot m_i - r_{t+1}(m_i - sw_t) \geq r_{t+1} sw_t, \tag{20}$$

that is:

$$r_{t+1} \leq R_i \cdot MPK_i \text{ for } i = 1, ..., J.$$

(19) can be rewritten as follows:

$$r_{t+1} \leq \frac{R_i \cdot MPK_i}{\left(1 - \frac{sw}{m_i}\right) / \lambda_i} \text{ for } i = 1, ..., J.$$

Let us define:

$$\Phi_i \equiv \frac{R_i \cdot MPK_i}{\max\left\{1, \left(1 - \frac{sw_t}{m_i}\right) / \lambda_i\right\}}.$$

Then, (19) and (20) can be summarized as follows:

$$r_{t+1} \leq \Phi_i \text{ for } i = 1, ..., J.$$

Suppose that $r_{t+1} < \Phi_i$. Then, everyone becomes an entrepreneur and employs type i technology, and this economy has no lender. This situation cannot be an equilibrium, and thus we have $r_{t+1} \geq \Phi_i$. Next, suppose that $r_{t+1} > \Phi_i$ for some i. Then, at least one of (19) and (20) for i does not hold, and thus type i is not employed. Because, there must be a positive investment in equilibrium, we have:

$$r_{t+1} = \max\left\{\Phi_1, ..., \Phi_J\right\}. \tag{21}$$

This indicates that the technology exhibiting the highest value on the right-hand side of (21) is employed.

In this study, we consider a special case of (21):

$$J = 2,\ R_1 = R_2 = 1,\ \lambda_1 = \lambda_2 = \lambda \text{ and } d_1 = d_2 = d.$$

In this case, (21) reduces to:

$$r_{t+1} = \max\left\{\frac{R \cdot MPK_1}{\max\left\{1, \left(1 - \frac{sw_t}{m}\right)/\lambda\right\}}, \frac{R \cdot MPK_2}{\max\left\{1, \left(1 - \frac{sw_t}{m}\right)/\lambda\right\}}\right\}$$

$$= \frac{R}{\max\left\{1, \left(1 - \frac{sw_t}{m}\right)/\lambda\right\}} \max\left\{MPK_1, MPK_2\right\}.$$

Thus, we can confirm that the technology with the higher marginal productivity of capital is adopted (as we assume in our analysis). It should be noted that, although the presence of credit constraint lowers the interest rate, it will not cause any significant change of our results since in our model the saving rate is constant and independent of the interest rate.

Appendix B: Proofs

Proof of Claim 2. Let $\alpha_1 A_1/\alpha_2 A_2 = 1$ and $1 > \alpha_2 > \alpha_1$. Then, all we need to show is that the following inequalities are possible:

$$s(1-\alpha_2)A_2 < 1 < s(1-\alpha_1)\alpha_2 A_2/\alpha_1.$$

Rewriting the above expression as:

$$\frac{\alpha_1}{\alpha_2(1-\alpha_1)} < sA_2 < \frac{1}{1-\alpha_2},$$

we notice that $\alpha_2/\alpha_1(1-\alpha_1) < 1/(1-\alpha_2)$ always holds because $\alpha_2 > \alpha_1$. As sA_2 can take any positive value, the claim is proven. □

Proof of Lemma 2 . Let $\alpha_1 A_1/\alpha_2 A_2 = 1$ and $1 > \alpha_2 > \alpha_1$. Then, the first inequality in condition (9) can be rewritten as:

$$\left(\frac{1}{1-\alpha_2}\right)^{\frac{\beta_1 - 1}{\beta_1}} \left(\frac{\alpha_1}{\alpha_2(1-\alpha_1)}\right)^{\frac{1}{\beta_1}} < sA_2.$$

Similarly, the second inequality in condition (9) is expressed as

$$sA_2 < \left(\frac{1}{1-\alpha_2}\right)^{\frac{1}{\beta_2}} \left(\frac{\alpha_1}{\alpha_2(1-\alpha_1)}\right)^{\frac{\beta_2 - 1}{\beta_2}}.$$

Letting $V = 1/(1-\alpha_2)$ and $W = \alpha_1/\alpha_2(1-\alpha_1)$, we observe that $V > W$ because $\alpha_2 > \alpha_1$. Because sA_2 can take any positive value, it suffices to show that $V^{1-1/\beta_1} W^{1/\beta_1} < V^{\beta_2} W^{1-1/\beta_2}$ or $V^\gamma < W^\gamma$, where $\gamma = 1 - 1/\beta_1 - 1/\beta_2$. Thus, the last inequality holds if $\gamma < 0$ or $1/\beta_1 + 1/\beta_2 > 1$. □

Proof of Lemma 3. Using the same notations as in the proof of Claim 2, it suffices to show that $V^{1-1/\beta_1}W^(1/\beta_1) > V^{\beta_2}W^{1-1/\beta_2}$ or $(V/W)^\gamma > 1$, where $\gamma = 1-1/\beta_1-1/\beta_2$. Because $V/W > 1$, the last inequality holds if we take β_1 and β_2 ($\beta_2 > \beta_1 > 1$) such that $\gamma > 0$ or $1/\beta_1 + 1/\beta_2 < 1$. □

Proof of Lemma 4. Let $s \in (0,1)$ and $\alpha_2 \in (0,1)$ (hence, $\eta_2 = \beta_2 - \alpha_2$) be fixed. Let a_i $(i = 1,2)$ be any numbers such that $1 < a_1 < a_2$. Let $\alpha_1 A_1/\alpha_2 A_2 = 1$. Then, inequalities (15) and (16) can be reduced to:

$$1 < s(1-\alpha_1)\alpha_2 A_2/\alpha_1 < (s(1-\alpha_2)A_2)^{\frac{1}{1-\beta_2}}.$$

Solving the following simultaneous equations for A_2 and α_1,

$$\begin{aligned} a_1 &= s(1-\alpha_1)\alpha_2 A_2/\alpha_1, \\ a_2 &= (s(1-\alpha_2)A_2)^{\frac{1}{1-\beta_2}}, \end{aligned}$$

we obtain:

$$A_2 = \frac{1}{\left(s(1-\alpha_2)a_2^{\beta_2-1}\right)} > 0 \quad \text{and} \quad \alpha_1 = \frac{1}{1+\left(\frac{1-\alpha_2}{\alpha_2}\right)a_1 a_2^{\beta_2-1}} \in (0,1),$$

which verifies the assertion. □

Proof of Lemma 8. Let us begin with $\tau_1(0) = 1-c\beta_1 > c$, which implies from Lemma 7 that any trajectory visits I_R at least once immediately after visiting I_L. To ensure that the trajectory stays successively twice in I_R, we require that:

$$\tau_2(\tau_1(0)) = \beta_2(1-c\beta_1-c) = \beta_2 - c\beta_1\beta_2 - c\beta_2 > c.$$

To ensure that the trajectory stays in I_R successively at least three times, we have:

$$\tau_2^2(\tau_1(0)) = \beta_2(\beta_2 - c\beta_1\beta_2 - c\beta_2 - c) = \beta_2^2 - c\beta_1\beta_2^2 - c\beta_2^2 - c\beta_2 > c.$$

Repeating this up to n times, we obtain:

$$\begin{aligned} \tau_2^{n-1}(\tau_1(0)) &= \beta_2^{n-1} - c\beta_1\beta_2^{n-1} - c\beta_2^{n-1} - c\beta_2^{n-2} - \cdots - c\beta_2^2 - c\beta_2 \\ &= \beta_2^{n-1} - c\beta_1\beta_2^{n-1} - c\beta_2\left(\sum_{j=0}^{n-2}\beta_2^j\right) > c. \end{aligned}$$

Solving the last inequality for c yields the result. □

Proof of Proposition 8. Note that for the value of the threshold c in the assumption of the proposition to be taken, it must hold that:

$$\frac{\beta_2 - 1}{\beta_2} < \frac{1}{1 + \beta_1}.$$

This inequality is equivalent to $\beta_1\beta_2 < 1 + \beta_1$, which is assured by assumption. Because Lemma 7 indicates that any trajectory (i.e., irrelevant to the initial conditions) of τ visits I_L successively at most once, it follows for any initial condition $x_0 \in (0, 1)$ that:

$$(\tau^2)'(x_0) \geq \beta_1\beta_2 > 1,$$

where the last inequality follows by assumption. Thus, τ is eventually expanding and hence chaotic in the sense of Lasota and Yorke (1973). By conjugacy h, T is also chaotic. □

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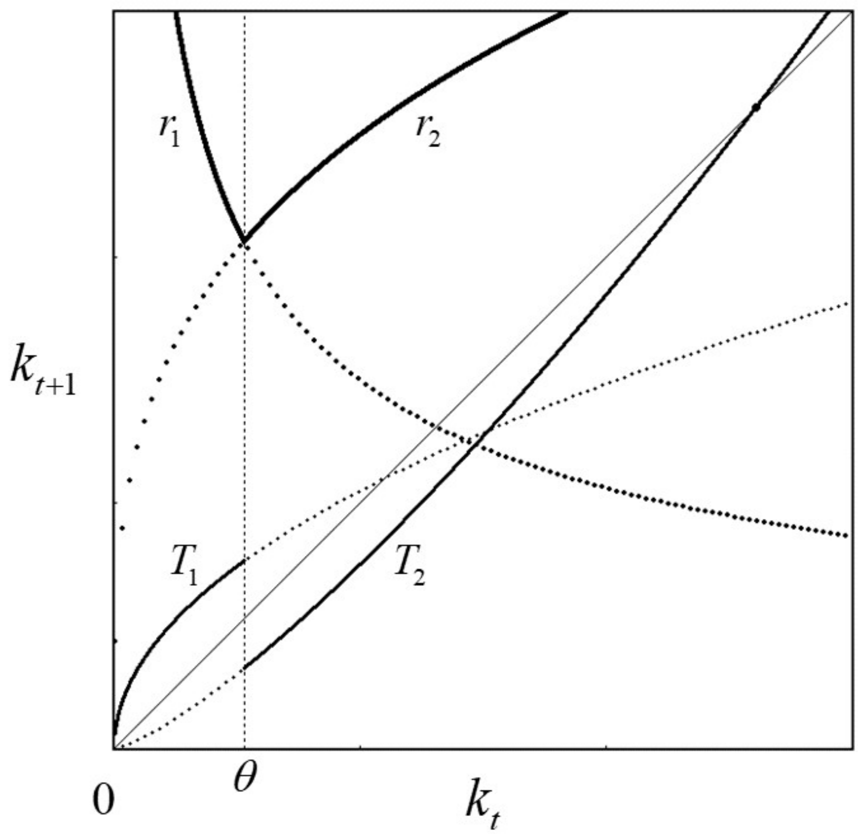

Figure 1: Graphs of r_1, r_2, T_1, and T_2 with $\beta_2 > 1 > \beta_1$. The r_2-curve is upward-sloping due to externality.

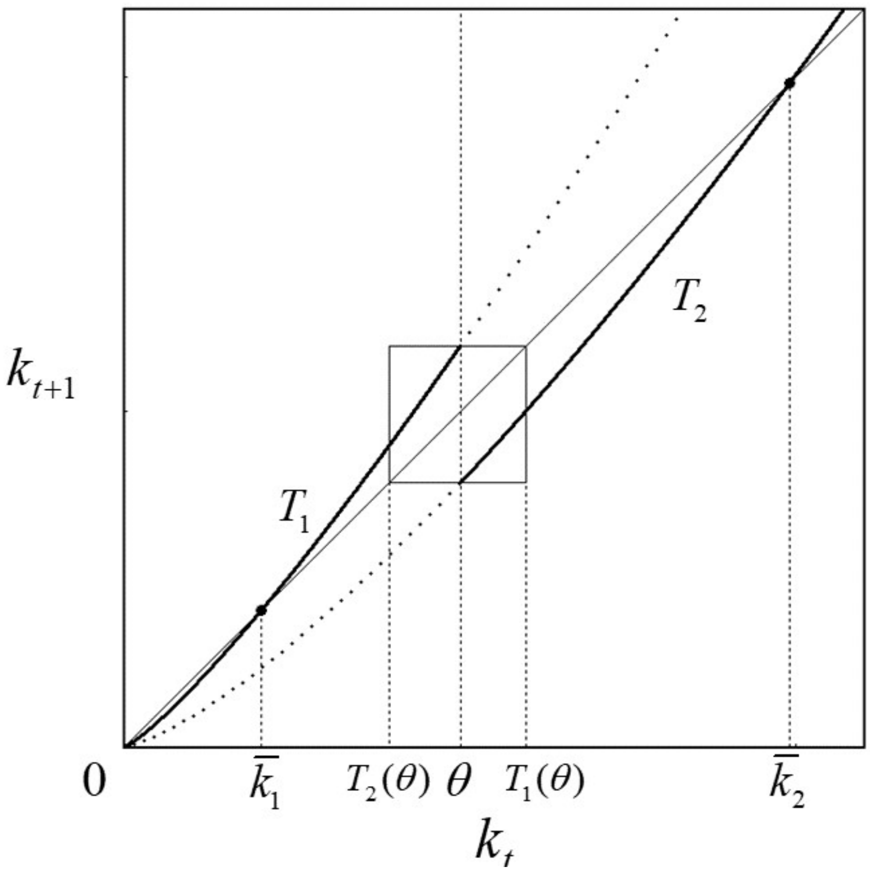

Figure 2: Coexistence of a poverty trap, middle-income trap, and perpetual growth paths. $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 > \beta_1\beta_2$. Parameters: $A_2 = 5$, $\alpha_1 = 0.55$, $\alpha_2 = 0.65$, $\eta_1 = 0.65$, $\eta_2 = 0.7$, $s = 0.45$, $A_1 = \alpha_2 A_2/\alpha_1 \approx 5.91$, $\beta_1 = \alpha_1 + \eta_1 = 1.2$, and $\beta_2 = \alpha_2 + \eta_2 = 1.35$.

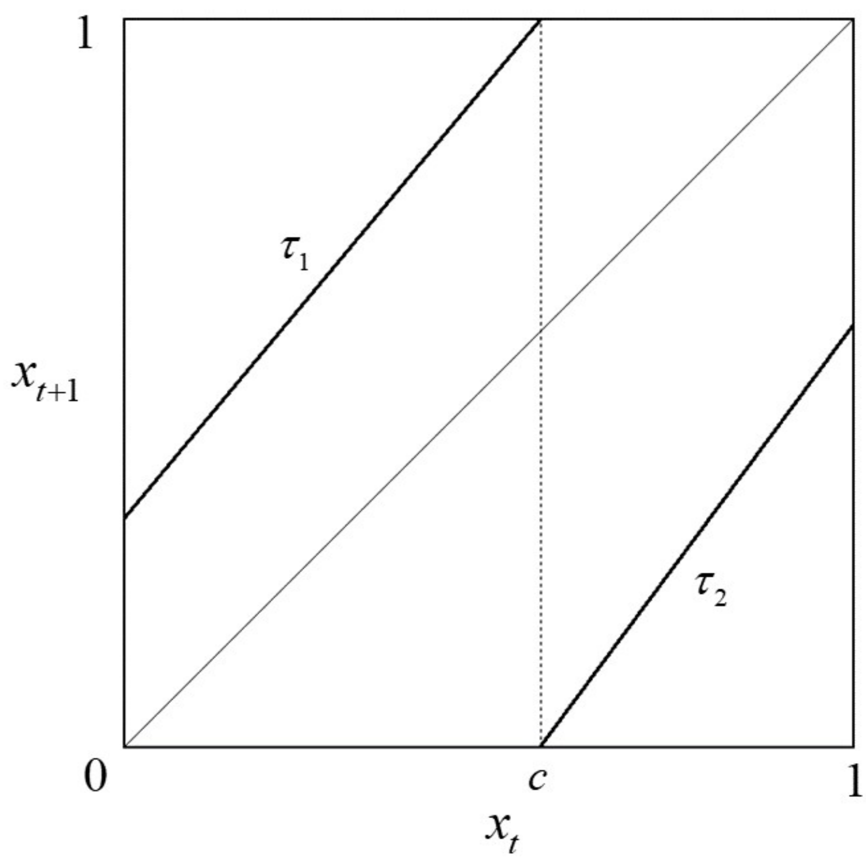

Figure 3: Piecewise-linearization on the middle-income trap for $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 > \beta_1\beta_2$. The parameter values are the same as in Figure 2.

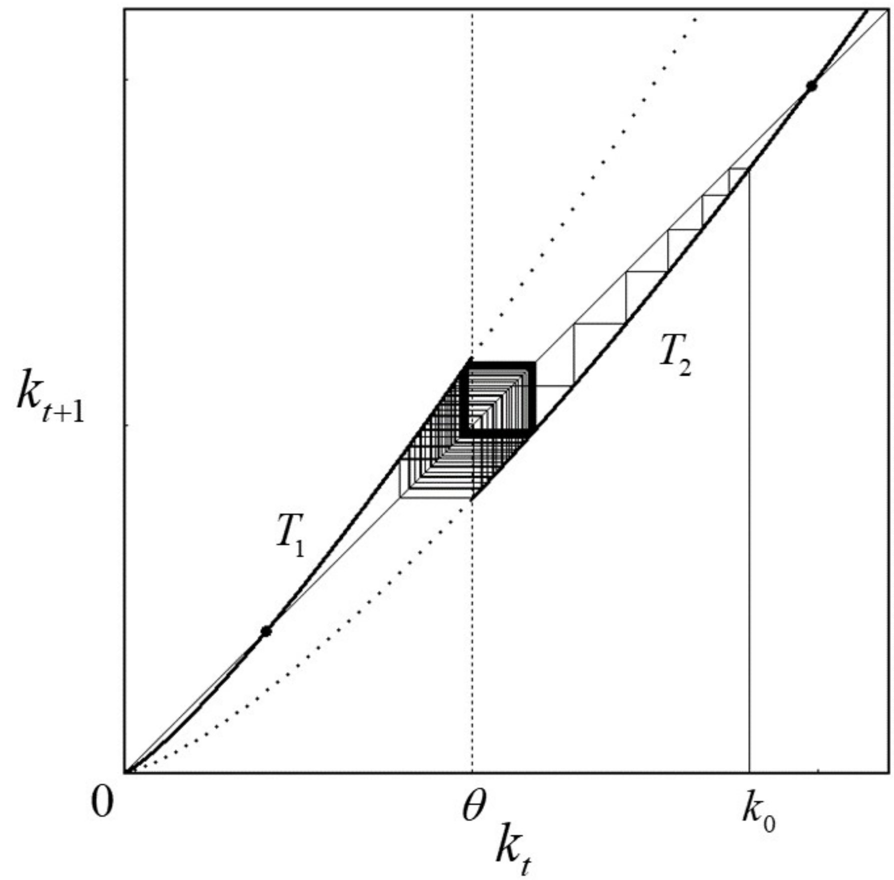

Figure 4: A trajectory converging into the middle-income trap and eventually fluctuating in that region in a chaotic manner. The parameters are the same as in Figure 2.

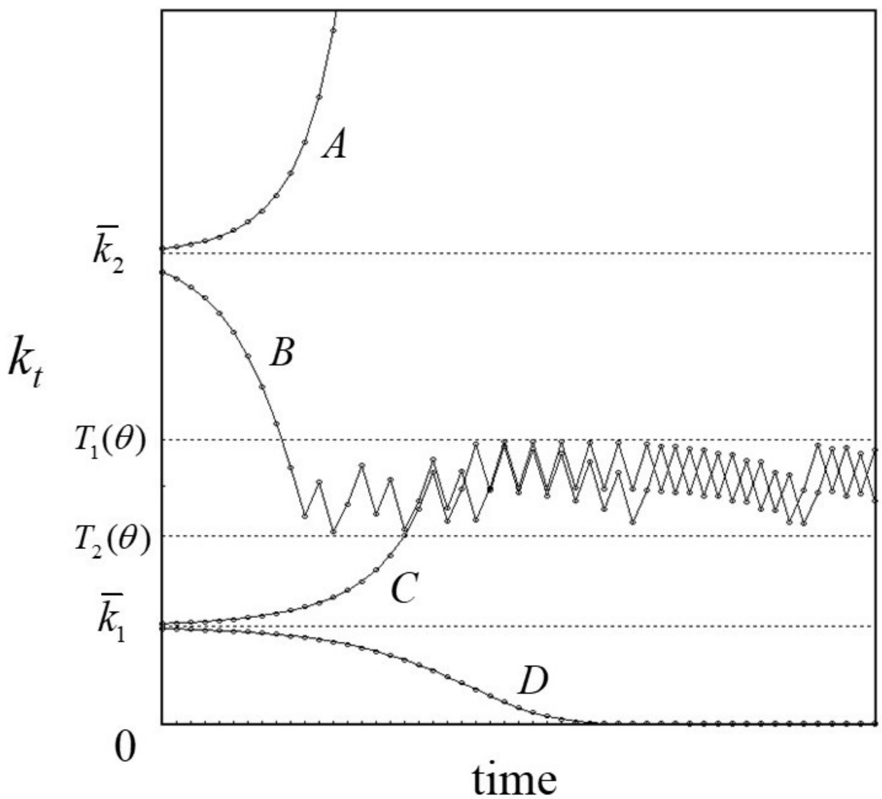

Figure 5: Time series corresponding to Proposition 4. A: a perpetual growth path. B and C: trajectories getting caught into the middle-income trap from above and below, respectively. D: a trajectory to the poverty trap. The parameters are the same as in Figure 2.

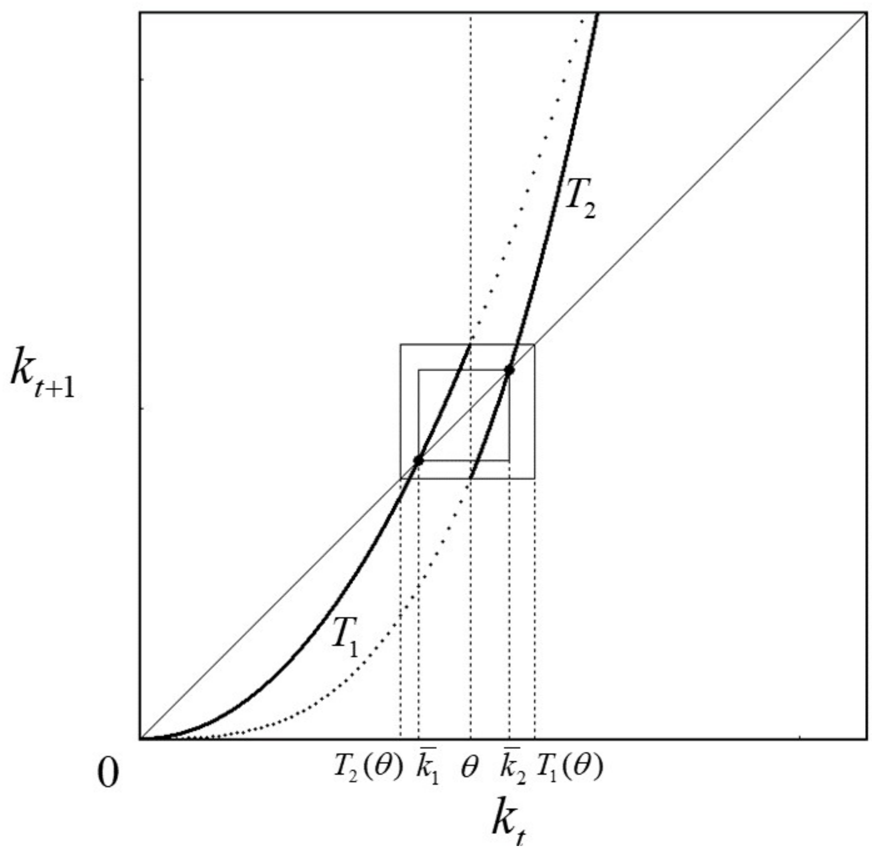

Figure 6: Collapse of the middle-income trap in Case 2. $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 < \beta_1\beta_2$. In this case, a typical trajectory starting in $[\bar{k}_1, \bar{k}_2]$ eventually gets caught in the poverty trap or goes onto a perpetual growth path. Parameters: $A_2 = 5$, $\alpha_1 = 0.45$, $\alpha_2 = 0.65$, $\eta_1 = 0.1$, $\eta_2 = 0.7$, $s = 0.45$, $A_1 = \alpha_2 A_2/\alpha_1 \approx 5.91$, $\beta_1 = \alpha_1 + \eta_1 = 2.05$, and $\beta_2 = \alpha_2 + \eta_2 = 3.15$

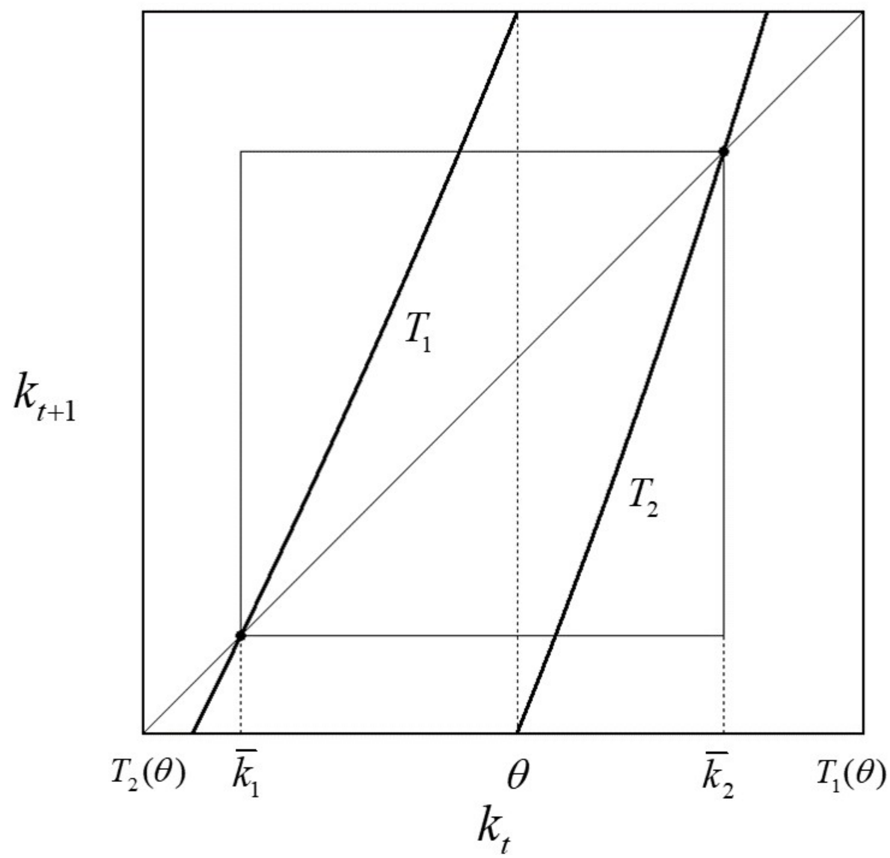

Figure 7: Enlargement of Figure 6.

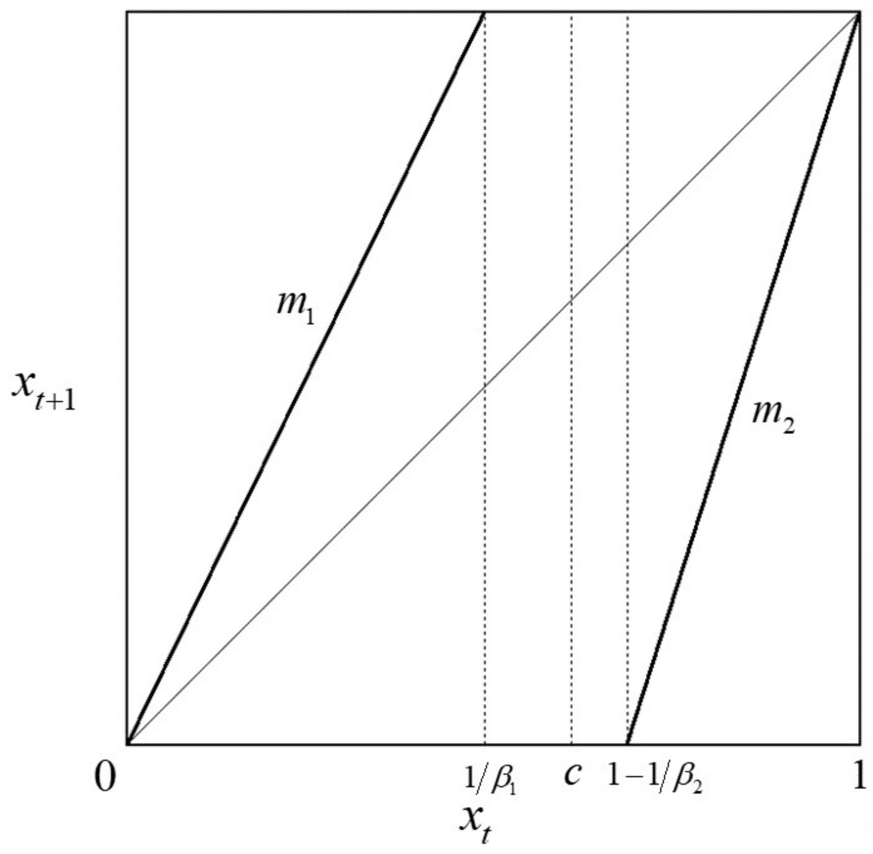

Figure 8: Piecewise-linearization of Figure 6 on the collapsed middle-income trap. The chaotic invariant set Λ is contained in $I_0 \cup I_1$. The iteration of the mapping brings any initial point that finally falls into the interval $(1/\beta_1, c)$ to the perpetual growth path and any initial point that finally falls into $(c, 1-\beta_2)$ to the poverty trap.

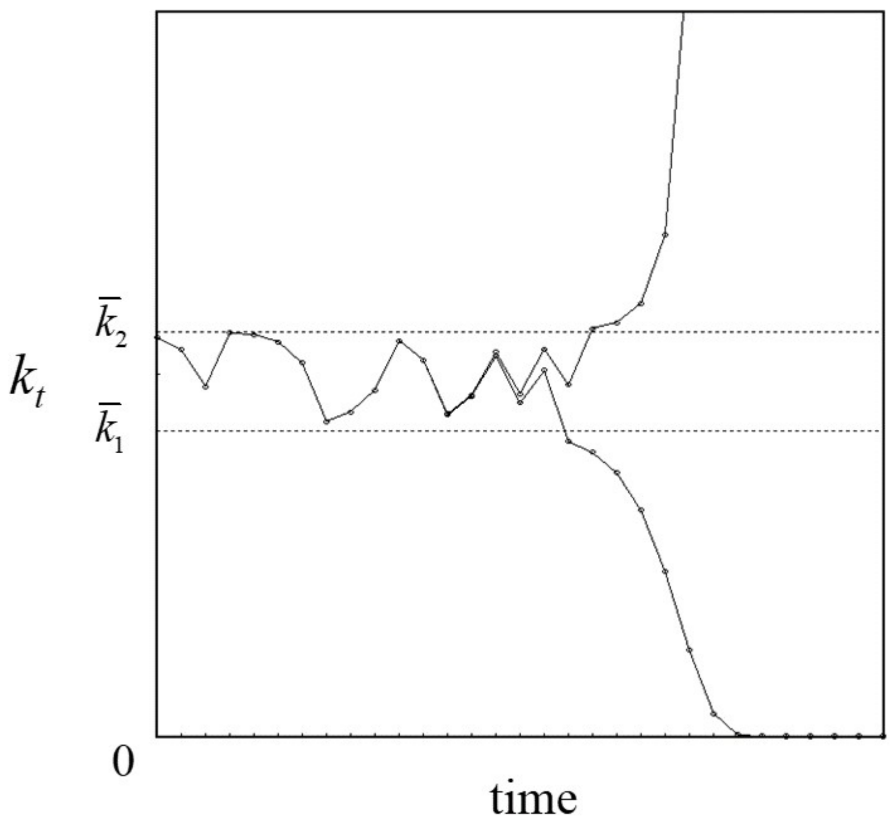

Figure 9: Two different but close to each other initial states near the chaotic invariant set Λ lead to different final states with transiently chaotic fluctuations. The parameters are the same as in Figure 6.

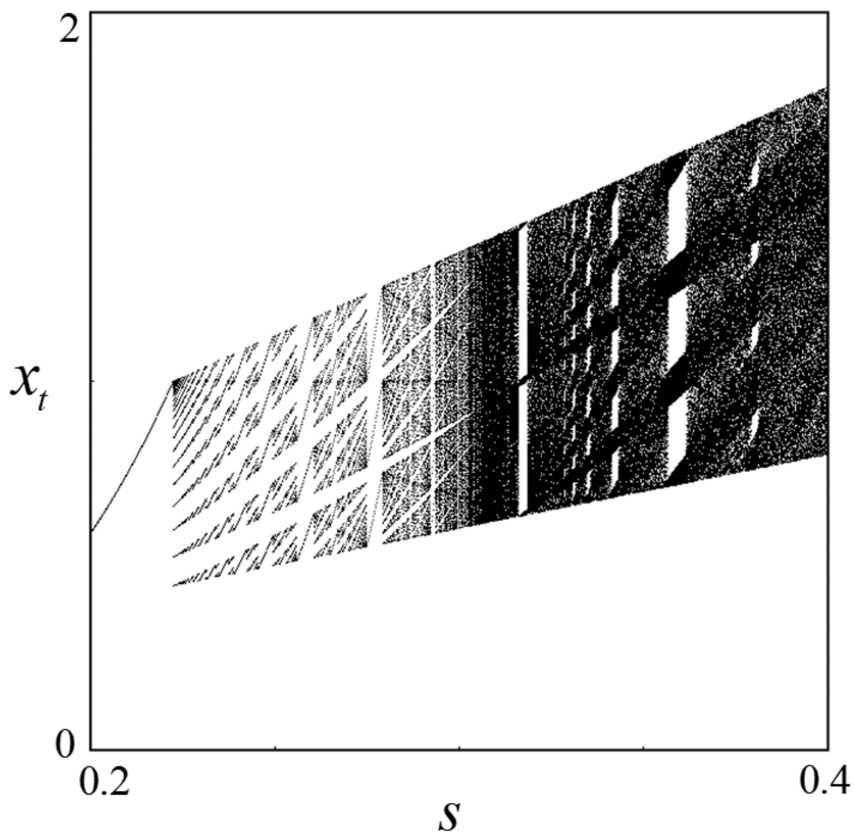

Figure 10: Bifurcation diagram with respect to $s \in (0.2, 0.4)$ for Case 3: $\beta_2 > 1 > \beta_1 > 0$. Parameters: $A_2 = 5$, $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $\eta_1 = 0.4$, $\eta_2 = 0.7$, $A_1 = \alpha_2 A_2/\alpha_1 = 7.5$, $\beta_1 = \alpha_1 + \eta_1 = 0.8$, and $\beta_2 = \alpha_2 + \eta_2 = 1.3$.

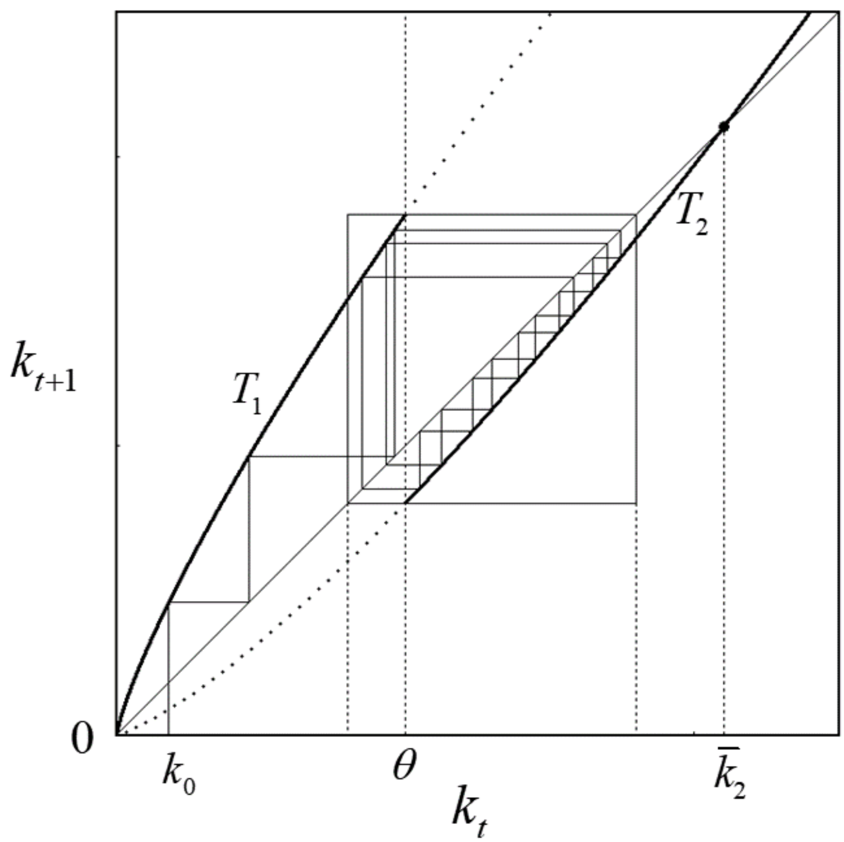

Figure 11: Coexistence of a middle-income trap and perpetual growth paths. There is no poverty trap associated with the origin, which is a repeller in Case 3: $\beta_2 > 1 > \beta_1 > 0$. All the trajectories starting near the origin fall into the middle-income trap. The parameters are the same as in Figure 10 except for $s = 0.4$.

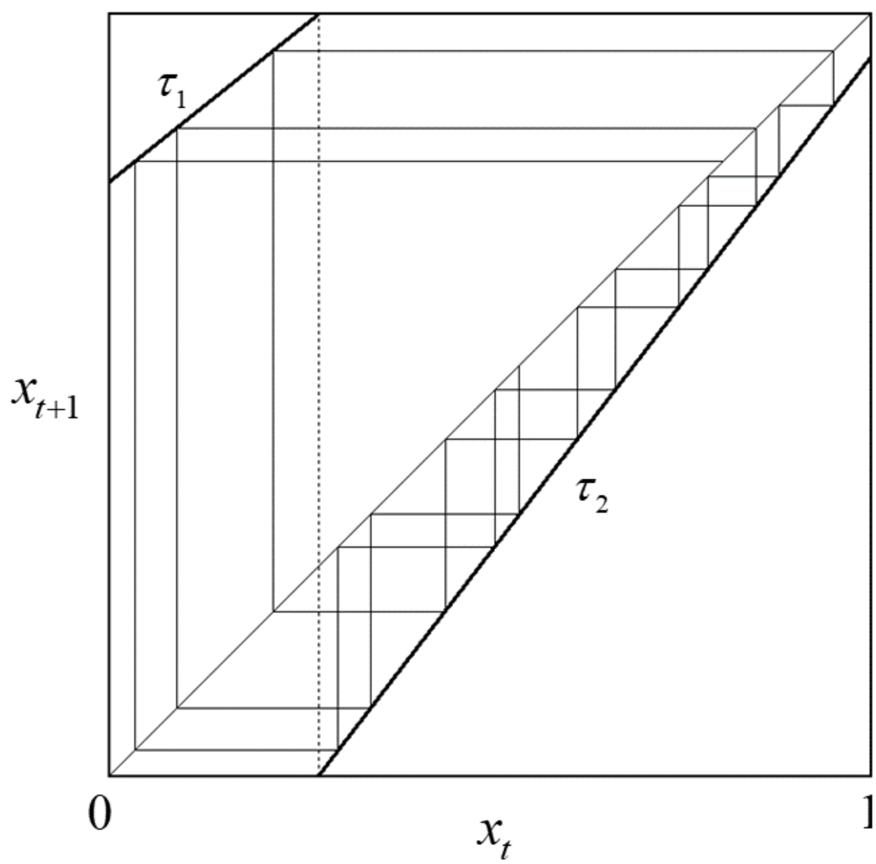

Figure 12: Piecewise-linearization on the middle-income trap for $\beta_2 > 1 > \beta_1 > 0$. After transients being omitted, only 15 iterations are displayed. The parameter values are the same as in Figure 11.

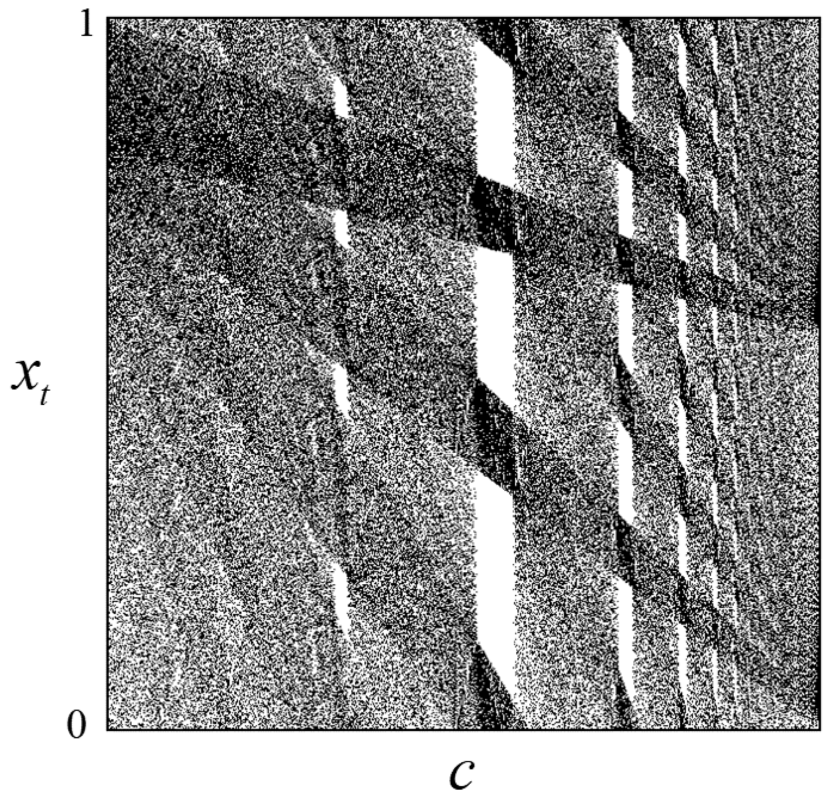

Figure 13: Bifurcation diagram of τ with respect to $c \in J_1 = ((\beta_2 - 1)/\beta_2, 1/(1+\beta_1))$. For each $c \in J_1$, chaotic behavior is observed in the middle-income trap, according to Propositions 8 and 9.