KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE
OF
ECONOMIC RESEARCH

Discussion Paper No.1075

"Technology Choice, Externalities in Production, and Chaotic Middle-Income Traps"

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March 2022

KYOTO UNIVERSITY

KYOTO, JAPAN

Technology Choice, Externalities in Production, and Chaotic Middle-Income Traps*

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March 10, 2022

Abstract

We incorporate external effects of capital on production and endogenous technology choice into the standard overlapping generations model. We demonstrate that our model can exhibit poverty traps, middle-income traps, and perpetual growth paths. We also show that these three phenomena coexist for some set of parameters and the economy caught in the middle-income trap can exhibit chaotic fluctuations in the long run. In obtaining these results in the standard overlapping generations model, the combination of technology choice and externalities in production plays a crucial role.

Keywords: External effect; Technology choice; Overlapping generations model, Middle-income trap; Chaos

*The authors would like to thank Real Arai, Keita Kamei, and seminar participants at Seinan Gakuin University for their helpful comments and suggestions. This research was financially supported by International Joint Research Center of Advanced Economic Research of KIER and the Grants-in-Aid for Scientific Research, JSPS (17K03806, 20H05631, 20K01745, 20H01507, and 21K01388).

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1 Introduction

Business cycle theory can be broadly divided into two categories: exogenous and endogenous business cycle theories. Exogenous business cycle theory attributes the fundamental source of economic fluctuations to stochastic shocks. Exogenous business cycle theory, especially the dynamic stochastic general equilibrium approach, has a dominant role in the business cycle research for decades. However, after the global financial crisis, a renewed interest in endogenous business cycle theory has appeared (e.g., Beaudry et al. 2020 and Schmitt-Grohé and Uribe 2021), in which economic fluctuations occur spontaneously due to factors within the economy without any shocks. At the almost same time, theoretical research on the complexity of business cycle fluctuations is gaining momentum (e.g., Matsuyama et al. 2016).

In this line of research, the role of technology choice in endogenous business cycles has attracted much attention. From the end of 1990s, this role in business cycles has been analyzed by several authors, such as Aghion et al. (1999), Iwaisako (2002), and Matsuyama (2007). These studies show the possibility of various patterns of dynamics in their models. However, they basically relied on graphical analysis and did not characterize the properties of the equilibrium dynamics in detail. Mathematically rigorous characterizations have recently been made by Asano et al. (2012), Asano et al. (2021), Matsuyama et al. (2016) and Umezuki and Yokoo (2019a). These studies assume neoclassical, constant-returns-to-scale technologies.[1] In reality, as Caballero and Lyons (1990) and many other studies have found,[2] there would be external effects in production, especially in manufacturing. Thus, the role of the external effects in business cycles should be considered.

The existence of these effects allows the cases of increasing marginal productivity of capital or increasing-returns-to-scale, which, combined with technology choice, can be a source of a rich variety of complex dynamics.[3] In fact, constructing an

[1]Iwaisako (2002) is an exception. He considers two possible technologies: a constant-returns-scale and an increasing-returns-to-scale. However, his analysis exclusively relies on the graphical one.

[2]For example, see Baxter and King (1991), Caballero and Lyons (1992), and Lindström (2000).

[3]The role of increasing-returns-to-scale has long been analyzed in the field of international trade (Negishi, 1969). Since the 1990s, the role of increasing returns has attracted attention in various fields. For example, the field of economic growth has shown that increasing-returns-to-scale (or

overlapping generations (OLG) model with external effects and two technologies (one of which is chosen endogenously), we show that our model can generate poverty traps, middle-income traps, and perpetual growth paths and these three phenomena can exist simultaneously. We also show that, if the external effect is mildly large in at least one technology enough to generate a slight degree of increasing marginal productivity of capital, then the economy can exhibit chaotic business cycles.[4] Under the standard Cobb-Douglas technologies, in which externalities are absent, whenever we observe long-run fluctuations, they are almost certainly periodic, as shown in Umezuki and Yokoo (2019a). It should be emphasized that, in obtaining long-run chaotic fluctuations in the Diamond model with Cobb-Douglas technology choice, the introduction of externalities in production plays a crucial role.

The remainder of this paper is organized as follows. Section 2 presents the settings of our model. Section 3 provides the main results, and Section 4 concludes this paper. Most of the proofs are relegated to appendices.

2 Settings of the model

This section describes the structure of our model: households' and firms' behavior and equilibrium dynamics.

2.1 Household's behavior

The basic setup follows the standard Diamond-type OLG model. Time is discrete and extends from 0 to infinity. Population is assumed to be constant over time and normalized to 1. Each generation lives two periods, supplying one unit of labor

external effects) is an engine of the long-run economic growth, as started by Romer (1986) and Lucas (1988), and has become one of the foundations of the modern economic growth theory. Furthermore, the field of urban economics has shown that increasing-returns-to-scale exists in the background of the phenomenon of urban agglomeration (Fujita and Thisse, 1996; Fujita et al., 1999). For example, Fujita and Thisse (1996) state that "We can therefore safely conclude that increasing returns to scale are essential for explaining the geographical distribution of economic activities." In the current urban economics, increasing-returns-to-scale has become one of its fundamental components.

[4]If both external effects of the two technologies are sufficiently small, our model can exhibit periodic fluctuations, which have been extensively studies, for example, by Ishida and Yokoo (2004), Asano et al. (2012), and Umezuki and Yokoo (2019a).

inelastically only when young. She maximizes her Cobb-Douglas utility according to the following problem:

$$\max_{c_t^y, c_{t+1}^o, s_t} (1-s)\log c_t^y + s \log c_{t+1}^o, \qquad s \in [0,1]$$

$$\text{s.t.} \qquad s_t + c_t^y = w_t, \qquad c_{t+1}^o = r_{t+1} s_t.$$

Here, c_t^y denotes the consumption when young, c_{t+1}^o the consumption when old, s_t savings, w_t the real wage rate, r_{t+1} the real rate of return on the loan maturing at $t+1$, and the subscription t for time. Utility maximization implies that

$$s_t = s w_t.$$

2.2 Firm's behavior

We introduce two additional factors into our model: externalities in production and multiple technologies. The firm is assumed to behave as its owner and as its manager.[5] This economy has two available production technologies. We assume that the firm as the owner has to choose one technology that maximizes the return on capital, whereas the firm manager attempts to maximize the firm's profit, which is driven away by competition. To capture our idea in the simplest possible settings, we employ Cobb-Douglas technologies as follows:

$$F_i(k, K, L) = A_i k^{\eta_i} K^{\alpha_i} L^{1-\alpha_i}, \; i \in \{1, 2\}, \; A_i > 0, \; \alpha_i \in (0,1), \text{ and } \eta_i \geq 0,$$

where subscript i denotes the i-th technology, K capital, L labor, and k the capital-labor ratio. Each manager regards k as given. This formulation follows that of Azariadis and Reichlin (1996).[6] The first argument of F_i is related to externalities. If

[5] Regarding another possible interpretation, we may assume that the firm chooses its production technology in a discrete manner in the first stage and then choose optimal inputs in the second stage.

[6] Several studies measure external effects by estimating the percentage increase in a firm's output that is caused by a 1% increase in aggregate inputs (or aggregate output), keeping an individual firm's input intact. Caballero and Lyons (1989, 1992) estimated the external effect in the US manufacturing and obtained the values from 0.49 to 0.89 and from 0.32 to 0.49, respectively. Caballero and Lyons (1990) also provided estimates for European countries ranging from 0.29 to 1.40. Moreover, the estimated values by Lindström (2000) for Swedish manufacturing range from 0.16 to 0.53. In contrast, using the industry-level UK manufacturing date, Oulton (1996) found evidence neither for external effects nor for increasing-returns-to-scale. These results show that the degree of external effects would vary across countries and industries.

$\eta_i > 0$, then positive externalities exist in production such as knowledge spillover. If $\eta_i = 0$, externality are absent, and F_i is a standard Cobb-Douglas production function. To avoid unnecessary complications, we ignore the case of negative externalities, that is, $\eta_i < 0$. Given the first argument in F_i and $L = 1$, in a symmetric equilibrium, competition implies the following first order conditions:

$$\begin{aligned} r_t &= \frac{\partial F_i(k_t, k_t, 1)}{\partial K_t} \equiv r_i(k_t) = \alpha_i A_i k_t^{\eta_i+\alpha_i-1}, \\ w_t &= \frac{\partial F_i(k_t, k_t, 1)}{\partial L_t} \equiv w_i(k_t) = (1-\alpha_i) A_i k_t^{\eta_i+\alpha_i}. \end{aligned} \tag{1}$$

Thus, the shape of the marginal productivity of capital depends on the value of $\eta_i + \alpha_i$. Note that $r(k)$, given by (1), is an increasing function with respect to k if the external effect is sufficiently large, that is, $\eta + \alpha > 1$.

Upon entering the market, the representative firm's owner in period t, who was born in period $t-1$, chooses a technology that, k_t being given, yields the highest return in a discrete manner (see Appendix). Thus, the owner's maximization problem is given by

$$\max_{i \in \{1,2\}} r_i(k_t).$$

For notational simplicity, we sometimes write

$$\beta_i = \eta_i + \alpha_i.$$

2.3 Equilibrium dynamic model

Considering the market equilibrium and optimization results in the previous subsections, we can represent our model in a little general form:

$$k_{t+1} = s w_m(k_t), \tag{2}$$

$$m = \arg\max_{i \in \{1,2\}} r_i(k_t), \tag{3}$$

$$k_0 > 0 : \text{given, and } t = 0, 1, 2, \cdots . \tag{4}$$

Without loss of generality, we assume throughout the paper that

$$\beta_2 > \beta_1. \tag{5}$$

We claim the following.

Claim 1. *If (5) is satisfied, then* $r_1(k) > r_2(k)$ *if and only if* $0 < k < \theta$*, where the threshold* θ *is the unique positive solution of* $r_1(\theta) = r_2(\theta)$*, that is,*

$$\theta = \left[\frac{\alpha_1 A_1}{\alpha_2 A_2}\right]^{1/(\beta_2 - \beta_1)}.$$

Proof. A simple calculation reveals that $r_1'(\theta) < r_2'(\theta)$ if and only if $\beta_2 > \beta_1$. □

Using this claim, we can rewrite our model given by (2)-(4) as the following mapping from $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$ into itself.

$$T : \mathbb{R}_+ \to \mathbb{R}_+, \tag{6}$$

$$k_{t+1} = T(k_t) = \begin{cases} T_1(k_t) = s(1-\alpha_1)A_1 k_t^{\beta_1} & \text{if} \quad k_t \leq \theta, \\ T_2(k_t) = s(1-\alpha_2)A_2 k_t^{\beta_2} & \text{if} \quad k_t > \theta. \end{cases}$$

For simplicity, we have assumed that, if $k_t = \theta$, then technology 1 is chosen. Note that T is a piecewise continuous mapping with one discontinuity. Figure 1 shows a typical case where the r_1-curve downward-sloping, while the r_2-curve is upward sloping and accordingly T_1 is chosen for $k_t \leq \theta$ and T_2 for $k_t > \theta$.

INSERT Figure 1 around here.

3 Analysis of the model

In this section, we demonstrate that the model given by (6) can exhibit poverty traps, middle-income traps, and perpetual growth paths. Moreover, we show that an economy caught in a middle-income trap can exhibit chaotic fluctuations in the long run.

In order to characterize the dynamics of the model given by (6), we consider the following three generic cases:

$$\begin{aligned} \text{Case 1} &: \quad 1 > \beta_2 > \beta_1, \\ \text{Case 2} &: \quad \beta_2 > \beta_1 > 1, \\ \text{Case 3} &: \quad \beta_2 > 1 > \beta_1. \end{aligned}$$

In Case 1, the external effects in both technologies are mild. The dynamics here has been extensively investigated. (See Ishida and Yokoo, 2004;, Asano et al., 2012; and Umezuki and Yokoo, 2019a.) Note that their models are generically not capable of generating chaotic dynamics, which would be in distinctive contrast with Cases 2 and 3. Case 2 is an extreme case where externality is strong enough for both of the technologies. Note that the condition in this case implies that each T_i is strictly convex. Consequently, we find that chaotic behavior is a ubiquitous feature for this case. Case 3 is the intermediate case where the external effect is weak or absent in one technology but strong in the other.

3.1 Case 1: Periodic fluctuations

As mentioned above, Case 1 reduces to the model ever studied by Umezuki and Yokoo (2019a). Therefore, we do not repeat this in detail here. The assumption that $1 > \beta_1 > \beta_2$ corresponds to the case wherein the external effects for both technologies $(i = 1, 2)$ are not so high and the marginal productivity of capital is decreasing. Thus, the main results in Umezuki and Yokoo (2019a) apply to Case 1 of our model, and are summarized in the following proposition:

Proposition 1. *Suppose that $1 > \beta_2 > \beta_1 > 0$. Then the map (6) exhibits a periodic attractor of arbitrarily large period by choosing other parameters appropriately. Furthermore, aperiodic motions occur only for parameter values of measure zero.*

Proof. See Umezuki and Yokoo (2019a). □

Note that by this proposition, whenever we observe a fluctuating behavior in the long run in a computer simulation under the condition of Case 1, this is almost certainly a periodic cycle, including an attracting steady state.

3.2 Case 2: Chaotic middle-income trap coexisting with poverty traps and perpetual growth paths

In this case, because $\beta_2 > \beta_1 > 1$, each T_i $(i = 1, 2)$ is strictly increasing and strictly convex. Note that the mapping T given by (6) has a trivial steady state at the origin, that is, $T(0) = 0$. For steady states other than the origin, T has two candidates for positive ones:

$$T_i(\bar{k}_i) = \bar{k}_i, \qquad i = 1, 2.$$

Solving these equations yields

$$\bar{k}_i = [s(1-\alpha_i)A_i]^{1/(1-\beta_i)}, \qquad i = 1, 2.$$

Note that each potential positive steady state is a repeller, while the origin is an attractor. For later use, we restate this in the following lemma:

Lemma 1. *If $\beta_2 > \beta_1 > 1$, then the origin of (6) is an attractor. Furthermore, any positive steady state, if it exists, is a repeller.*

Proof. The first statement holds by $T'(0) = T_1'(0) = 0$, and the second statement holds by $T_i'(\bar{k}_i) = \beta_i > 1$. □

By drawing the graph of T, one can recognize that unless

$$\lim_{k \to \theta+} T_2(k) \equiv T_2(\theta) < \theta < T_1(\theta) \tag{7}$$

the threshold has little effect on the dynamics of T. Therefore, we require (7) or, equivalently,

$$s(1-\alpha_2)A_2 \left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{\beta_2-1}{\beta_2-\beta_1}} < 1 < s(1-\alpha_1)A_1 \left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{\beta_1-1}{\beta_2-\beta_1}} \tag{8}$$

We first check that such a set of parameter values is not empty and see how to find such parameters:

Claim 2. *The set of parameter values that satisfy (8) is not empty.*

Proof. See Appendix. □

We further intend to specify a closed *trapping interval* $M \subset \mathbb{R}_+$ such that $T(M) \subset M$ and $0 \notin M$. By strict monotonicity of T_i, if $T(M) \subset M$, then $\theta \in M$. Such an interval M would be regarded as a *middle-income trap*. Thus, if

$$\bar{k}_1 < T_2(\theta) \quad \text{and} \quad T_1(\theta) < \bar{k}_2, \tag{9}$$

then $M = [T_2(\theta), T_1(\theta)]$ is such a trapping interval, and so is $M' = [\bar{k}_1, \bar{k}_2]$ with $M \subset M'$. We can show that such parametric restrictions are indeed possible:

Lemma 2. *The set of parameters values that satisfy (9) is not empty. In fact, let* $\alpha_1 A_1 = \alpha_2 A_2$ *with* $\alpha_2 \in (\alpha_1, 1)$ *and, let* $\beta_2 > \beta_1 > 1$ *with* $1/\beta_1 + 1/\beta_2 > 1$. *Then, the inequalities (9) hold.*

Proof. See Appendix. □

Lemma 2 states that the middle-income trap is more likely to occur when β's are large but not too large; that is, the external effect for each technology is "moderately" large.

Note that (9) implies (8) by convexity of T_i $(i = 1, 2)$. By drawing the graph of T, we can summarize our observations into the following proposition.

Proposition 2. *Assume that* $\beta_2 > \beta_1 > 1$ *and* $\beta_1 + \beta_2 > \beta_1 \beta_2$. *Then, the economy represented by (6) simultaneously exhibits poverty traps, middle-income traps, and perpetual growth paths for an open set of parameter values.*

Proof. By Lemma 2, (6) has a middle-income trap for some specific parameter values. The co-existence of the poverty trap and perpetual growth paths follows directly from Lemma 1. As any slight perturbations of all parameters preserve the inequalities in (9), the assertion is proved. □

This situation is depicted in Figure 2.

INSERT Figure 2 around here.

The above proposition is interesting from two perspectives. First, for some set of parameter values, poverty traps, middle-income traps, and perpetual growth paths emerge simultaneously. Second, which of the three economic phenomena in Proposition 2 will actually occur depends only on the initial conditions, which is explained in Proposition 4.

Subsequently, we focus on the dynamics on the trapping interval; that is, the middle-income trap case in Proposition 2. First, suppose that all the conditions in Proposition 2 are satisfied. Second, we restrict mapping T to M. Note that mapping T can be log-linearized as follows:

$$\log k_{t+1} = \begin{cases} \log s(1-\alpha_1)A_1 + \beta_1 \log k_t & \text{if} \quad \log k_t \le \log \theta, \\ \log s(1-\alpha_2)A_2 + \beta_2 \log k_t & \text{if} \quad \log \theta < \log k_t. \end{cases}$$

Next we define a variable change such that

$$x_t = h(k_t) = \frac{\log(k_t/T_2(\theta))}{\log(T_1(\theta)/T_2(\theta))}. \tag{10}$$

By (10), the restriction mapping $T_{|_M} : M \to M$ can be transformed into the following topologically equivalent piecewise linear mapping from the unit interval $I = [0, 1]$ to itself:

$$\tau : I \to I, \tag{11}$$

$$x_{t+1} = \tau(x_t) = \begin{cases} \tau_1(x_t) = 1 + \beta_1(x_t - c) & \text{if} \quad 0 \le x_t \le c, \\ \tau_2(x_t) = \beta_2(x_t - c) & \text{if} \quad c < x_t \le 1, \end{cases}$$

where $c = h(\theta)$, and for any $k \in M$, it holds that $h \circ T_{|_M}(k) = \tau \circ h(k)$. Note that c cannot take all the values between 0 and 1.

Claim 3. *If $\beta_2 > \beta_1 > 1$ and $1/\beta_1 + 1/\beta_2 > 1$, then the threshold c in (11) is located in the interval $(1 - 1/\beta_2, 1/\beta_1) \subset I$.*

Proof. From (9), we require $\tau_1(0) \in (0, 1)$ and $\tau_2(1) \in (0, 1)$. From $\tau_1(0) \in (0, 1)$, it follows that $0 < 1 - c\beta_1 < 1$, implying $c < 1/\beta_1$. From $\tau_2(1) \in (0, 1)$, $0 < \beta_2(1-c) < 1$ implies $c > 1 - 1/\beta_2$. □

Figure 3 depicts the graph of τ corresponding to Figure 2.

INSERT Figure 3 around here.

Let I be a closed interval and $f : I \to I$ be a piecewise smooth mapping. If there is an integer $n \ge 1$ such that $\inf |df^n(x)/dx| > 1$ whenever the derivative exists, then f is said to be *eventually expanding.*

If the above assumption holds for $n = 1$, f is just said to be expanding. It is known by, for example, Lasota and Yorke (1973) that an (eventually) expanding mapping on the interval can have absolutely continuous invariant measures, implying that there is observable *chaos* in the long run.

Proposition 3. *Suppose that the parameters are as in Proposition 2. Let M be the trapping interval for T and let $T_{|_M}$ be the restriction of T to M. Then, $T_{|_M} : M \to M$ is chaotic in the sense that it admits an absolutely continuous invariant measure.*

Proof. By conjugacy, it suffices to show that τ in (11) admits an absolutely continuous invariant measure. As $\inf|\tau'(x)| = \beta_1 > 1$, that is, τ is expanding, the assertion follows from Lasota and Yorke (1973). □

Note that the chaotic behavior described in Proposition 3 is robust in the sense that it persists for any perturbations of parameters provided that they are as in Proposition 2.

Let us summarize our findings for Case 2 in the following proposition:

Proposition 4. *Suppose that the parameters for the model given by (6) are as in Proposition 2. Then, three cases typically emerge depending on the initial condition:*
(i) *Poverty trap; for $k_0 < \bar{k}_1$, the economy converges to 0.*
(ii) *Chaotic middle-income trap; for $k_0 \in (\bar{k}_1, \bar{k}_2)$, the economy gets trapped in an interval, where it keeps fluctuating in a chaotic manner.*
(iii) *Perpetual growth; for $k_0 > \bar{k}_2$, the economy grows unboundedly.*

Note that, in the case of the chaotic middle-income trap, periodic points exist in the region. However, they are always unstable (i.e., repellers) and not observable.

The implication of Proposition 4 is as follows. In Case (i), if the economy starts from a sufficiently small initial value of capital k_0 with $k_0 < \bar{k}_1$, then the economy gets caught in a poverty trap, that is, k_t is attracted to the origin. This is because the low return from capital due to increasing marginal productivity of capital obstructs capital accumulation. In Case (iii), if the economy starts from a sufficiently large initial value of capital k_0 with $k_0 > \bar{k}_2$, then the economy exhibits perpetual growth. This is because the high marginal productivity of capital accelerates economic growth by the reverse logic to that of Case (i). In Case (ii), if the initial value of capital k_0 lies in the middle range, then the economy is caught by the middle-income trap. The intuition behind this might be explained as follows. If the economy starts from a value greater than $\bar{k}_1$ but smaller than θ, then technology 1 is chosen, and the marginal productivity of capital becomes large due to increasing marginal productivity, which accelerates economic growth until the threshold is crossed and the regime switches from technology 1 to technology 2. Then, the per capita capital stock is not sufficiently large for technology 2 to keep the economy growing. Thus, it begins to shrink, which brings it back to a point near the initial value, and the story repeats itself. Such a mechanism creates middle-income traps. Furthermore, the expanding property of the underlying dynamic system causes chaotic motions. This is an intriguing case because not only the economy becomes trapped in a middle-income trap but also the economy fluctuates chaotically in the trap. However, this cannot occur when the external effects of both technologies are weak (see Proposition 1).

Figure 4 depicts a typical trajectory that is eventually caught and chaotically fluctuates in a middle-income trap, as described in Proposition 4. Figure 5 plots four trajectories in the time series described in Proposition 4. Trajectory A in Figure 5 corresponds to a perpetual growth path. Trajectories B and C stand for trajectories caught into middle-income traps from above and from below, respectively. Finally, trajectory D is a typical path caught into the poverty trap.

INSERT Figures 4 and 5 around here.

3.3 Another situation in Case 2: Breakdown of the middle-income trap

When the trapping interval collapses due to some possible change in parameters, the economy is expected to escape the middle-income region in the long run so that it eventually either gets caught in the poverty trap or goes onto a perpetual growth path. Such cases occur, rather than (9), if

$$T_2(\theta) < \bar{k}_1 \quad \text{and/or} \quad \bar{k}_2 < T_1(\theta).$$

In the rest of this subsection, we focus on the following case:

$$T_2(\theta) < \bar{k}_1 \quad \text{and} \quad \bar{k}_2 < T_1(\theta). \tag{12}$$

See Figure 6 for this situation and Figure 7 for enlargement.

INSERT Figure 6 and Figure 7 around here.

Lemma 3. *The set of parameter values that satisfy (12) is not empty.*

Proof. See Appendix. □

Proposition 5. *Let $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 < \beta_1\beta_2$ be given. Then, for some open set of parameter values, $T : [\bar{k}_1, \bar{k}_2] \to \mathbb{R}_+$ is topologically chaotic in the sense that there exists an invariant Cantor set $\Lambda \subset [\bar{k}_1, \bar{k}_2]$ such that $T_{|_\Lambda} : \Lambda \to \Lambda$ is topologically conjugate to the one-sided full-shift on two symbols. Furthermore, for such Λ and any $k_0 \in [\bar{k}_1, \bar{k}_2] \setminus \Lambda$, either $\lim_{n\to\infty} T^n(k_0) = 0$ or $\lim_{n\to\infty} T^n(k_0) = \infty$ holds.*

Proof. From Lemma 3, we can take a set of parameter values satisfying (12). Using variable transformation

$$x_t = v(k_t) = \frac{\log(k_t/\bar{k}_1)}{\log(\bar{k}_2/\bar{k}_1)}, \tag{13}$$

we obtain a piecewise linear mapping

$$m : \mathbb{R} \to \mathbb{R},$$

$$x_{t+1} = m(x_t) = \begin{cases} m_1(x_t) = \beta_1 x_t & \text{if} \quad x_t \le c, \\ m_2(x_t) = 1 + \beta_2(x_t - 1) & \text{if} \quad c < x_t, \end{cases}$$

where $v \circ T(k_t) = m \circ v(k_t)$ and $c = v(\theta) \in (0, 1)$, indicating that T is topologically equivalent to m. Consider points in the unit interval $I = [0, 1]$ that remain under iteration of m. Because $1/\beta_1 + 1/\beta_2 < 1$, there are two closed subintervals $I_0 = [0, 1/\beta_1]$ and $I_1 = [1 - \beta_2, 1]$ such that $I_0 \cap I_1 = \phi$ and $I_0 \cup I_1 \subset \tau(I_i)$ $(i = 0, 1)$ (horseshoe condition). Furthermore, it holds that $\tau'(x) \ge \beta_1 > 1$ for all $x \in I_0 \cup I_1$ (hyperbolicity). See Figure 8. Thus, according to the standard argument of elementary dynamical systems theory (see e.g. Guckenheimer and Holmes 1983), there exists a closed m-invariant subset $\Lambda = \cap_{n\ge 0} T^{-n}(I_0 \cup I_1) = \{x \in I_0 \cup I_1 \,|\, T^n(x) \in I_0 \cup I_1,\, n \ge 0\} \subset I_0 \cup I_1$ as stated in the proposition. The trajectories that go to positive infinity correspond to high growth paths, whereas the trajectories thatgo to negative infinity correspond to those caught in the poverty trap. □

INSERT Figure 8 around here.

The invariant set Λ in Proposition 5 corresponds to the one-dimensional version of Smale's horseshoe. This suggests the possibility that the economy starting in the middle range exhibits a transiently chaotic behavior before it either gets caught in the poverty trap or goes onto a perpetual growth path. The destination that the economy ends up with can be highly random because the chaotic invariant set scrambles the nearby points. In their numerical study on endogenous business cycles, Asano et al. (2020) called this the "pinball effect" in the middle-income trap. Figure 9 shows how two initial states that are different but close to each other lead to different final states with transiently chaotic fluctuations.

INSERT Figure 9 around here.

3.4 Case 3: The occurrence of chaotic behaviors in the middle-income trap

Case 3 is the intermediate case between Cases 1 and 2. As in Case 2, the map given by (6) is valid for Case 3. The situation in Case 3 differs from that in Case 2 in that branch T_1 of map (6) becomes concave, whereas T_2 remains convex because $\beta_2 > 1 > \beta_1 > 0$. Consequently, several situations occur depending on the configuration of potential steady states $\bar{k}_i$ $(i = 1, 2)$ and the threshold θ.

Because we are interested in the occurrence of the middle-income trap, we focus on the situations wherein a trapping interval appears. By the concavity of T_1 and T_2, we can observe that the following inequality suffices to assure the existence of such a trapping interval for Case 3:

$$\theta < \bar{k}_1 \quad \text{and} \quad T_1(\theta) < \bar{k}_2 \tag{14}$$

or equivalently,

$$\left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{1}{\beta_2-\beta_1}} < (s(1-\alpha_1)A_1)^{\frac{1}{1-\beta_1}} \quad \text{and} \tag{15}$$

$$s(1-\alpha_1)A_1\left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{\frac{\beta_1}{\beta_2-\beta_1}} < (s(1-\alpha_2)A_2)^{\frac{1}{1-\beta_2}}\,. \tag{16}$$

Lemma 4. *Let $\beta_2 > 1 > \beta_1 > 0$ be fixed. Then the parameters that satisfy the inequalities given by (14) is not empty. Thus, there exists a trapping interval (or middle-income trap) $M = [T_2(\theta), T_1(\theta)]$ such that $T(M) \subset M$.*

Proof. See Appendix. □

As $T(0) = T_1(0) = 0$ and $\lim_{k \to +0} T_1'(k) = +\infty$, the origin is an unstable (i.e., repelling) steady state in Case 3. This implies that the poverty trap associated with the origin does not exist in this case.

Lemma 5. *For $\beta_2 > 1 > \beta_1 > 0$, the origin of T given by (6) is always a repelling steady state.*

Let us summarize what we have observed thus far.

Proposition 6. *Assume that $\beta_2 > 1 > \beta_1 > 0$. Then there exists some open set of parameter values for which the economy represented by (6) simultaneously exhibits middle-income traps and perpetual growth paths, but without a poverty trap.*

Proof. From Lemma 4 we can find parameters that satisfy the inequalities given by (14), which implies the existence of a trapping interval (middle-income trap). As all inequalities appearing in Lemma 4 are strict, any mapping T with slightly perturbed parameters also exhibits a middle-income trap. The non-existence of poverty traps follows from Lemma 5. For perpetual growth paths, just consider that $T'(\bar{k}_2) = T_2'(\bar{k}_2) > 1$ because of convexity of T_2. □

Figure 10 graphically represents the meaning of Proposition 6.

INSERT Figure 10 around here.

The relationship between the final states and initial conditions presented in Proposition 6 can be roughly summarized by the following proposition:

Proposition 7. *Assume that $\beta_2 > 1 > \beta_1 > 0$ and let the parameters satisfy (14). Then, two cases typically occur depending on the initial condition:*

(i) *Persistent fluctuations in the middle-income trap; for $k_0 \in (0, \bar{k}_2)$, the economy becomes trapped in an interval where it exhibits persistent fluctuation.*

(ii) *Perpetual growth; for $k_0 > \bar{k}_2$, the economy diverges to infinity.*

The following is the intuition behind Proposition 7. In this case two threshold values of k exist: the first one is θ representing the switching point of technology, and the second one is $\bar{k}_2$ standing for the unstable steady state under the technology with increasing marginal productivity of capital. For $k < \theta$, the economy's behavior is essentially the same as the standard Solow type model. For $k \in [\theta, \bar{k}_2)$, because the marginal productivity of capital is low, the economy shrinks. However, for $k > \bar{k}_2$ the marginal productivity of capital is high enough to promote capital accumulation. Further, the marginal productivity increases as capital accumulates, and therefore the economy grows perpetually.

Next, the study examines, in detail, what happens in the middle-income trap. We show that chaotic dynamics in the middle-income trap are possible in Case 3. To verify this, the same variable transformation is conducted as performed for Case 2 to obtain the mapping $\tau : I \to I$ given by (11), with only difference in $\beta_1 \in (0, 1)$ rather than $\beta_1 > 1$.

Similar to Case 2, the range of threshold c for τ in Case 3 is limited to some subinterval of $I = [0, 1]$.

Lemma 6. *Let $\beta_2 > 1 > \beta_1 > 0$ be fixed. Then, the threshold $c = h(\theta)$ of mapping (11) is in $(1 - 1/\beta_2, 1) \subset (0, 1)$.*

Proof. Translating (14) through the conjugacy h implies that $h(T_1(\theta)) = 1 < h(\bar{k}_i)$ for $i = 1, 2$. As $h(\bar{k}_1) = (1 - c\beta_1)/(1 - \beta_1)$ and $h(\bar{k}_2) = c\beta_2/(\beta_2 - 1)$, rearranging the inequalities above yields $(\beta_2 - 1)/\beta_2 < c < 1$. □

Let us consider the mapping $\tau : I = [0, 1] \to I$ given by (11). Let $I_L = [0, c]$ (left interval) and $I_R = (c, 1]$ (right interval) with $c \in (0, 1)$. We consider some simplest possible patterns of trajectories generated by τ. Specifically, we find a trajectory that visits the left interval successively only once and the right interval successively at least n times.

Lemma 7. *Any trajectory generated by τ stays successively at most once in the left interval I_L if $c < 1/(1 + \beta_1)$.*

Proof. Requiring $\tau_1(0) = 1 - c\beta_1 > c$, we obtain the result. □

Note that this condition implies that $\tau(I_L) \subset I_R$, which assures that any trajectory visits I_R at least once immediately after it has visited I_R.

Lemma 8. *Any trajectory generated by τ stays successively at least n times ($n \geq 2$) in the right interval I_R, if*

$$c < \frac{\beta_2^{n-1}}{1 + \beta_1 \beta_2^{n-1} + \sum_{j=1}^{n-1} \beta_2^j}. \tag{17}$$

Proof. See Appendix. □

For the first step, we identify the condition under which the chaotic behavior occurs when the trajectory of τ successively visits I_L at most once and I_R at least once.

Proposition 8. *Let $\beta_2 > 1 > \beta_1 > 0$ and $1 < \beta_1\beta_2 < 1 + \beta_1$. Then τ is chaotic for any $c \in ((\beta_2 - 1)/\beta_2, 1/(1 + \beta_1))$, and so is $T_{|_M} : M \to M$, where M is the middle-income trap.*

Proof. See Appendix. □

Next, we extend the above result to a slightly general pattern where the trajectory stays more often in the right interval.

Proposition 9. *Let $\beta_2 > 1 > \beta_1 > 0$ and $1 < \beta_1\beta_2^n < 1 + \beta_1\beta_2^{n-1}$ for $n \geq 2$. Then τ is chaotic for any*

$$c \in \left(\frac{\beta_2 - 1}{\beta_2}, \frac{\beta_2^{n-1}}{1 + \beta_1 \beta_2^{n-1} + \sum_{j=1}^{n-1} \beta_2^j} \right),$$

and so is $T_{|_M} : M \to M$.

Proof. Similar to Proposition 8, we have (17) from Lemma 8, and we have $(\beta_2 - 1)/\beta_2 < c < 1$ from Lemma 6. In order that such c can be taken, it must hold that

$$\frac{\beta_2 - 1}{\beta_2} < \frac{\beta_2^{n-1}}{1 + \beta_1 \beta_2^{n-1} + \sum_{j=1}^{n-1} \beta_2^j},$$

which is equivalent to $\beta_1\beta_2^n < 1 + \beta_1\beta_2^{n-1}$. Furthermore, as any trajectory of τ visits I_L successively at most once and I_R at least n times, it follows for any initial condition $x_0 \in (0, 1)$ that

$$(\tau^{n+1})'(x_0) \geq \beta_1\beta_2^n > 1$$

by assumption. Thus, τ is eventually expanding. □

INSERT Figure 11 around here.

Proposition 7, along with Propositions 8 and 9, suggests that even when only one out of the two technologies exhibits moderately strong externalities, chaotic dynamics in the middle-income trap can be observed for a large set of parameter values. This result is in sharp contrast to Proposition 1, where no chaotic behavior occurs virtually.

4 Concluding remarks

This paper introduced externalities in production into an OLG model with endogenous technology choice. Then, how the introduction affected macroeconomic fluctuations was analyzed. Specifically, we considered two types of production technologies that allow for the existence of external effects of capital and specified them as the Cobb-Douglas type. Umezuki and Yokoo (2019a) showed that, under the Cobb-Douglas specification, technology choice can generate periodic fluctuation with any lengths but never create chaotic fluctuations. In contrast, in the present model, if one of either technology exhibits a moderate degree of increasing marginal productivity of capital, the economy can exhibit a chaotic behavior.

The present analysis has some limitations. In analyzing technology choice, two production technologies are specified as the Cobb-Douglas type. However, this assumption may be slightly restrictive, and adopting a broader class of production technology, for example, the CES type, would be worthwhile. Asano et al. (2020, 2021) analyzed the dynamic implications of technology choice under the setting of CES technologies. However, they did not consider external effects in production. Thus, an analysis using CES technologies with external effects will be our future task. Moreover, Umezuki and Yokoo (2019b) analyzed the case of a continuum of Cobb-Douglas type technologies and showed that chaotic dynamics can appear for a wide set of parameters. An interesting extension is to incorporate into our model a continuum of technologies with production externalities.

Appendix

Appendix A: Microfoundation of technology choice behavior

This appendix provides a microfoundation for technology choice behavior in our model.

Our basic setup follows that of Matsuyama (2007). The economy begins in period 1, and continues over time toward infinity. The goods and factor markets are competitive. This economy has J types of production technologies. A type i technology converts m_i units of the final goods into $m_i R_i$ units of capital, and the final good is produced by $Y_{it} = F_i(k_t, K_t, L_t)$. Here, K_t and L_t are capital and labor at time t, respectively, and k_t is the capital-labor ratio capturing capital deepening externalities. The private marginal return of capital is

$$\frac{\partial}{\partial K_t} F_i(k_t, K_t, L_t) \equiv MPK_i.$$

We assume that capital depreciates completely in one period.

In each period, a unit of a new generation is born and lives in two periods: young and old periods. Each agent is assumed to have a long-linear utility. Thus, the saving rate is constant and independent of the real interest rate. We denote the savings rate by s. Young agents have two options in managing their saving: becoming either a lender or an entrepreneur. An agent who chooses to become a lender lends savings and obtains $r_{t+1} s w_t$ when old, where r_{t+1} denotes the real interest rate. An agent who chooses to become an entrepreneur selects one technology from the two types of technologies. Because an entrepreneur's wealth is equal to their saving, if $m_i > s w_t$, they have to borrow $m_i - s w_t$. However, due to the presence of capital market frictions, each entrepreneur can pledge only up to a constant fraction of the project revenue for the repayment, $\lambda_i m_i R_i \cdot MPK_i$, where $0 \leq \lambda_i \leq 1$. The fraction, λ_i, differs between the two types of projects. Specifically, the entrepreneur's borrowing constraint is represented by

$$\lambda_i m_i R_i \cdot MPK_i \geq r_{t+1}(m_i - s w_t) \text{ for } i = 1, ..., J. \quad (18)$$

As λ_i becomes smaller, the credit constraint becomes stronger.

Because an entrepreneur is always able to choose to become a lender, earnings from investment should not to be smaller than those from lending:

$$R_i \cdot MPK_i \cdot m_i - r_{t+1}(m_i - sw_t) \geq r_{t+1} sw_t, \tag{19}$$

that is,

$$r_{t+1} \leq R_i \cdot MPK_i \text{ for } i = 1, ..., J.$$

(18) can be rewritten as follows:

$$r_{t+1} \leq \frac{R_i \cdot MPK_i}{\left(1 - \frac{sw}{m_i}\right) / \lambda_i} \text{ for } i = 1, ..., J.$$

Defining

$$\Phi_i \equiv \frac{R_i \cdot MPK_i}{\max\left\{1, \left(1 - \frac{sw_t}{m_i}\right) / \lambda_i\right\}},$$

we can summarize (18) and (19) as

$$r_{t+1} \leq \Phi_i \text{ for } i = 1, ..., J.$$

Let us assume here that $r_{t+1} < \Phi_i$. Then, all agents become entrepreneurs and adopt type i technology, and this economy has no lender. Clearly, this cannot be an equilibrium, and we have $r_{t+1} \geq \Phi_i$. Let us next suppose that $r_{t+1} > \Phi_i$ for some i. Then, at least one of (18) and (19) for i is not satisfied; thus, type i is not adopted. In equilibrium, there must be a positive investment; it follows

$$r_{t+1} = \max\left\{\Phi_1, ..., \Phi_J\right\}. \tag{20}$$

Evidently, the technology yielding the highest value in the right-hand-side of (20) is adopted.

In this study, we consider a special case of (20):

$$J = 2,\ R_1 = R_2 = 1,\ \lambda_1 = \lambda_2 = \lambda \text{ and } d_1 = d_2 = d.$$

In this case, (20) reduces to

$$r_{t+1} = \max\left\{\frac{R \cdot MPK_1}{\max\left\{1, \left(1 - \frac{sw_t}{m}\right)/\lambda\right\}}, \frac{R \cdot MPK_2}{\max\left\{1, \left(1 - \frac{sw_t}{m}\right)/\lambda\right\}}\right\}$$

$$= \frac{R}{\max\left\{1, \left(1 - \frac{sw_t}{m}\right)/\lambda\right\}} \max\{MPK_1, MPK_2\}.$$

Thus, it is confirmed that the technology with higher marginal productivity of capital is selected (which is our technology choice assumption in the main text). Note that, although the interest rate may be reduced by the existence of credit constraints, it does not affect our analysis because the saving rate in the present model is independent of the interest rate.

Appendix B: Proofs

Proof of Claim 2. Let $\alpha_1 A_1/\alpha_2 A_2 = 1$ and $1 > \alpha_2 > \alpha_1$. Then, all we need to show is that the inequalities

$$s(1-\alpha_2)A_2 < 1 < s(1-\alpha_1)\alpha_2 A_2/\alpha_1$$

are possible. Rewriting the above expression as

$$\frac{\alpha_1}{\alpha_2(1-\alpha_1)} < sA_2 < \frac{1}{1-\alpha_2},$$

we notice that $\alpha_2/\alpha_1(1-\alpha_1) < 1/(1-\alpha_2)$ always holds because $\alpha_2 > \alpha_1$. As sA_2 can take any positive value, the claim is proven. □

Proof of Lemma 2 . Let $\alpha_1 A_1/\alpha_2 A_2 = 1$ and $1 > \alpha_2 > \alpha_1$. Then, the first inequality in condition (9) can be rewritten as

$$\left(\frac{1}{1-\alpha_2}\right)^{\frac{\beta_1-1}{\beta_1}} \left(\frac{\alpha_1}{\alpha_2(1-\alpha_1)}\right)^{\frac{1}{\beta_1}} < sA_2.$$

Similarly, the second inequality in condition (9) is expressed as

$$sA_2 < \left(\frac{1}{1-\alpha_2}\right)^{\frac{1}{\beta_2}} \left(\frac{\alpha_1}{\alpha_2(1-\alpha_1)}\right)^{\frac{\beta_2-1}{\beta_2}}.$$

Letting $V = 1/(1-\alpha_2)$ and $W = \alpha_1/\alpha_2(1-\alpha_1)$, we observe that $V > W$ as $\alpha_2 > \alpha_1$. Because sA_2 can be taken as any positive value, it suffices to show that $V^{1-1/\beta_1}W^{1/\beta_1} < V^{\beta_2}W^{1-1/\beta_2}$ or $V^\gamma < W^\gamma$, where $\gamma = 1 - 1/\beta_1 - 1/\beta_2$. Thus, the last inequality holds if $\gamma < 0$ or $1/\beta_1 + 1/\beta_2 > 1$. □

Proof of Lemma 3. Using the same notations as in the proof of Claim 2, it suffices to show that $V^{1-1/\beta_1}W^(1/\beta_1) > V^{\beta_2}W^{1-1/\beta_2}$ or $(V/W)^\gamma > 1$, where $\gamma = 1-1/\beta_1-1/\beta_2$. As $V/W > 1$, the last inequality holds if we take β_1 and β_2 ($\beta_2 > \beta_1 > 1$) such that $\gamma > 0$ or $1/\beta_1 + 1/\beta_2 < 1$. □

Proof of Lemma 4. Let $s \in (0,1)$ and $\alpha_2 \in (0,1)$ (hence, $\eta_2 = \beta_2 - \alpha_2$) be fixed. Let a_i ($i = 1,2$) be any numbers such that $1 < a_1 < a_2$. Let $\alpha_1 A_1/\alpha_2 A_2 = 1$. Then, inequalities (15) and (16) can be reduced to

$$1 < s(1-\alpha_1)\alpha_2 A_2/\alpha_1 < (s(1-\alpha_2)A_2)^{\frac{1}{1-\beta_2}}.$$

Solving the following simultaneous equations for A_2 and α_1,

$$\begin{aligned} a_1 &= s(1-\alpha_1)\alpha_2 A_2/\alpha_1, \\ a_2 &= (s(1-\alpha_2)A_2)^{\frac{1}{1-\beta_2}}, \end{aligned}$$

we obtain

$$A_2 = \frac{1}{\left(s(1-\alpha_2)a_2^{\beta_2-1}\right)} > 0 \quad \text{and} \quad \alpha_1 = \frac{1}{1+\left(\frac{1-\alpha_2}{\alpha_2}\right)a_1 a_2^{\beta_2-1}} \in (0,1),$$

which verifies the assertion. □

Proof of Lemma 8. Let us begin with $\tau_1(0) = 1 - c\beta_1 > c$ for at least once in I_R. To ensure that the trajectory to stay successively twice in I_R, we require

$$\tau_2(\tau_1(0)) = \beta_2(1 - c\beta_1 - c) = \beta_2 - c\beta_1\beta_2 - c\beta_2 > c.$$

To ensure that the trajectory to stay successively at least three times in I_R, we have

$$\tau_2^2(\tau_1(0)) = \beta_2(\beta_2 - c\beta_1\beta_2 - c\beta_2 - c) = \beta_2^2 - c\beta_1\beta_2^2 - c\beta_2^2 - c\beta_2 > c.$$

Repeating this up to n times, we obtain

$$\begin{aligned} \tau_2^{n-1}(\tau_1(0)) &= \beta_2^{n-1} - c\beta_1\beta_2^{n-1} - c\beta_2^{n-1} - c\beta_2^{n-2} - \cdots - c\beta_2^2 - c\beta_2 \\ &= \beta_2^{n-1} - c\beta_1\beta_2^{n-1} - c\beta_2\left(\sum_{j=0}^{n-2}\beta_2^j\right) > c. \end{aligned}$$

Solving the last inequality for c yields the result. □

Proof of Proposition 8. From Lemma 7, we have $c < 1/(1+\beta_1)$. Furthermore, from Lemma 6, $(\beta_2 - 1)/\beta_2 < c$. For such a c to be taken, we require

$$\frac{\beta_2 - 1}{\beta_2} < \frac{1}{1+\beta_1},$$

which is equivalent to $\beta_1\beta_2 < 1 + \beta_1$. Furthermore, because any trajectory (i.e., irrelevant to the initial conditions) of τ visits I_L successively at most once and I_R at least once, it follows for any initial condition $x_0 \in (0,1)$ that

$$(\tau^2)'(x_0) \geq \beta_1\beta_2 > 1,$$

where the last inequality follows by assumption. Thus, τ is eventually expanding and hence chaotic in the sense of Lasota and Yorke (1973). By conjugacy h, T is also chaotic. □

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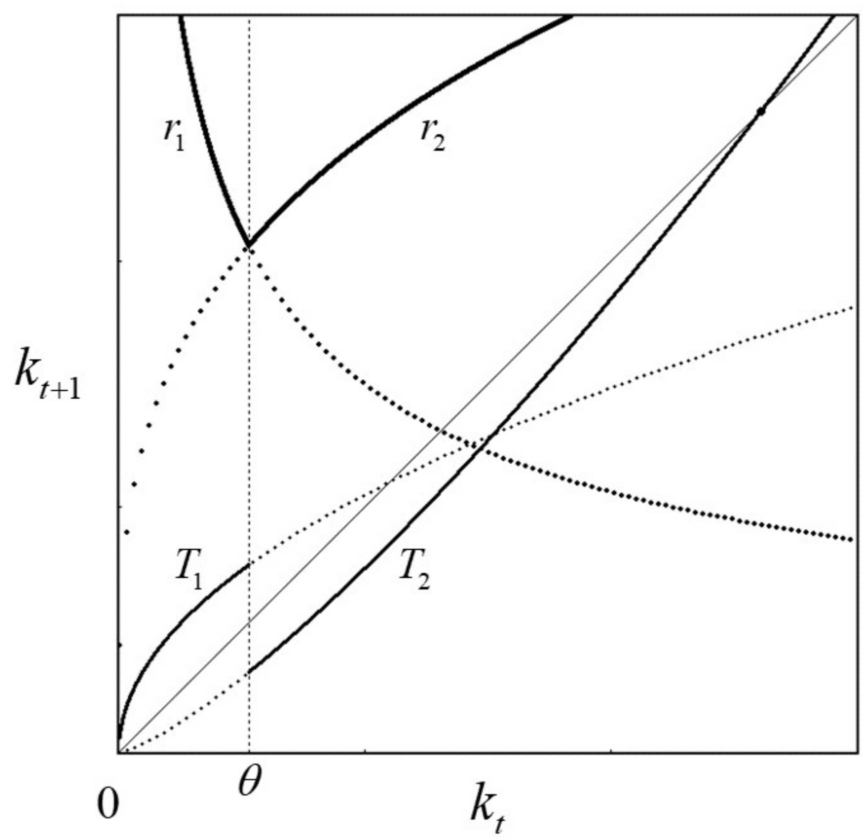

Figure 1: Graphs of r_1, r_2, T_1, and T_2 with $\beta_2 > 1 > \beta_1$. The r_2-curve is upward-sloping due to externality.

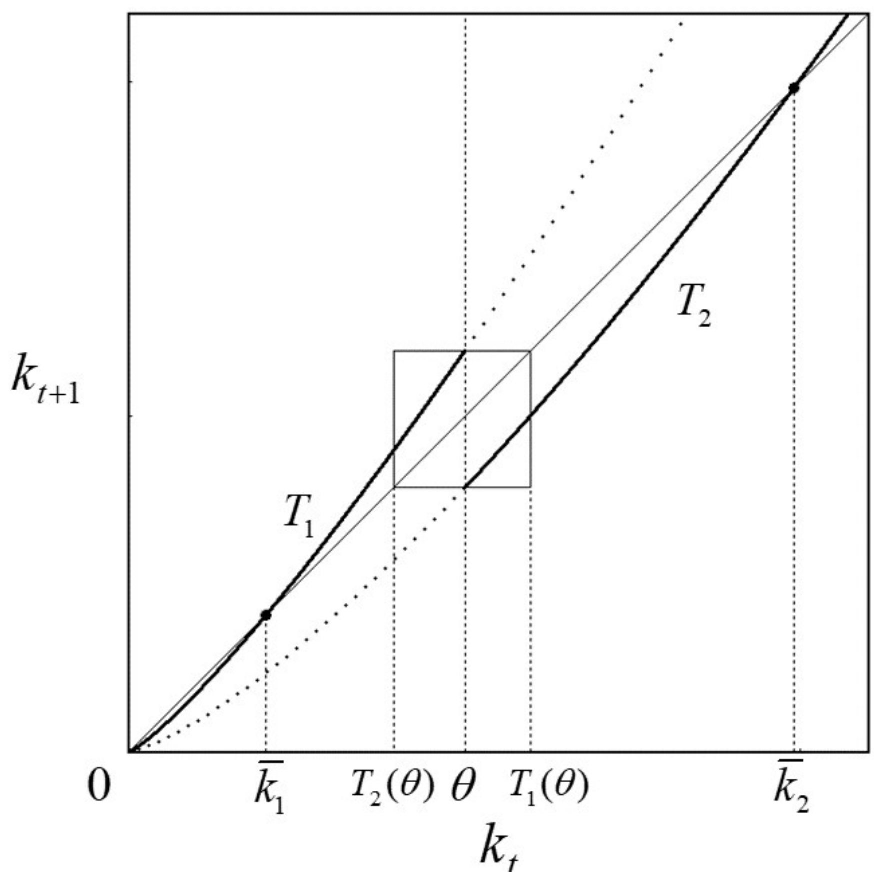

Figure 2: Coexistence of a poverty trap, middle-income trap, and high growth paths. $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 > \beta_1\beta_2$. Parameters: $A_2 = 5$, $\alpha_1 = 0.55$, $\alpha_2 = 0.65$, $\eta_1 = 0.65$, $\eta_2 = 0.7$, $s = 0.45$, $A_1 = \alpha_2 A_2/\alpha_1 \approx 5.91$, $\beta_1 = \alpha_1 + \eta_1 = 1.2$, and $\beta_2 = \alpha_2 + \eta_2 = 1.35$.

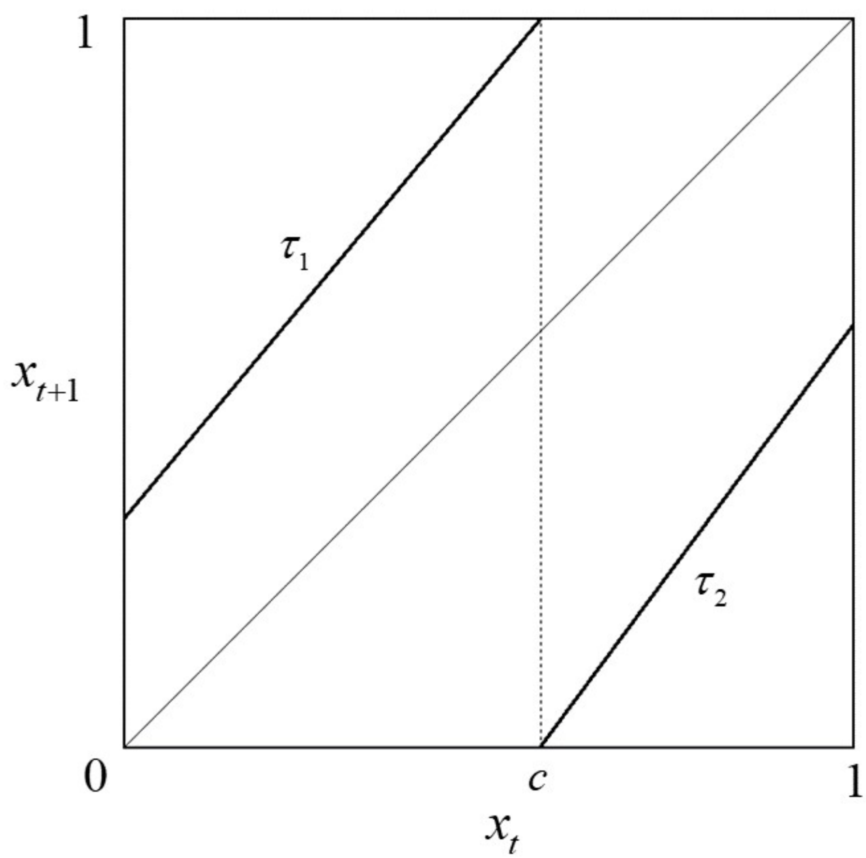

Figure 3: Piecewise-linearization on the middle-income trap for $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 > \beta_1\beta_2$. The parameter values are the same as in Figure 2.

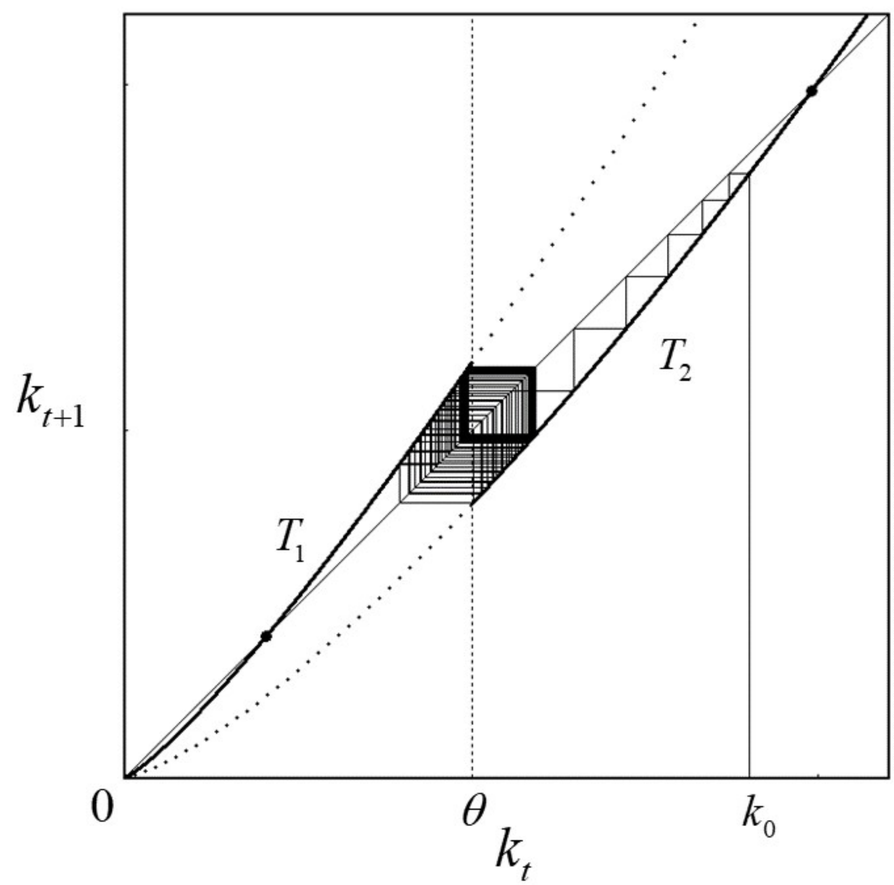

Figure 4: A trajectory converging into the middle-income trap and eventually fluctuating in that region in a chaotic manner. The parameters are the same as in Figure 2.

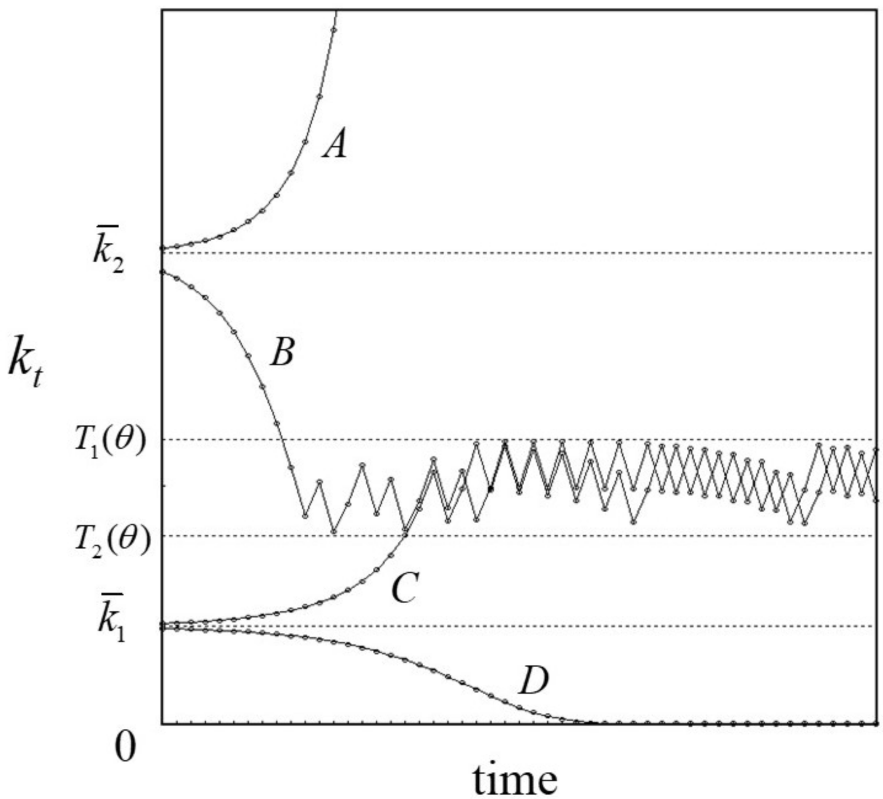

Figure 5: Time series corresponding to Proposition 4. A: a perpetual growth path. B and C: trajectories getting caught into the middle-income trap from above and below, respectively. D: a trajectory to the poverty trap. The parameters are the same as in Figure 2.

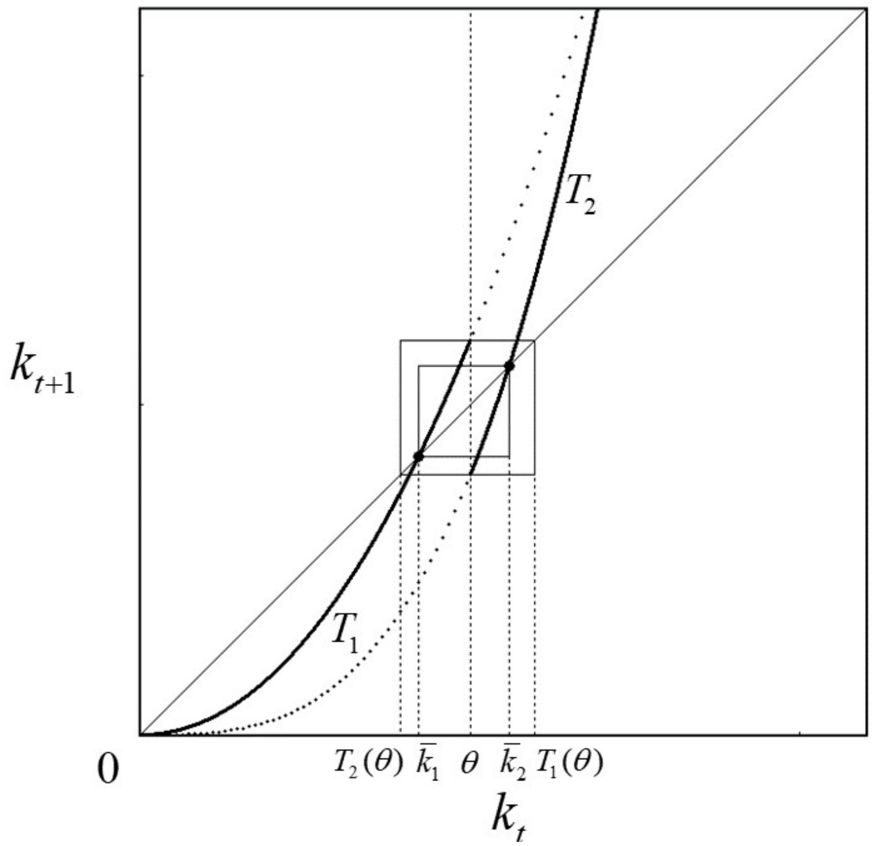

Figure 6: Collapse of the middle-income trap in Case 2. $\beta_2 > \beta_1 > 1$ and $\beta_1 + \beta_2 < \beta_1\beta_2$. In this case, a typical trajectory starting in $[\bar{k}_1, \bar{k}_2]$ eventually gets caught in the poverty trap or goes onto a high growth path. Parameters: $A_2 = 5$, $\alpha_1 = 0.45$, $\alpha_2 = 0.65$, $\eta_1 = 0.1$, $\eta_2 = 0.7$, $s = 0.45$, $A_1 = \alpha_2 A_2/\alpha_1 \approx 5.91$, $\beta_1 = \alpha_1 + \eta_1 = 2.05$, and $\beta_2 = \alpha_2 + \eta_2 = 3.15$

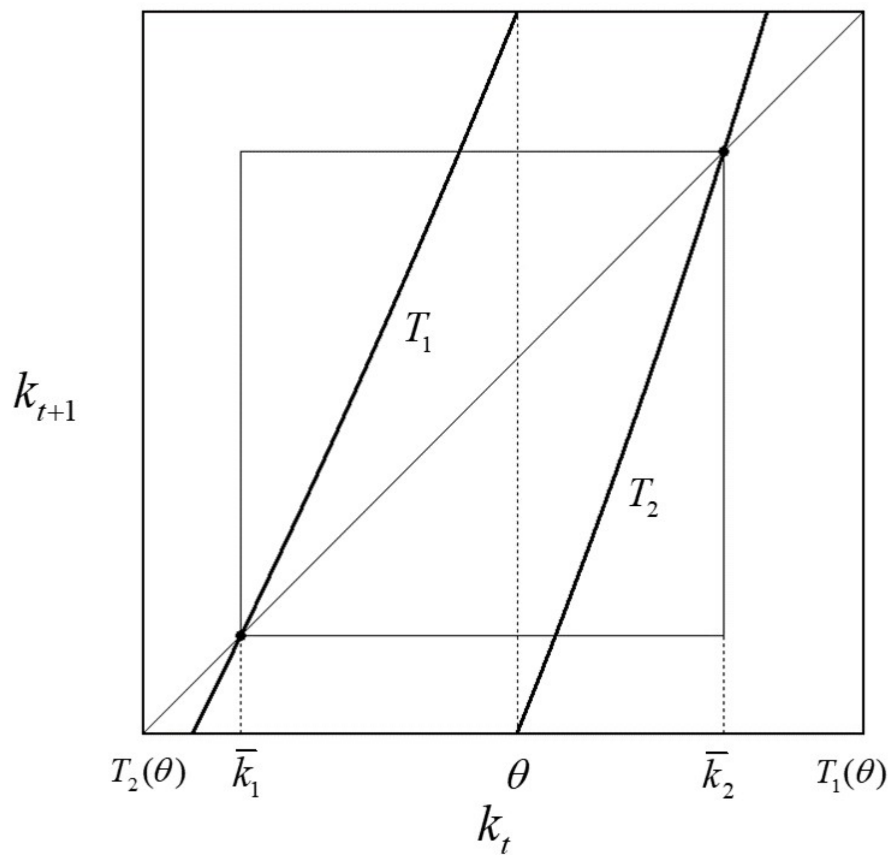

Figure 7: Enlargement of Figure 6.

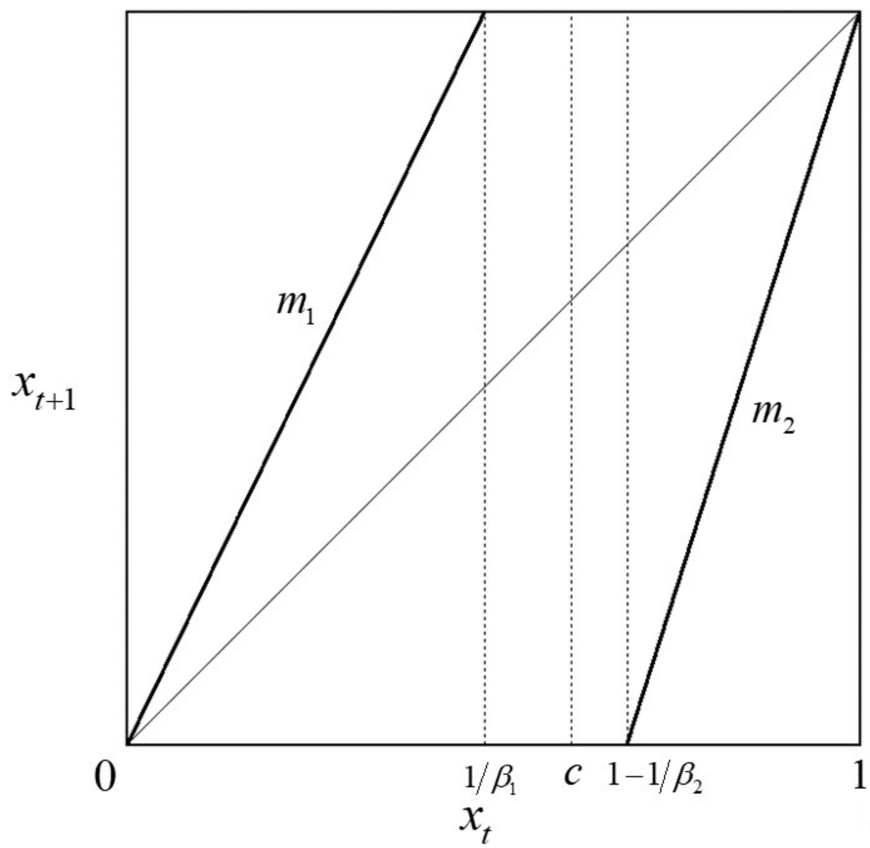

Figure 8: Piecewise-linearization of Figure 6 on the collapsed middle-income trap. The chaotic invariant set Λ is contained in $I_0 \cup I_1$. The iteration of the mapping brings any initial point that finally falls into the interval $(1/\beta_1, c)$ to the high growth path and any initial point that finally falls into $(c, 1-\beta_2)$ to the poverty trap.

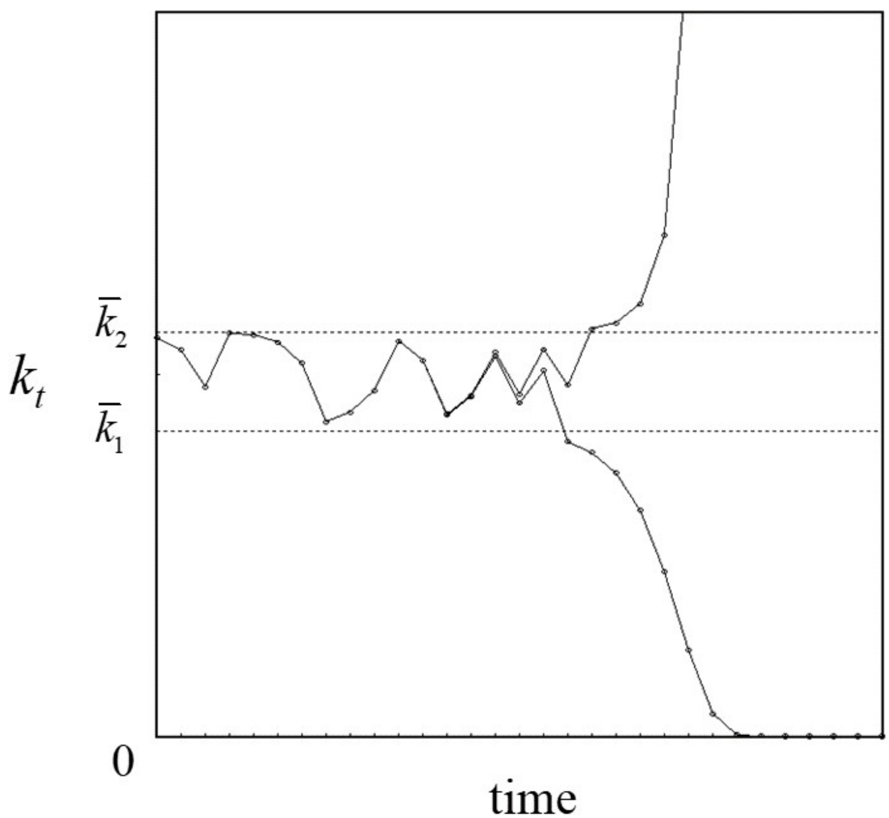

Figure 9: Two different but close to each other initial states near the chaotic invariant set Λ lead to different final states with transiently chaotic fluctuations. The parameters are the same as in Figure 6.

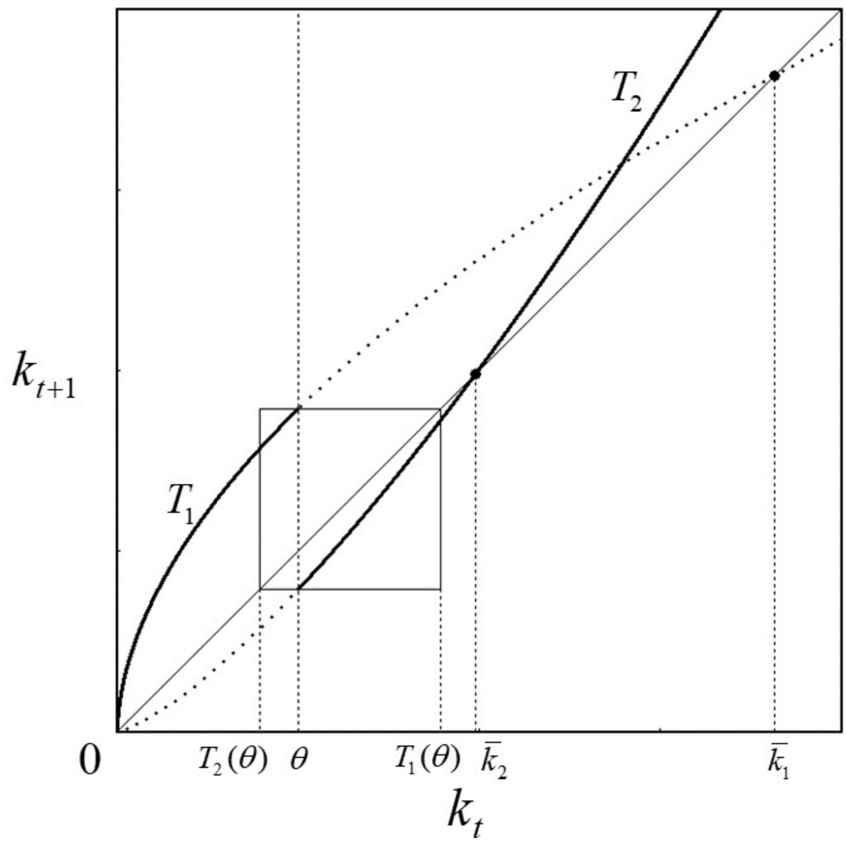

Figure 10: Coexistence of a middle-income trap and high growth paths. There is no poverty trap associated with the origin, which turns to a repeller in Case 3: $\beta_2 > 1 > \beta_1 > 0$. Parameters: $A_2 = 5$, $\alpha_1 = 0.55$, $\alpha_2 = 0.65$, $\eta_1 = 1.5$, $\eta_2 = 2.5$, $s = 0.45$, $A_1 = \alpha_2 A_2/\alpha_1 \approx 7.22$, $\beta_1 = \alpha_1 + \eta_1 = 0.55$, and $\beta_2 = \alpha_2 + \eta_2 = 1.35$.

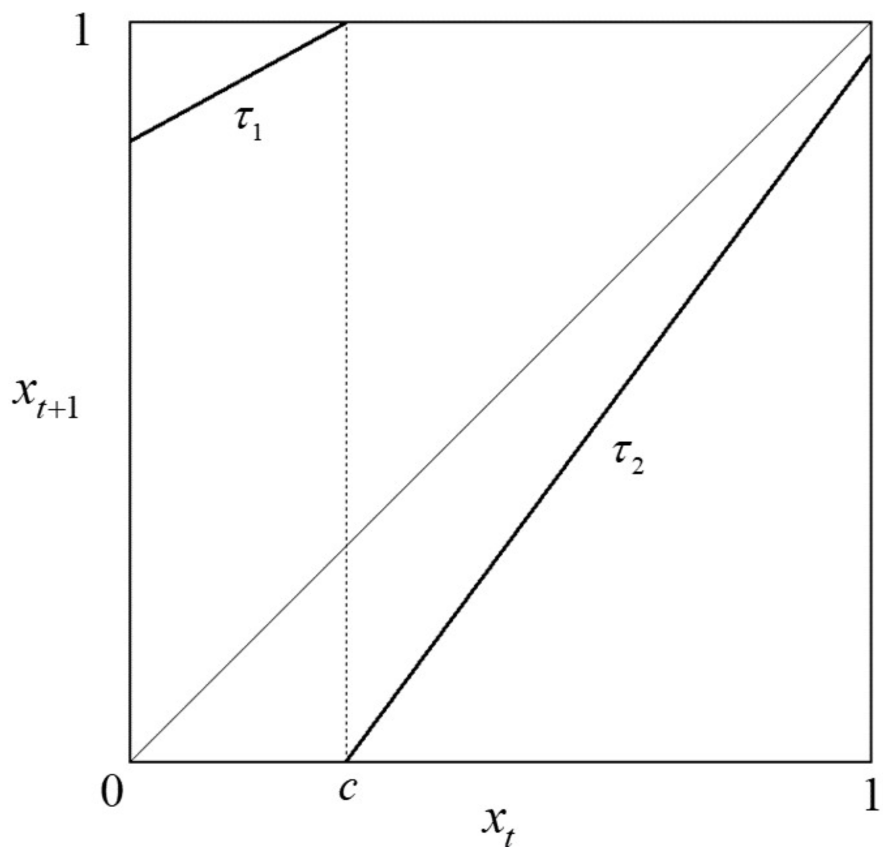

Figure 11: Piecewise-linearization on the middle-income trap for $\beta_2 > 1 > \beta_1 > 0$. The parameter values are the same as in Figure 10.