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Takao Asano, Akihisa Shibata and Masanori Yokoo

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Quasi-Periodic Motions in a Polarized Overlapping Generations Model with Technology Choice*

Takao Asano†, Akihisa Shibata‡ and Masanori Yokoo§

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Abstract

This paper constructs a simple overlapping generations (OLG) model with the working and capitalist classes and two types of production technologies. The behavior of agents belonging to the working class is basically the same as that in the standard Diamond (1965) type OLG model, whereas agents belonging to the capitalist class face two available technologies, select the one with a higher return on capital, and bequeath their assets to the next generation without supplying labor. Using techniques concerning the circle map in dynamical systems theory, we show that in an extreme case in which one technology is linear and the other is of the Leontief type, the economy exhibits bounded, non-periodic but non-chaotic motions for a large set of parameter values. We provide explicit formulas for the rotation number and the absolutely continuous invariant probability measure of our model.

Keywords: Endogenous business cycles; technology choice; quasi-periodic motion; OLG model; rotation number

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†Faculty of Economics, Okayama University, Tsushima-naka 3-1-1, Kita-ku, Okayama 700-8530, Japan. e-mail: takaoasano@okayama-u.ac.jp

‡Institute of Economic Research, Kyoto University, Yoshida, Sakyo-ku, Kyoto 606-8501, Japan, Phone:+81-75-753-7126, Fax:+81-75-753-7158, Email: shibata@kier.kyoto-u.ac.jp

§Faculty of Economics, Okayama University, Tsushima-naka 3-1-1, Kita-ku, Okayama 700-8530, Japan. e-mail: yokoo@e.okayama-u.ac.jp

1 Introduction

As summarized by Zucman (2019), income and wealth inequality in the United States have worsened significantly since the 1980s, and at the global level, the wealth holdings of the middle class have been squeezed, implying a polarization of the wealth distribution. These facts show that current economies are well represented by our model, in which we specify two types of agents: one that holds most of the wealth of the economy (the capitalist class) and another that has much a smaller share of the wealth (the working class). This global trend is widely recognized by researchers, including Moll (2014) and Mattauch et al. (2018), who also construct two-class dynamic models. In this paper, we construct a two-class overlapping generations (OLG) model with two possible technologies and show that an endogenous technology switch can generate perpetual fluctuations.

The endogenous occurrence of business cycles has attracted attention from many researchers. A typical factor that causes endogenous fluctuations is the existence of financial market imperfections. Examples of relevant studies in this literature include Woodford (1986), Azariadis and Smith (1989), Matsuyama (2007), Kunieda and Shibata (2014, 2017), and Vachadze (2020). One of the most recent studies on an open economy with a collateral constraint (that is, with limited external debts) is by Schumit-Grohé and Uribe (2021). They show that the mere presence of financial frictions can generate cyclical fluctuations of any periodicity and chaotic behavior. In contrast to these studies, the existence of credit market imperfections does not play any important role in this paper.

Another stream in the literature emphasizes the role of public debt issuance. For example, Farmer (1986) derives a necessary condition for a two-dimensional OLG model with public debt to generate persistent cycles around the golden rule steady state due to the Neimark–Sacker bifurcation. Proposition 2 in the paper indicates that such persistent cycles emerge only if the net worth of the government is positive (that is, only if the private sector is a net debtor) in the golden rule steady state. Introducing production externalities and public debt into the standard OLG model, Azariadis and Reichlin (1996) show by local bifurcation analysis that observable endogenous cycles can exist, and that the existence of public debt may cause an economy—which could have grown unboundedly under no public debt—to stagnate. Compared with the studies by these authors, who analyze local bifurcations and persistent cycles, Yokoo (2000), based on Farmer (1986), investigates the complicated global dynamics of a two-dimensional OLG model arising from homoclinic bifurcations. Moreover, Menuet et al. (2017) construct a continuous-time endogenous growth model with public debt and derive a two-dimensional dynamic system. They show that the model can have two balanced growth paths (high and low) and produce limit cycles along with local and global bifurcations.

Because public debt is a stock variable, its introduction into a model adds an extra dimension to the model. In a discrete-time formulation, a change from a one-dimensional to a two-dimensional model can generate richer patterns of economic dynamics, whereas in a continuous-time formulation, a change from a two-dimensional model to a three-dimensional one creates possibilities of complex dynamics in the economy.

In our model, because of the presence of two types of assets, and hence two asset accumulation equations, the dimension of the dynamics becomes higher. We face a discrete-time, two-dimensional OLG model. The theory of higher-dimensional nonlinear dynamical systems has been developed in mathematics (for textbook presentations, see, for example, Guckenheimer and Holmes (1983) and Palis and Takens (1993)). Its application to economics has been recognized as useful, especially in analyzing the *global* dynamics of nonlinear economic models. In higher-dimensional dynamic economic models, it is often the case that the dynamic process is ultimately attributed to the dynamics of a *circle map*. A typical example in economics occurs when the trajectories of a discrete-time dynamic model are attracted to a closed invariant curve that appears after the Neimark–Sacker bifurcation. However, it is rare that the explicit forms of the maps of such invariant curves can be obtained; therefore, in most cases, the nature of the circle dynamics cannot be rigorously characterized, and numerical investigation is required. Hommes' (1991, 1993, 1995) pioneering works are exceptions. Assuming the Hicksian nonlinearity, he succeeds in presenting a tractable model, giving a clue to analytical investigation of the circle dynamics.

Recent important extensions of this line of research are collected in Puu and Sushko (2006).

In fact, circle dynamics is more important than it may seem at first glance; a map with a certain type of discontinuity on an interval can be viewed as a map on a circle by identifying the endpoints of the interval. We will exploit the theory of the circle map to investigate our two-dimensional OLG model. In this paper, we slightly modify a standard textbook OLG model to obtain endogenously a kind of Hicksian nonlinearity, and present a simple model in which we can rigorously characterize the nature of the circle dynamics.[1] More concretely, by focusing on an extreme case, we first reduce the model to the dynamics on the interval. Then, the existence of a discontinuity caused by technology switching allows us to identify the interval as a circle, which is the key to our analysis. Thus, we use the theory of the circle map to investigate our two-dimensional economic model.

The basic structure of our model follows Diamond (1965). However, there are two significant modifications. In our model, two classes exist, which we refer to as the working and capitalist classes, and there are two types of production technologies from which to choose.[2] The agent belonging to the working class behaves like the agent in the standard settings of the Diamond (1965) type OLG model, whereas the agent belonging to the capitalist class, without supplying labor, bequeaths his/her assets to the next generation. We consider two state variables: capitalists' and worker's assets. Based on the theory of higher-dimensional nonlinear dynamical systems (in particular, a two-dimensional nonlinear dynamical system), we show that in an extreme case in which one technology is linear and the other technology is of the Leontief type, the dynamics of the capital supplied by the capitalist class is characterized by a rigid rotation on the circle, giving rise to quasi-periodic motions for typical parameter values.

In this paper, the assumption of a *sustainability condition* plays an important role. From the viewpoint of realism, we ignore the case in which the capitalists' wealth

[1]Although Pintus et al. (2000) analyze the circle dynamics using a standard infinite horizon agent model, their analyses depend on numerical simulations.

[2]Woodford (1986) and Pintus et al. (2000) also assume that the economy consists of two classes: workers and capitalists.

becomes negative. In addition, because our focus is not on the case of perpetual growth, we restrict our analysis to the case in which capital and output remain finite. We call the condition that guarantees this situation the *sustainability condition*.

Finally, we briefly explain differences between Yokoo's (2000) and our results. Yokoo's paper (2000) is the most closely related to ours in the sense that both investigate the global dynamics of two-dimensional OLG models with public "debt". However, there are significant differences. In Yokoo (2000), there is only one type of technology, and it is assumed to be of the constant elasticity of substitution (CES) type, for which the elasticity of substitution is small (i.e., close to the Leontief type) but finite. Under this setting, Yokoo (2000) shows that when the propensity to save is sufficiently low and the elasticity of the marginal production function is high, the model can generate chaotic dynamics due to the existence of a transverse homoclinic point. In contrast, our model incorporates two technologies, and the endogenous change of the technology choice can create quasi-periodic fluctuations even under production technologies that are as simple as possible analytically, e.g., linear and Leontief technologies. Furthermore, unlike Yokoo (2000), our model does not require that the workers' propensity to save be sufficiently small for such a permanent fluctuation pattern to occur.

2 Settings of the model

We examine an OLG model where there are two classes and a choice between two production technologies. Time is discrete and runs from 0 to infinity. Populations are constant over time. Only one good is produced. For the technology choice, we follow the formulation of Umezuki and Yokoo (2019) and Asano et al. (2020), among others.[3] Umezuki and Yokoo (2019) study the case of two Cobb–Douglas technologies, whereas Asano et al. (2020) deal with the case of two CES technologies. For an illustrative argument, we mainly focus here on an extreme case where there are only

[3]See also Aghion et al. (1999), Iwaisako (2002), and Matsuyama (2007) for graphical analyses of endogenous business cycle models due to endogenous technology choice, and Kunieda and Shibata (2003), Asano et al. (2012), Matsuyama et al. (2018), Asano and Yokoo (2019), and Umezuki and Yokoo (2019) for more rigorous mathematical analyses.

two technologies: one is of the Leontief type and the other is linear among the CES technology class.

Unlike Umezuki and Yokoo (2019) and Asano et al. (2020), we consider an agent group that solely aims at asset formation, leaving the assets to the next generation as bequests. We call this agent group the *capitalist class*. We assume that the capitalist class does not supply labor but invests capital in the industry to maximize returns. We refer to the conventional agents, who supply labor to the market, as the *working class*. Thus, our model involves two classes: the working and capitalist classes. We assume that the population of the capitalist class relative to the working class is $l > 0$. If we consider the situation where a few people own most of the wealth, then l should be small, say, $l \in (0, 1)$, but we do not impose such a restriction for generality.

2.1 Optimizing behavior of the agents

We consider a representative competitive firm. The production functions in the intensive form, f_i, are of the CES type of constant returns to scale. As usual, the firm's behavior is characterized by the first order conditions:

$$r_t = f_i'(k_t), \qquad w_t = f_i(k_t) - k_t f_i'(k_t) \equiv w_i(k_t), \qquad i \in M = \{1, 2, \dots, N\},$$

where k denotes capital per worker and M is the set of technologies. To be more specific, the functional forms are given by

$$\begin{aligned}
f_i(k) &= A_i k \left[\alpha_i + (1-\alpha_i)k^{\rho_i}\right]^{-1/\rho_i}, \\
f_i'(k) &= \alpha_i A_i \left[\alpha_i + (1-\alpha_i)k^{\rho_i}\right]^{-(1+\rho_i)/\rho_i}, \\
w_i(k) &= A_i k \left[\left[\alpha_i + (1-\alpha_i)k^{\rho_i}\right]^{-1/\rho_i} - \alpha_i \left[\alpha_i + (1-\alpha_i)k^{\rho_i}\right]^{-(1+\rho_i)/\rho_i}\right],
\end{aligned}$$

where $\alpha_i \in (0, 1)$ and $A_i > 0$ are constants. Note that if $\rho_i = -1$, then f_i becomes the linear production function, whereas if $\rho_i \to +\infty$, then f_i tends to the Leontief function. Furthermore, if $\rho_i \to 0$, then f_i approaches the Cobb–Douglas function.

We next consider the saving behavior of the working class. The representative agent of the working class is assumed to live for two periods, supplying one unit of labor inelastically only when he/she is young. This agent, born at time t, maximizes his/her Cobb–Douglas utility as follows:

$$\max_{c_t^w, d_{t+1}, s_t} (1-s)\log c_t^w + s\log d_{t+1}, \qquad s \in [0,1]$$

$$\text{s.t.} \qquad s_t + c_t^w = w_t, \qquad d_{t+1} = (1 + r_{t+1} - \delta)s_t.$$

Here, c_t^w denotes the consumption of the working class when young, d_{t+1} denotes consumption when old, s_t denotes saving in the form of capital, w_t denotes the real wage rate, r_{t+1} is the real rate of return on capital, δ is the depreciation rate of capital, and the subscript t denotes time. Utility maximization yields the agent's optimal saving:

$$s_t = sw_{t.}$$

Because the old workers are capital owners and d_{t+1} is an increasing function of r_{t+1}, we assume that they choose the technology giving the higher r_{t+1}. A microfoundation of this behavior is given in Appendix A.

The agent belonging to the capitalist class is assumed to live for two periods but consumes only when young. When old, he/she manages savings and leaves the amount of savings, with interest added, to his/her child; that is, the agent born at t obtains utility from his/her temporary consumption c_t^c (noting that c_t^c denotes the consumption of the capitalist class when young) and the bequest with interest added for his/her offspring. His/her utility maximization problem with Cobb–Douglas utility could be formulated as follows:

$$\max_{c_t^c, m_{t+1}} (1-\sigma)\log c_t^c + \sigma \log\left(1 + r_{t+1} - \delta\right) m_{t+1}, \;\; \sigma \in [0,1]$$

$$\text{s.t.} \qquad c_t^c + m_{t+1} = \left(1 + r_t - \delta\right) m_t,$$

which leads to the optimal bequest represented by

$$m_{t+1} = \sigma\left(1 + r_t - \delta\right) m_t.$$

In the old period, the capitalist chooses the technology that yields the higher return to maximize the bequest. Because $r_{t+1} = f_i'(k_{t+1})$, an old capitalist's problem is given by

$$\max_{i \in M} f_i'(k_{t+1}),$$

which shows that capitalists' technology choice behavior is the same as that of the working class. Because the depreciation rate affects neither the workers' saving behavior nor the capitalists' technology choice, in the following analysis we set $\delta = 1$; that is, capital depreciates completely once used.

2.2 A general model

Taking the optimization results in the previous subsection into account, we can represent the model that we utilize in what follows in a slightly general form:

$$k_{t+1} = sw_\gamma(k_t) + x_{t+1}, \tag{1}$$

$$x_{t+1} = \sigma f'_\gamma(k_t)x_t, \tag{2}$$

$$\gamma = \arg\max_{i \in M} f'_i(k_t). \tag{3}$$

Here, we have introduced a variable change: $m_t = x_t/l$. Note that if the agent of the working class is extremely impatient; that is, if $s = 0$, then the economy is perfectly polarized in the sense that the entire capital stock in the economy is solely owned by the capitalist class.

2.3 Extreme functional forms

For illustrative purposes, we consider the extreme case where there are only two production technologies, one of which is of the Leontief type (technology 1) and the other is linear (technology 2). The intensive form production functions (and their derivatives) and the real wage functions are represented as follows:

$$f(k) = \begin{cases} A[\alpha k + (1-\alpha)] \quad \text{if} \quad \rho = -1 \quad \text{(linear)} \\ Ak \quad \text{if} \quad k < 1,\ \rho = \infty, \quad \text{(Leontief)} \\ A \quad \text{if} \quad k \geq 1,\ \rho = \infty, \quad \text{(Leontief)} \end{cases}$$

$$f'(k) = \begin{cases} \alpha A \quad \text{if} \quad \rho = -1 \quad \text{(linear)} \\ A \quad \text{if} \quad k < 1,\ \rho = \infty, \quad \text{(Leontief)} \\ 0 \quad \text{if} \quad k \geq 1,\ \rho = \infty, \quad \text{(Leontief)} \end{cases}$$

$$w(k) = f(k) - kf'(k) = \begin{cases} (1-\alpha)A \quad \text{if} \quad \rho = -1 \quad \text{(linear)} \\ 0 \quad \text{if} \quad k < 1,\ \rho = \infty, \quad \text{(Leontief)} \\ A \quad \text{if} \quad k \geq 1,\ \rho = \infty. \quad \text{(Leontief)} \end{cases}$$

Note that the subscripts (1 for Leontief and 2 for linear) are omitted above. Of course, when $k = 1$, the derivative of f_1 does not exist. In such cases, we replace $f_1'(1)$ by the right derivative of f_1 for simplicity. To avoid the situation where technology 1 is never chosen, we assume that for $k_t < 1$,

$$f_1'(k_t) = A_1 > f_2'(k_t) = \alpha_2 A_2. \tag{4}$$

Then, technology choice is represented by

$$\gamma = \begin{cases} 1 \quad \text{(Leontief)}, & \text{if} \quad k < 1, \\ 2 \quad \text{(linear)}, & \text{if} \quad k \geq 1. \end{cases}$$

2.4 The Leontief–linear model

When the two technologies are of the Leontief and linear types under the condition (4), the dynamics of the economy given by (1)–(3) is represented by the following simultaneous difference equations with given initial conditions $k_0 > 0$ and $x_0 > 0$:

$$\text{If } k_t < 1, \text{ then} \qquad \begin{cases} k_{t+1} = \sigma A_1 x_t, \\ x_{t+1} = \sigma A_1 x_t. \end{cases} \tag{5}$$

$$\text{If } k_t \geq 1, \text{ then} \qquad \begin{cases} k_{t+1} = s(1-\alpha_2)A_2 + \sigma\alpha_2 A_2 x_t, \\ x_{t+1} = \sigma\alpha_2 A_2 x_t. \end{cases} \tag{6}$$

For simplicity of presentation, we introduce some new parameters as follows:

$$\begin{aligned} a &= \sigma A_1 \geq 0, \quad b = \sigma\alpha_2 A_2 \geq 0 \quad \text{(the equalities hold only if } \sigma = 0). \\ c &= s(1-\alpha_2)A_2 \geq 0 \quad \text{(the equality holds only if } s = 0). \end{aligned}$$

Note that c represents the amount of capital stock that the working class contributes to the economy.

Let us denote that $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$. Let $X_t = (k_t, x_t) \in \mathbb{R}_+^2$. Then, Eqs. (5)–(6) simplify to a map from $\mathbb{R}_+^2$ into itself, as follows:

$$F : \mathbb{R}_+^2 \to \mathbb{R}_+^2, \tag{7}$$

where

$$X_{t+1} = \begin{pmatrix} k_{t+1} \\ x_{t+1} \end{pmatrix} = F(X_t) = \begin{cases} F_L(X_t) & \text{if} \quad k_t < 1, \\ F_R(X_t) & \text{if} \quad k_t \geq 1 \end{cases}$$

with

$$F_L(X_t) = \begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix} X_t = \begin{pmatrix} ax_t \\ ax_t \end{pmatrix}, \tag{8}$$

$$F_R(X_t) = \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} X_t + \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} bx_t + c \\ bx_t \end{pmatrix}. \tag{9}$$

It is important to notice that each Jacobian matrix of F_j ($j = L, R$) is not of full rank. Thus, the dynamic behavior of the model will seem more "one-dimensional" even though the model itself is formally two-dimensional.

3 Analysis of the Leontief–linear model

3.1 Sustainability

Notice that the parameters a, b, and c in (8)-(9) can take any nonnegative values by choosing α_i, A_i, s, and σ appropriately. In what follows, we conduct the analysis under the following parametric restrictions:

$$a > 1 > b > 0. \tag{10}$$

Some comments are in order. As noted earlier, in the Introduction, from the viewpoint of realism it is appropriate to ignore the case in which the capitalists' wealth becomes negative. In addition, because our focus is not on the case of perpetual growth, we restrict our analysis to the case in which capital and output remain finite. Indeed, under (10), the economy neither explodes to infinity nor vanishes into nothing. Condition (10) guarantees that the capitalists' assets will not explode to infinity and that the economy-wide capital stock never vanishes. Therefore, this condition could be called the *sustainability condition.* Let us summarize this fact in the following proposition:

Proposition 1. *Let the initial conditions $k_0 > 0$ and $x_0 > 0$ be given. If the sustainability condition given by (10) holds, then one of the following two cases occurs depending on the value of c, that is:*
(i) if $c \in [0, 1)$, then x_t enters the interval $[(1-c)b, a]$ in some finite iterates and never leaves it.
(ii) if $c \geq 1$, then x_t converges to zero, while k_t converges to c, as t goes to infinity.

Proof. See the Appendix. $\square$

Case (ii) is rather dull because the capitalist class vanishes in the end and the economy settles down to a steady state where the entire capital stock is owned by the working class. In what follows, we will focus on case (i).

3.2 Some preliminaries

Before proceeding to the detailed analysis of the model, we briefly review some mathematical notions used. For more detail, see Guckenheimer and Holmes (1983). Let us consider the unit circle denoted by S^1. There are three ways of defining the circle: $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ (the Euclidean circle), $\{z \in \mathbb{C} \,|\, z = e^{2\pi i\theta},\, \theta \in \mathbb{R}\}$ (the complex circle), and $\mathbb{R}/\mathbb{Z}$ (the real numbers modulo the integers).[4] For our purpose, it is most convenient to define that $S^1 = \mathbb{R}/\mathbb{Z}$, which can be thought of as the interval $[0,1)$ with the endpoints connected. The circle map that we examine in this paper is restricted to a homeomorphism (one-to-one, onto, continuous, and with a continuous inverse) of the circle. Instead of the notion of order on the real line, we need the notion of *orientation* for S^1. Let $a, b \in S^1$; then, (a,b) means the interval wrapping forward from a to b. Let $f : S^1 \to S^1$ be a homeomorphism. We say that f is *orientation-preserving* if for any $a, b \in S^1$ and every point $c \in (a,b)$, $f(c) \in (f(a), f(b))$ holds. This corresponds to the situation where f is increasing on $\mathbb{R}$. The simplest orientation-preserving circle homeomorphism is the *rotation* $R_\alpha : S^1 \to S^1$ given by

$$R_\alpha(x) = x + \alpha,$$

where $\alpha \in \mathbb{R}$ is a fixed number. It is known that if α is rational, then every point in S^1 is periodic. To be more specific, if $\alpha = p/q$ is an irreducible fraction with $p, q \in \mathbb{Z}$, then for every $x \in S^1$,

$$R^q_{p/q}(x) = x + p \bmod 1 = x.$$

On the other hand, if α is irrational, it is known that the orbit of any point is dense in S^1. Equivalently, for any x, $y \in S^1$ and any neighborhood of y, $N(y) \subset S^1$, there

[4] For this argument, see, for example, Turer (2019).

is an integer n such that $R_\alpha^n(x) \in N(y)$. Compared with the rational case, we will call this irrational case of the dynamics *quasi-periodic*.[5]

Next, we define the projection $\pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ by $\pi(x) = x$ mod 1. In the rest of this subsection, let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. We say that the map $F : \mathbb{R} \to \mathbb{R}$ is a *lift* of $f : S^1 \to S^1$ if $\pi \circ F = f \circ \pi$. For a given f, there are infinitely many lifts, any two of which differ only by an integer. Note that f is orientation-preserving if and only if F is orientation-preserving (i.e., increasing).

The most important notion of the circle homeomorphism is the *rotation number*. The rotation number of f, $\rho(f)$, is well defined by

$$\rho(f) = \left(\lim_{n\to\infty} \frac{F^n(x)}{n} \right) \bmod 1, \tag{11}$$

where x is any point in S^1 and F is any lift of f.

There is an equivalent but more practical definition of the rotation number. To see this, pick a point $x \in S^1$ and partition S^1 into two "arcs": $I_0 = [x, f(x))$ and $I_1 = [f(x), x)$. For any point $y \in S^1$, we define the rotation number as follows:

$$\rho(f) = \lim_{n\to\infty} \frac{1}{n} \left(\text{cardinality}\{f^i(y) \mid 0 \le i < n \,\text{and}\, f^i(y) \in I_0\} \right).$$

It is known that $\rho(f)$ exists and is independent of y (see Guckenheimer and Holmes (1983, Proposition 6.2.1)). It is also known that $\rho(f)$ is rational if and only if f has a periodic orbit (see Guckenheimer and Holmes (1983, Proposition 6.2.4)). For the rotation above, it is known that $\rho(R_\alpha) = \alpha$ (see Guckenheimer and Holmes (1983, p. 297)). Intuitively, the rotation number is the asymptotic proportion of the points on the trajectory that visit I_0.

Let X and Y be topological spaces. Let $f : X \to X$ and $g : Y \to Y$ be two maps. We say that f and g are *topologically conjugate* if there exists a homeomorphism $\varphi : X \to Y$ such that for all $x \in X$, $\varphi \circ f(x) = g \circ \varphi(x)$. Topological conjugacy means that g inherits many properties of f. For instance, if f has a dense orbit, so does g. It is also known that the rotation number is a topological invariant; that is, if f and g are topologically conjugate, then $\rho(f) = \rho(g)$.

[5] In the literature, it is also called *almost periodic*.

Let $I = [0, 1]$ and let $\mathcal{B}$ be the Borel σ-algebra of $[0, 1]$. Given a measurable function $\tau : I \to I$, a measure μ is said to be *τ-invariant* if $\mu(\tau^{-1}(E)) = \mu(E)$ for all measurable sets $E \in \mathcal{B}$. We say that a measure μ is *absolutely continuous* with respect to a measure ν if $\nu(E) = 0$ implies that $\mu(E) = 0$. The existence of an absolutely continuous invariant measure plays an important role in economics because it ensures the observability of recurrent but not periodic fluctuations in the long run and describes the asymptotic distribution of economic states over the course of a business cycle. A measurable function $\tau : I \to I$ is *ergodic* with the τ-invariant measure μ if for any $E \in \mathcal{B}$ with $\tau^{-1}(E) = E$, $\mu(E) = 0$ or $\mu(I \backslash E) = 0$. This implies that a set E with $\tau^{-1}(E) = E$ is a zero-measure set or is of full measure; that is, the measure can no longer be decomposed.

3.3 Periodic and quasi-periodic motions

We investigate the typical dynamics of the system described by (7)-(9). We begin with a simpler case where $c = 0$ or $s = 0$. In other words, this case corresponds to a situation where the working class agents are extremely impatient, so that they consume their entire income when they are young and, as a result, capital is entirely owned by the capitalist class. In this case, for $t \geq 1$, $k_t = x_t$ holds. Furthermore, the two-dimensional system (7)-(9) reduces to a (one-dimensional) *piecewise linear* map of the interval:

$$T_{a,b} : I_{a,b} \to I_{a,b}, \qquad I_{a,b} = [b, a],$$

$$x_{t+1} = T_{a,b}(x_t) = \begin{cases} ax_t, & \text{if } x_t < 1, \\ bx_t, & \text{if } x_t \geq 1. \end{cases} \tag{12}$$

It is crucial to recognize that this type of map of the interval can be regarded as a piecewise linear homeomorphism (i.e., a homeomorphism with piecewise constant derivatives) of the circle. As a circle map, our model given by (12) is *continuous* because $T_{a,b}(b) = T_{a,b}(a) = ab$, irrespective of the choice of a and b. It is important to notice that the threshold $x = 1$ in (12) is not a discontinuity[6] for a circle map.

[6] Several economic models in which the dynamics can be characterized by a piecewise continuous map with a discontinuity of the interval can be identified with some circle maps, but they are not necessarily continuous as circle maps. For instance, every economic model developed in Ishida and

Proposition 2. *The map $T_{a,b}$ given by (12) is identified with an orientation-preserving piecewise-linear homeomorphism of the circle.*

Proof. It suffices to identify the endpoints of the interval $[b, a]$ and to see that $T_{a,b}(b) = T_{a,b}(a) = ab$, irrespective of the choice of a and b. $\square$

To understand this situation, see Figure 1 for a circle map of $\mathbb{R}/\mathbb{Z}$ given by (12) and Figure 2 for its lifts on $\mathbb{R}$, where the map is modified so as to be defined on $[0, 1]$ using the trivial linear conjugacy: $\phi(x) = (x - b)/(a - b)$.

[insert Figures 1 and 2 around here]

Fortunately, the dynamics of a class of piecewise linear circle homeomorphisms has been well studied in the mathematics literature (although not necessarily the economics literature). Among others, Coelho et al. (1995) and de Faria and Tresser (2014) deal with some classes of piecewise linear circle homeomorphisms including $T_{a,b}$ in (12) as a special case with a different parametrization. For a more general class of piecewise linear maps of the circle, see Liousse (2004).

Based on the results of the abovementioned authors, it can readily be understood that the map $T_{a,b}$ in (12) exhibits a unique absolutely continuous invariant measure (and thus ergodic behavior) on the interval $I_{a,b}$ as long as its rotation number is irrational. Moreover, the formulas for the rotation number and the invariant measure can be computed explicitly:

Proposition 3. *(Abundance of quasi-periodic motions for $c = 0$) Let the sustainable condition (10) be satisfied and let $c = 0$. Then, the map $T_{a,b} : I_{a,b} \to I_{a,b}$ given by (12) is topologically conjugate to the rotation $R_\alpha : S^1 \to S^1$ with the rotation number:*

$$\alpha = \frac{\log a}{\log a - \log b} = \log_{a/b} a. \tag{13}$$

When α is rational, every point in $I_{a,b}$ is a periodic point of some period $q > 1$. Moreover, when α is irrational, $T_{a,b}$ is uniquely ergodic, having an absolutely continuous

Yokoo (2004), Asano et al.(2012), and Umezuki and Yokoo (2019) could be treated as a circle map. However, in contrast to ours, all these models have a discontinuity as a circle map. For its dynamic consequences, see Keener (1980).

invariant probability measure μ *given by*

$$d\mu(x) = \frac{dx}{x\log(a/b)}. \tag{14}$$

Proof. See the Appendix. □

Now, we want to see what happens if $c \neq 0$. We will show that if c is not very large and ab is not very small, then there is no qualitative difference in the behavior of the capital stock provided by the capitalist class when $c = 0$ or $c \neq 0$. To this end, for $c \in [0, 1)$, let $I_{a,b,c}$ be a closed interval defined by

$$I_{a,b,c} = [(1-c)b, (1-c)a].$$

Furthermore, we define the map $\tau_{a,b,c} : I_{a,b,c} \to I_{a,b,c}$ to be:

$$x_{t+1} = \tau_{a,b,c}(x_t) = \begin{cases} ax_t & \text{if} \quad x_t < 1-c, \\ bx_t & \text{if} \quad x_t \geq 1-c. \end{cases} \tag{15}$$

Proposition 4. *(Abundance of quasi-periodic motions for* $c \geq 0$*) Let the sustainable condition (10) be satisfied. Furthermore, assume that* $ab > 1$ *and let* c *be such that:*

$$0 \leq c < \theta, \quad \textit{where} \quad \theta = \frac{ab-1}{ab}. \tag{16}$$

Then, for any initial condition $(k_0, x_0) \in \mathbb{R}^2_+$*, there exists some integer* $t_0 = t_0(k_0, x_0) \geq 0$ *such that for any* $t \geq t_0$*, the sequence of the capitalists' assets* $\{x_t\}$ *generated by (7) coincides with a trajectory of the map* $\tau_{a,b,c} : I_{a,b,c} \to I_{a,b,c}$ *given by (15), which is topologically conjugate to the rotation, with the rotation number given by (13).*

Hence, when α *is rational, every trajectory of* $\{x_t\}$ *is eventually periodic*[7] *with some period* q*, irrespective of initial condition* (k_0, x_0)*. Furthermore, when* α *is irrational,* $\tau_{a,b,c}$ *has an absolutely continuous invariant probability measure on* $I_{a,b,c}$*, given by (14), and therefore every trajectory of* $\{x_t\}$ *is eventually quasi-periodic and dense in* $I_{a,b,c}$*.*

Proof. See the Appendix. □

[7] The definition of eventual periodicity in Deng et al. (2021) is different from ours. In Deng et al. (2021), the term *eventual periodicity* is used interchangeably with the *global stability of a periodic orbit*.

Remark 1. The above proposition implies that if the set of parameter values (a, b, c) is randomly chosen by nature so that they satisfy the conditions in Proposition 4, then quasi-periodic motions are observed with probability 1 because the graph of the function of α given by (13) is strictly monotonic with respect to a and b. See the next section for more detail.

Remark 2. If c violates the restriction given by (16), the capitalists' assets x_t may exceed the upper bound of $I_{a,b,c} = [(1-c)b, (1-c)a]$, although they will remain within the interval $[(1-c)b, a]$. See Proposition 1. The dynamics of this case may be intriguing, but more complex; thus, we will leave it to future research. See Figure 3, which shows a bifurcation diagram with respect to c, where one can observe that the capitalists' assets x_t exhibit a quasi-periodic cycle on $I_{a,b,c} = [(1-c)b, (1-c)a]$ for each $c \in [0, \theta)$. When c reaches θ, a kind of bifurcation occurs and, for $c \in (\theta, 1)$, x_t spreads over $[(1-c)b, a]$, giving rise to complicated limit sets comprised of disconnected bands.

Remark 3. Under our parametric restrictions above, one cannot observe chaotic behaviors even when the economy exhibits bounded, nonperiodic fluctuations for almost all cases. In fact, the Lyapunov exponent, which measures the sensitive dependence on initial conditions; that is, the most essential feature of chaotic dynamics, will point to zero as we point out in the Appendix.

The proposition above says that if the working class does not save much (that is, c is not very large), then the capitalists' assets will neither explode to infinity nor vanish, but will keep fluctuating within some range in a nonperiodic but almost periodic manner, irrespective of initial conditions. We will examine the quasi-periodicity in subsection 3.4 in detail with some parametric examples.

[insert Figure 3 around here]

3.4 Some parametric examples

For an exposition, we provide some examples for (7) with the parameters set as in Proposition 4. Let $ab = 2$ be fixed. Let a, b, and c satisfy the conditions given in Proposition 4. As $\rho(T_{a,2/a}) = \rho(\tau_{a,2/a,c})$, we write these rotation numbers as $\rho(a)$ as long as c is appropriately adjusted. For instance, if $a = 4$ and $b = 2/a = 1/2$, then the rotation number $\rho(4) = \alpha$ must satisfy:

$$a^{\alpha} b^{1-\alpha} = 4^{\alpha}(1/2)^{1-\alpha} = 1.$$

Solving this for α, we obtain $\rho(4) = \alpha = 1/3$. This means that for, say, $\tau_{a,b,c}$ in (15), any trajectory exhibits a period-three cycle, visiting the interval $[(1-c)b, 1-c]$ once and the interval $(1-c, (1-c)a]$ twice over one cycle. Similarly, $\rho(8) = 2/5$, which indicates the occurrence of a period-five cycle. Note that for $a \in [4, 8]$ and $ab = 2$, it suffices to assume that $0 \le c < 1/2$ for condition (16) is satisfied. See Figures 4 and 5 for a period-three cycle and a period-five cycle, respectively.

[insert Figures 4 and 5 around here]

Now, for two rational numbers p/q and p'/q', where $0 < p/q < p'/q' < 1$ and $\gcd(p, q) = \gcd(p', q') = 1$,[8] we define a new rational number p''/q'' in such a way that:

$$\frac{p''}{q''} = \frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}.$$

Then, $p/q < p''/q'' < p'/q'$. By creating a new rational number for two given neighboring rational numbers, we obtain the following sequence starting with $\{\rho(4), \rho(8)\}$:

$$\begin{aligned}
F_1 &= \left\{\frac{1}{3}, \frac{2}{5}\right\}, \\
F_2 &= \left\{\frac{1}{3}, \frac{3}{8}, \frac{2}{5}\right\}, \\
F_3 &= \left\{\frac{1}{3}, \frac{4}{11}, \frac{3}{8}, \frac{5}{13}, \frac{2}{5}\right\}, \\
F_4 &= \left\{\frac{1}{3}, \frac{5}{14}, \frac{4}{11}, \frac{7}{19}, \frac{3}{8}, \frac{8}{21}, \frac{5}{13}, \frac{7}{18}, \frac{2}{5}\right\}, \\
F_5 &= \left\{\frac{1}{3}, \frac{6}{17}, \frac{5}{14}, \frac{9}{25}, \frac{4}{11}, \frac{11}{30}, \frac{7}{19}, \frac{10}{27}, \frac{3}{8}, \frac{11}{29}, \frac{8}{21}, \frac{13}{34}, \frac{5}{13}, \frac{14}{31}, \frac{7}{18}, \frac{9}{23}, \frac{2}{5}\right\}, \\
&\vdots
\end{aligned}$$

This is reminiscent of the Farey sequence (Hardy et al. (2008) provide more detail on this). Taking account of the continuity of $\rho(a)$ with respect to a when $\tau_{a,2/a,c}$ is

[8] Note that $\gcd(p, q)$ denotes the greatest common divisor of p and q.

continuous with respect to a (see Guckenheimer and Holmes (1983)), we can find periodic cycles in the order of, say, F_4, as we continuously increase the parameter value of a from 4 to 8:

$$3 \to 14 \to 11 \to 19 \to 8 \to 21 \to 13 \to 18 \to 5, \tag{17}$$

where each number corresponds to the period of the cycle. Furthermore, by repeating the same argument, we can find a periodic cycle of an arbitrarily large period between, say, period-three and period-14 cycles in (17).

As the rotation number, which is a topological invariant, is given by (13), we can calculate the value of a for which the rotation number of $\tau_{a,2/a,c}$ is, say, 3/8 in F_2. In fact, solving:

$$\frac{3}{8} = \frac{\log a}{\log a - \log(2/a)}$$

yields $a = 4\sqrt{2}$. See Figure 6. For the corresponding period-eight cycle, see Figure 7.

[insert Figures 6 and 7 around here]

However, for the set of parameter values in Proposition 4, it is very rare to observe such periodic motions. As the rotation number $\rho(a)$ is in fact a smooth, strictly increasing function of a, as shown in Fig.6, the Lebesgue measure of the set of a for which $\rho(a)$ is rational is zero. For the set of parameter values in Proposition 4, it is most likely that we will observe quasi-periodic motions, instead to which irrational rotation numbers correspond. See Figures 8, 9, and 10 for quasi-periodic behaviors.

[insert Figures 8, 9, and 10 around here]

Contrary to the periodic case, the trajectory runs densely in the interval $[(1-c)b, (1-c)a]$ when the rotation number is irrational. Nonetheless, it is important to recognize that any irrational rotation number can arbitrarily be approximated by rational numbers because of their density in the interval $[0, 1]$. In other words, even when some quasi-periodic cycle occurs, it can be approximated by some periodic cycle. That may be more apparent in a time series. Compare Figure 11 with Figure

12. In this case, the quasi-periodic trajectory in Figure 12 may be viewed as a period-eight cycle with some "delay".

[insert Figures 11 and 12 around here]

Finally, we have seen that Proposition 4 suggests that the fluctuation of the capitalists' assets in the long run follows a certain distribution, irrelevant of initial conditions. The numerical simulations below demonstrate that the larger the number of iterations of our model is, the more accurately the histograms actually approximate the theoretic density curve given by (14) on $I_{a,b,c}$. See Figures 13 through 16.

[insert Figures 13, 14, 15, and 16 around here]

Interestingly, the lower levels of capitalist' assets occur more often than the higher counterparts do.

4 Discussion and concluding remarks

This paper examined an overlapping generations model in which there exist two classes: the working class and the capitalist class, and a choice between two types of production technologies. The behavior of the working class is the same as that in the standard Diamond (1965) type OLG model, whereas the capitalist class, without supplying labor, bequeaths its assets to the next generation. We showed that in an extreme case in which one technology is linear and the other technology is of the Leontief type, the dynamics of capital supplied by the capitalist class is characterized by a rigid rotation on the circle, giving rise to quasi-periodic motions for a very large set of parameter values.

We may regard capitalists' asset holdings as public debt outstanding. In this interpretation, our results suggest that even if the government manages to constrain the amount of public debt within a sustainable range, the dynamics of public debt can be fairly complex (non-periodic but non-chaotic) in return.

Our results, particularly the quasi-periodicity of the capitalists' assets, heavily rely on the assumption that both of the technologies available are extreme ones among CES technologies. However, examining the extreme cases has a great advantage in that it allows us to analytically investigate the global dynamic properties in detail without relying on numerical methods. Conducting the same analysis for the non-extreme cases would be much harder, if not impossible. Of course, the more general, less extreme, cases are certain to generate much more complex patterns of dynamics. In fact, we have observed by computer simulations that our model with nonextreme CES technologies can easily exhibit richer dynamic patterns including chaotic behaviors. We believe that our analysis here will be helpful in understanding these dynamic phenomena.

Appendices

Appendix A: A microfoundation of workers' technology choice behavior

This appendix provides a microfoundation for technology choice behavior in our model.

Our basic setup follows Matsuyama (2007). There are N types of production technologies in this economy. A type i technology converts e_i units of the final goods into $e_i R_i$ units of capital and the final good is produced by $Y_{it} = F_i(K_t, L_t)$, where K_t and L_t are capital and labor at time t, respectively. The final good production functions in per worker terms are

$$y_{it} = f_i(k_t), \ i = 1, ..., N$$

where $y_t = Y_t/L_t$ and $k_t = K_t/L_t$ and $f_i(k_t) = F_i(k_t, 1)$.

Because the workers have log-linear utility, their saving rate is constant and independent of the return from saving. Any saver has two options in managing their saving, namely becoming either a lender or an entrepreneur. An agent selecting to be a lender lends his/her saving a_t and obtains $r_{t+1}a_t$ when old, where r_{t+1} denotes the real interest rate. An agent becoming an entrepreneur selects one technology from the two types of technologies. Because an entrepreneur's wealth is equal to his/her saving, if $e_i > a_t$, the entrepreneur has to borrow $e_i - a_t$. However, due to the presence of capital market frictions, each entrepreneur can pledge only up to a constant fraction of the project revenue for the repayment, $\lambda_i e_i R_i f_i'(k_{t+1})$, where $0 \leq \lambda_i \leq 1$. The fraction, λ_i, differs between the two types of projects. The entrepreneur's borrowing constraint is represented by

$$\lambda_i e_i R_i f_i'(k_{t+1}) \geq r_{t+1}(e_i - a_t) \text{ for } i = 1, ..., N. \tag{18}$$

As λ_i becomes smaller, the credit constraint becomes stronger.

Because an entrepreneur is always able to choose to become a lender, earnings from investment will not be smaller than those from lending:

$$f_i'(k_{t+1}) e_i R_i - r_{t+1}(e_i - a_t) \geq r_{t+1} a_t, \tag{19}$$

that is:

$$r_{t+1} \leq f_i'(k_{t+1})R_i \text{ for } i = 1, ..., J.$$

(18) can be rewritten as:

$$r_{t+1} \leq \frac{R_i f_i'(k_{t+1})}{\left(1 - \frac{a_t}{e_i}\right)/\lambda_i} \text{ for } i = 1, 2.$$

By defining:

$$\Phi_i \equiv \frac{R_i f_i'(k_{t+1})}{\max\left\{1, \left(1 - \frac{a_t}{e_i}\right)/\lambda_i\right\}},$$

we can summarize (18) and (19) as:

$$r_{t+1} \leq \Phi_i \text{ for } i = 1, 2.$$

Let us assume here that $r_{t+1} < \Phi_i$. Then, all agents become entrepreneurs and adopt type i technology and there is no lender in this economy. Obviously, this cannot be an equilibrium as we have $r_{t+1} \geq \Phi_i$. Next, let us suppose that $r_{t+1} > \Phi_i$. Then, at least one of (18) and (19) for i is not satisfied and thus type i is not adopted. In equilibrium, because we must have positive investment, it follows that:

$$r_{t+1} = \max\left\{\Phi_1, \Phi_2\right\}, \tag{20}$$

showing that the technology yielding higher value on the right-hand side of (20) is adopted.

In this paper, we consider a special case of (20), that is:

$$N = 2,\ R_1 = R_2 = 1,\ \lambda_1 = \lambda_2 = \lambda \text{ and } e_1 = e_2 = e.$$

In this case, (20) reduces to

$$\begin{aligned} r_{t+1} &= \max\left\{\frac{Rf_1'(k_{t+1})}{\max\left\{1, \left(1 - \frac{a_t}{e}\right)/\lambda\right\}}, \frac{Rf_2'(k_{t+1})}{\max\left\{1, \left(1 - \frac{a_t}{e}\right)/\lambda\right\}}\right\} \\ &= \frac{R}{\max\left\{1, \left(1 - \frac{a_t}{e}\right)/\lambda\right\}} \max\left\{f_1'(k_{t+1}), f_2'(k_{t+1})\right\}. \end{aligned}$$

Thus, we can confirm that the workers select the technology with a higher marginal productivity of capital, which is our technology choice assumption in the main text.

It should be noted that the interest rate may be lowered by the existence of credit constraints but it does not affect our analysis because the saving rate in our model is independent of the interest rate.

Appendix B: Proofs

Proof of Proposition 1. For case (i), we first notice that $k_{t_0} = x_{t_0} \leq 1$ for some t_0. As $a > 1$ in F_L, which makes x_t increase; it follows that $k_{t_1} = x_{t_1} \in (1, a]$ for some $t_1 > t_0$. As $b \in (0, 1)$ in F_R, which makes x_t decrease, we have that $x_t \leq a$ for $t \geq t_1$. Moreover, there is the smallest integer $n \geq 1$ such that $k_{t_1+n} = c + x_{t_1+n} \leq 1$ or $x_{t_1+n} \leq 1 - c$ and that $k_{t_1+n-1} = c + x_{t_1+n-1} > 1$ or $x_{t_1+n-1} > 1 - c$. Thus, it follows that $x_t \geq b(1 - c)$ for $t \geq t_1 + n$. For case (ii), the statement is evident from (9). $\square$

Proof of Proposition 3. Since $T_{a,b}$ is a circle homeomorphism by Proposition 2, the rotation number exists, which we denote by α. We claim that it must satisfy:

$$a^{1-\alpha} b^{\alpha} = 1. \tag{21}$$

There are two cases to consider. First, when α is irrational, then we know from Theorem 1 in Coelho et al (1995) that $T_{a,b}$ is uniquely ergodic and we let its unique invariant measure be μ. By definition, $\alpha = \mu(b, T_{a,b}(b)) = \mu(b, ab)$. Since μ is invariant under $T_{a,b}$, it follows that $\alpha = \mu(b, ab) = \mu(T_{a,b}^{-1}(b, ab)) = \mu(1, a)$. By the Ergodic Theorem and the fact that $|DT_{a,b}^n|$, where D denotes the derivative, is bounded away from 0 and infinity (see for this point, Coelho et al. (1995), Proposition 2), we have

$$0 = \lim_{n \to \infty} \frac{1}{n} \log |DT_{a,b}^n(x)| = \int \log |DT_{a,b}| d\mu(x),$$

which implies

$$0 = \mu(b, 1) \log a + \mu(1, a) \log b = (1 - \alpha) \log a + \alpha \log b.$$

This[9] gives (21).

[9]This also says that the Lyapunov exponent $\lim_{n \to \infty} (1/n) \sum_{t=0}^{n-1} \log |T'_{a,b}(x_t)| = \log a^{1-\alpha} b^{\alpha}$ is zero, which indicates that there is neither expansion nor contraction on average for the trajectories generated by the map. This applies to the case where the rotation number of $T_{a,b}$ is rational as well. In fact, each periodic point of $T_{a,b}$ is neither repelling nor attracting.

Next, when α is rational or $\alpha = p/q$ where p and q are prime integers, there is some $x_0 \in [b, a]$ such that $T^q_{a,b}(x_0) = x_0$, implying $a^{(1-\alpha)q} b^{\alpha q} x_0 = x_0$, and thus we obtain (21). This means that $DT^q_{a,b}$ is identically unity and therefore, every point in $[b, a]$ is a periodic point of period q.

Taking logarithm of (21) and solving for α, we obtain:

$$\alpha = \log a / \log(a/b).$$

For the topological conjugacy (see e.g. the proof of Theorem 1 in de Faria and Tresser (2014)), let $h : I = [0, 1] \to I_{a,b} = [b, a]$ be given by:

$$h(t) = b(a/b)^t, \tag{22}$$

which is clearly a homeomorphism. To prove conjugacy, it suffices to check that $h \circ R_\alpha = T_{a,b} \circ h$. There are two cases to consider: (i) $0 \leq t < 1 - \alpha$, and (ii) $1 - \alpha < t \leq 1$.

For case (i), as $R_\alpha(t) = t + \alpha$, it follows that

$$\begin{aligned}
h \circ R_\alpha(t) &= h(t + \alpha) \\
&= b\left(\frac{a}{b}\right)^{t+\alpha} \\
&= \left(\frac{a}{b}\right)^{\alpha} h(t) \\
&= ah(t) = T_{a,b} \circ h(t).
\end{aligned}$$

Similarly, for case (ii), as $R_\alpha(t) = t + \alpha - 1$, it follows that

$$\begin{aligned}
h \circ R_\alpha(t) &= h(t + \alpha - 1) \\
&= b\left(\frac{a}{b}\right)^{t+\alpha-1} \\
&= \left(\frac{a}{b}\right)^{\alpha} \frac{b}{a} h(t) \\
&= bh(t) = T_{a,b} \circ h(t).
\end{aligned}$$

Thus, the topological conjugacy between R_α and $T_{a,b}$ is proven.

Finally, the absolutely continuous invariant measure μ for $T_{a,b}$ in $I_{a,b}$, when α is irrational, can be expressed as the push-forward of the Lebesgue measure λ in $[0, 1]$

via the homeomorphism h. See again the proof of Theorem 1 in de Faria and Tresser (2014) . That is, for a Borel measurable set $E \subset I_{a,b}$:

$$\mu(E) = \lambda(h^{-1}(E)) = \int_{h^{-1}(E)} dt = \int_E (h^{-1}(x))' dx.$$

From (22) we have:

$$(h^{-1})'(x) = \left(\frac{\log(x/b)}{\log(a/b)}\right)' = \frac{1}{x\log(a/b)}.$$

Thus, we obtain:

$$d\mu(x) = \frac{dx}{x\log(a/b)},$$

as desired. □

Proof of Proposition 4. We first show that for any initial condition (k_0, x_0), the trajectory of x_t $(t = 0, 1, 2, \dots)$ generated by the iteration of the map (7) is eventually trapped in the interval $I_{a,b,c} = [(1-c)b, (1-c)a]$ and that its dynamics is governed by (15). That is, there is some $t_0 \geq 0$ (depending on the initial condition) such that for $t \geq t_0$, $x_t \in [(1-c)b, 1-c)$ if and only if $k_t < 1$ and $x_t \in I_{a,b,c}$.

From Proposition 1 and the inevitable occurrence of technology change for $c \in [0, 1)$, it suffices to assume $x_0 = k_0 \in L = [(1-c)b, 1) \subset [(1-c)b, a]$. By partitioning $L = L_1 \cup L_2$, where $L_1 = [(1-c)b, (1-c))$ and $L_2 = [1-c, 1)$, we have two cases to examine.

Case (i): Let $x_0 = k_0 \in L_1$. Then, $F(k_0, x_0) = F_L(x_0, x_0) \in [ab(1-c), a(1-c)]^2 \subset [1, a(1-c)]^2$. The last inclusion is followed by the assumption that $c \leq 1 - 1/ab$ in (16). Thus, $x_1 \geq 1$ and $k_1 \geq 1$. Then there is the smallest integer $n \geq 1$ such that $F^{n+1}(k_0, x_0) = F_R^n \circ F_L(k_0, x_0) = F_R^n \circ F_L(x_0, x_0) = (b^n x_1 + c, b^n x_1) = (k_{n+1}, x_{n+1})$ with $x_{n+1} < 1$. There are two subcases to follow. That is, subcase (i-1): $k_{n+1} \geq 1$ and subcase (i-2): $k_{n+1} < 1$.

For subcase (i-1), from $k_{n+1} - x_{n+1} = c$, we have $x_{n+1} \in L_2$. As $k_{n+1} \geq 1$, we have $F(k_{n+1}, x_{n+1}) = F_R(k_{n+1}, x_{n+1}) = (bx_{n+1} + c, bx_{n+1}) = (k_{n+2}, x_{n+2})$. As $x_{n+1} \in [(1-c), 1) = L_2$, we have $x_{n+2} = bx_{n+1} \in L_1$. We also obtain $k_{n+2} \in L$, that is, $k_{n+2} < 1$, because $k_{n+2} = x_{n+2} + c$ and $|L_2| = c$. Thus, $F(k_{n+2}, x_{n+2}) =$

$F_L(x_{n+2}, x_{n+2})$, which brings us to the beginning of case (i) and we are done for $k_0 = x_0$ and $t \geq 0$. For subcase (i-2), $k_{n+1} < 1$ implies $x_{n+1} < 1 - c$ and hence $x_{n+1} \in L_1$. Thus, again, this subcase goes back to the beginning of case (i). Therefore, the argument above shows that the dynamics of x_t is governed by (15) for case (i) for any $t \geq 0$.

Case (ii): Let $x_0 = k_0 \in L_2$. As $k_0 < 1$, we have $F(k_0, x_0) = F_L(x_0, x_0) = (ax_0, ax_0)$. Thus, $k_1 = x_1 \in (a(1-c), a]$, which implies $x_1 \notin I_{a,b,c}$. However, as $c \leq 1 - 1/ab$ by (16), we have $a(1-c) > ab(1-c) \geq 1$ and hence $x_1 = k_1 > 1$. Therefore, there is the smallest integer $m \geq 1$ such that $F^{m+1}(k_0, x_0) = F_R^m \circ F_L(k_0, x_0) = F_R^m(x_1, x_1) = (b^m x_1 + c, b^m x_1) = (k_{m+1}, x_{m+1})$ with $x_{m+1} < 1$, which implies that the argument reduces to that of case (i). Thus, we are also done for $k_0 = x_0 \in L_2$ and for $t \geq m + 1$.

Thus, the above argument shows that for any initial conditions (k_0, x_0), $x_t \in I_{a,b,c}$ for $t \geq t_0$ for some t_0 and that the sequence of x_t can be described by (15) for $t \geq t_0$.

For conjugacy, let $h_c : I_{a,b} \to I_{a,b,c}$ with $h_c(x) = (1-c)x$. Then, we see that for $x \in I_{a,b}$, $h_c \circ T_{a,b}(x) = \tau_{a,b,c} \circ h_c(x) \in I_{a,b,c}$. Because $\tau_{a,b}$ is topologically conjugate to the rigid rotation by Proposition 3, so is $\tau_{a,b,c}$ by the chain of topological conjugacy.

For the invariant measure, we can take a homeomorphism $\varphi : [0,1] \to I_{a,b,c} = [(1-c)b, (1-c)a]$ such that $\varphi(t) = h_c \circ h(t) = (1-c)b(a/b)^t$ and use the same argument as in Proposition 3. This completes the proof. $\square$

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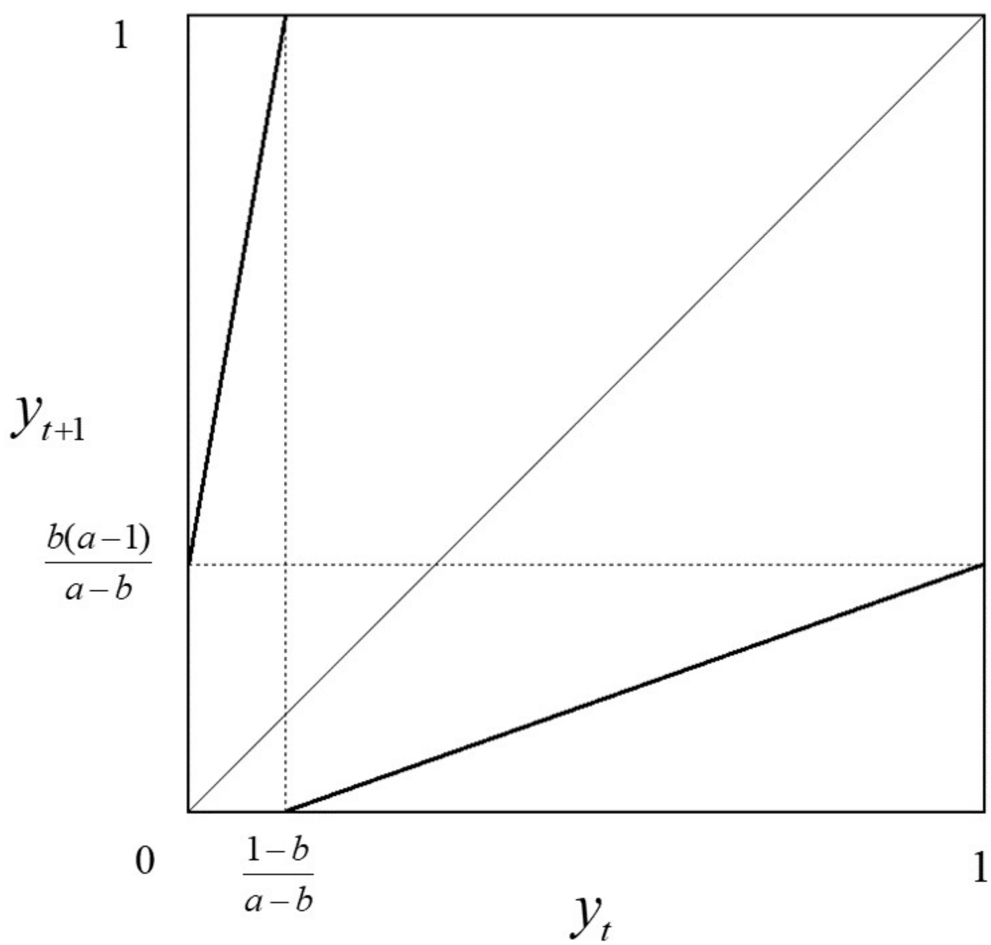

Figure 1: A piecewise linear circle homeomorphism of $\mathbb{R}/\mathbb{Z}$. In this figure, $y_t = \phi(x_t)$.

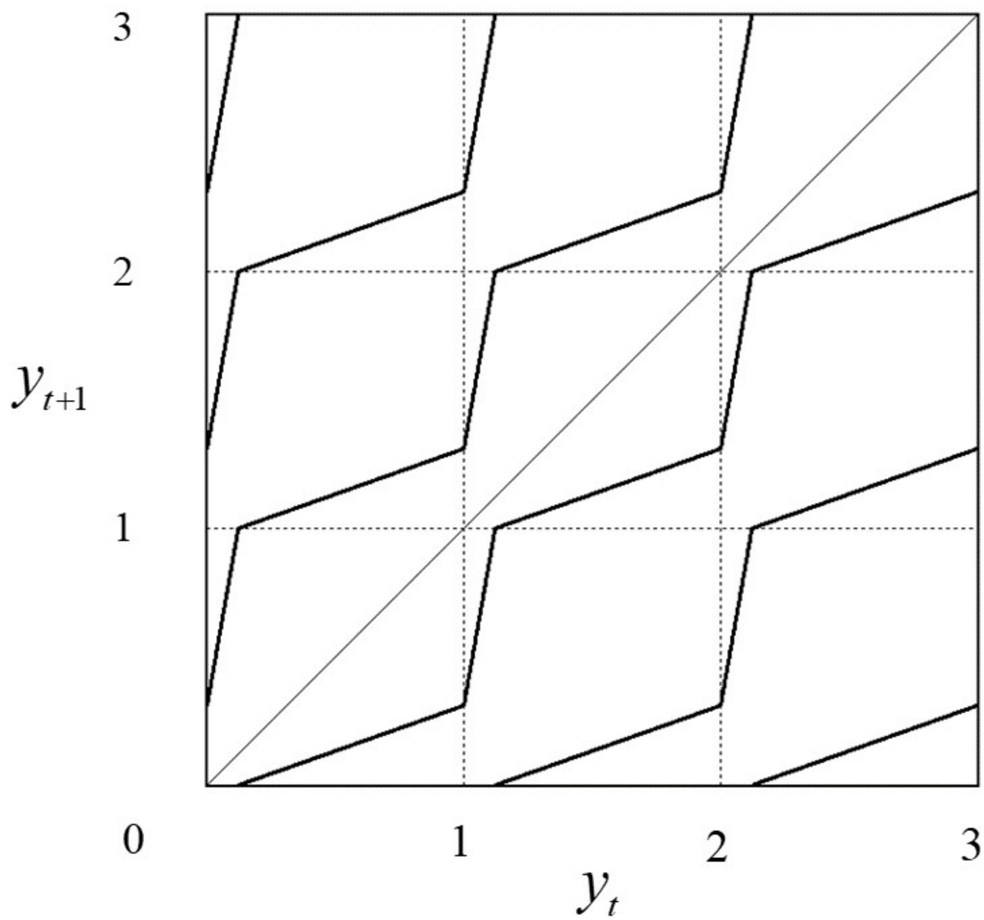

Figure 2: Its lifts in $\mathbb{R}$, corresponding to Figure 1.

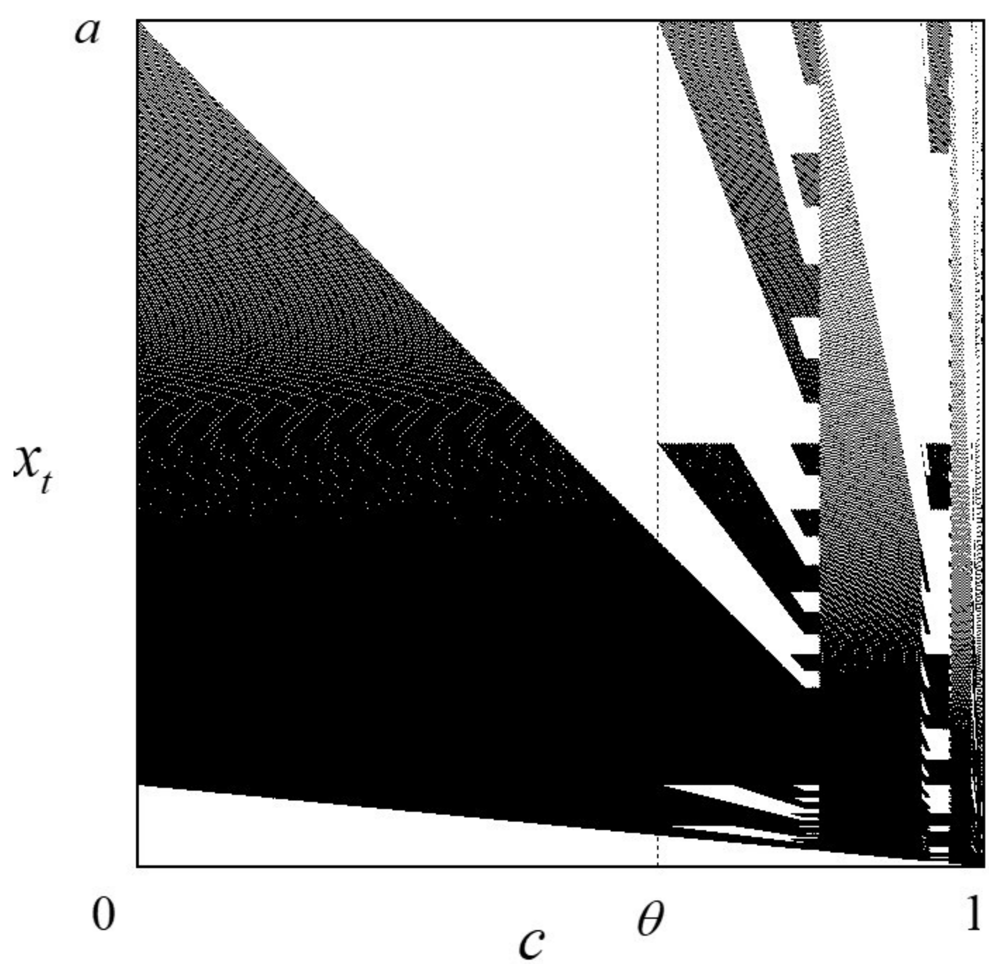

Figure 3: Bifurcation diagram of the capitalists' assets x_t with respect $c \in [0, 1)$. $a = 3\sqrt{3}, b = 1/2$, and $\theta \approx 0.6115$. Proposition 4 explains that the dynamics of x_t for $c \in [0, \theta)$ can be completely characterized by the rotation of $\mathbb{R}/\mathbb{Z}$.

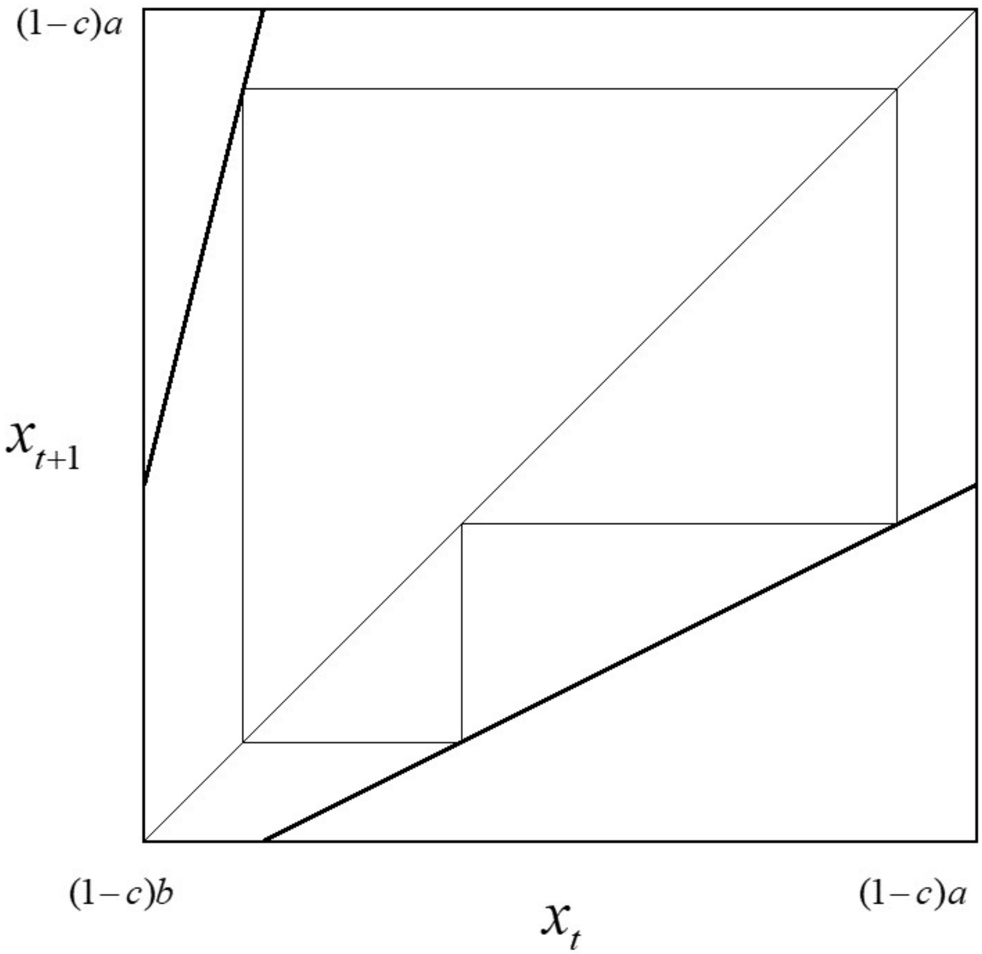

Figure 4: Period-three cycle with $\rho = 1/3$. $a = 4, b = 0.5$, and $c = 0.4$.

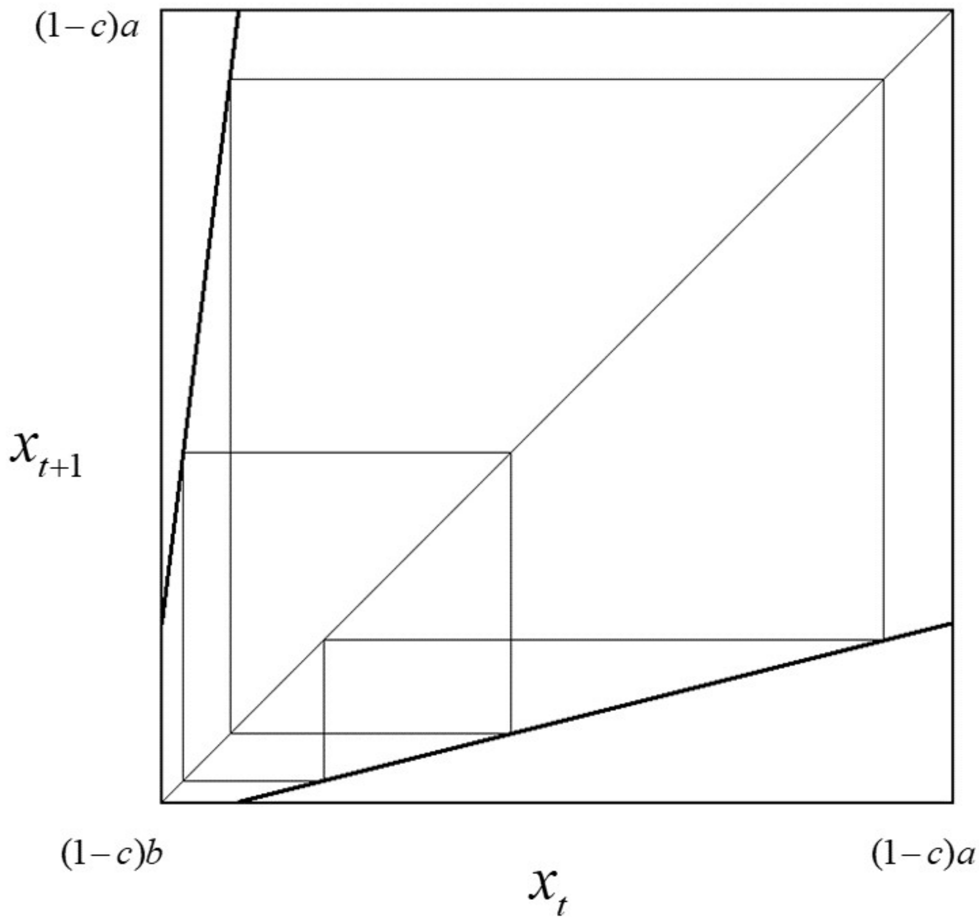

Figure 5: Period-five cycle with $\rho = 2/5$. $a = 8, b = 0.25$, and $c = 0.4$

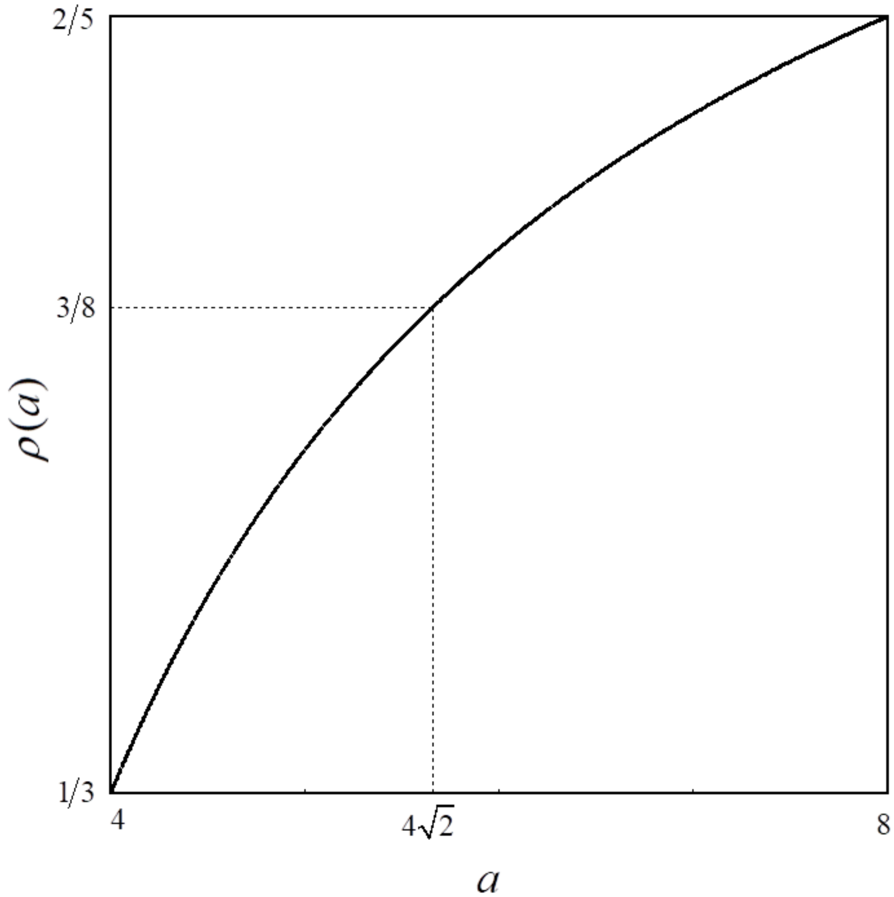

Figure 6: Rotation numbers for $a \in [4, 8]$ with $ab = 2$ and $c = 0.4$

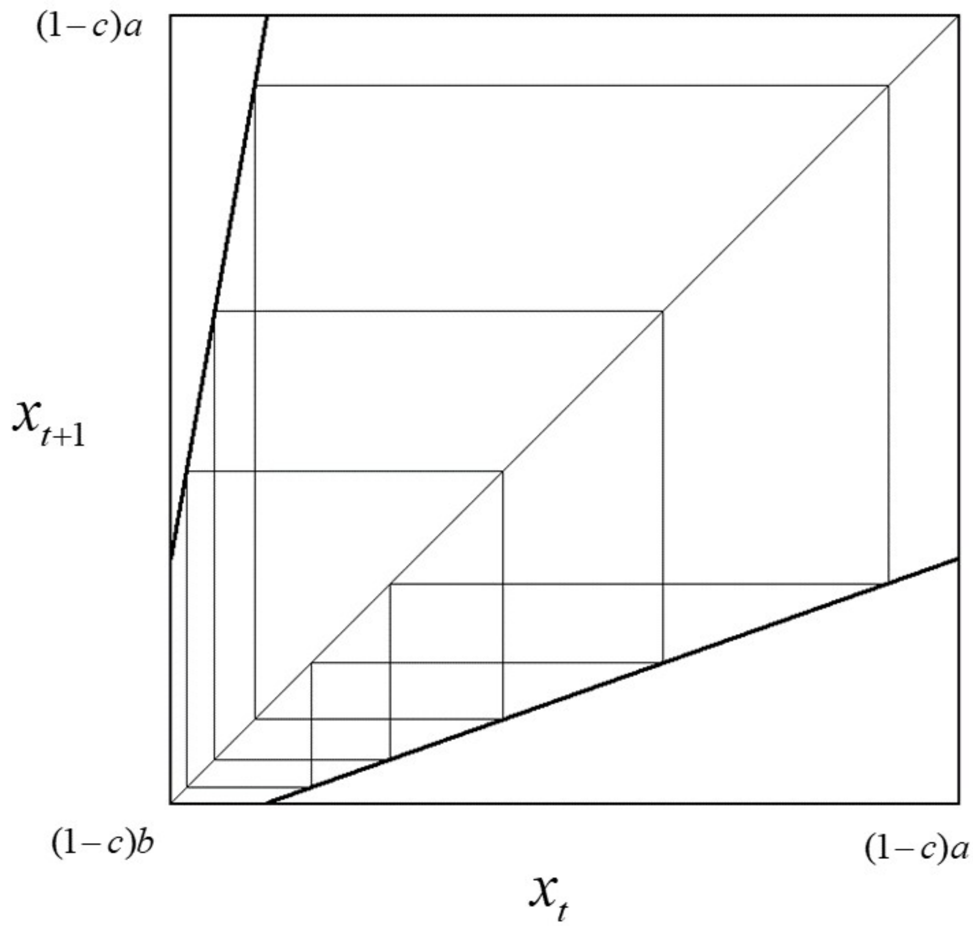

Figure 7: Period-eight cycle with $\rho = 3/8$. $a = 4\sqrt{2}, b = \sqrt{2}/4$, and $c = 0.4$

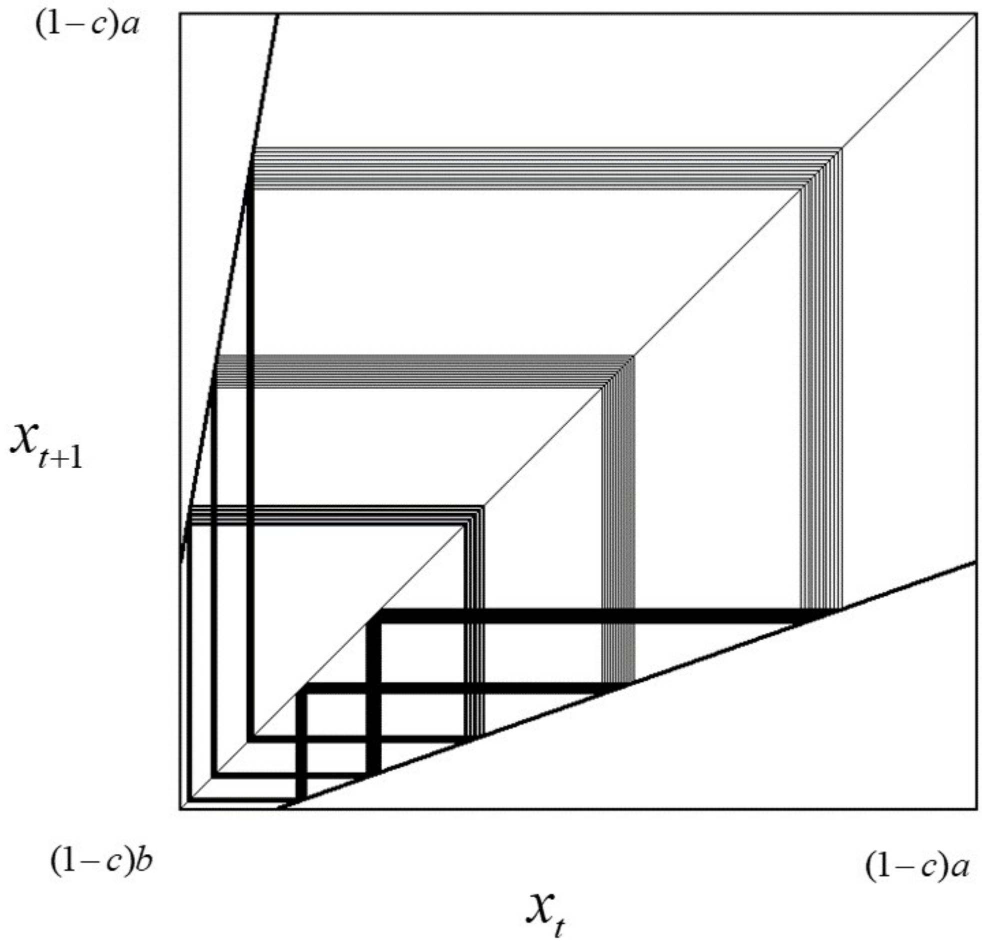

Figure 8: Quasi-periodic cycle with ρ that is irrational but close to 3/8. $a \approx 4\sqrt{2} + 0.01, b = \sqrt{2}/4$, and $c = 0.4$. 100 iterations.

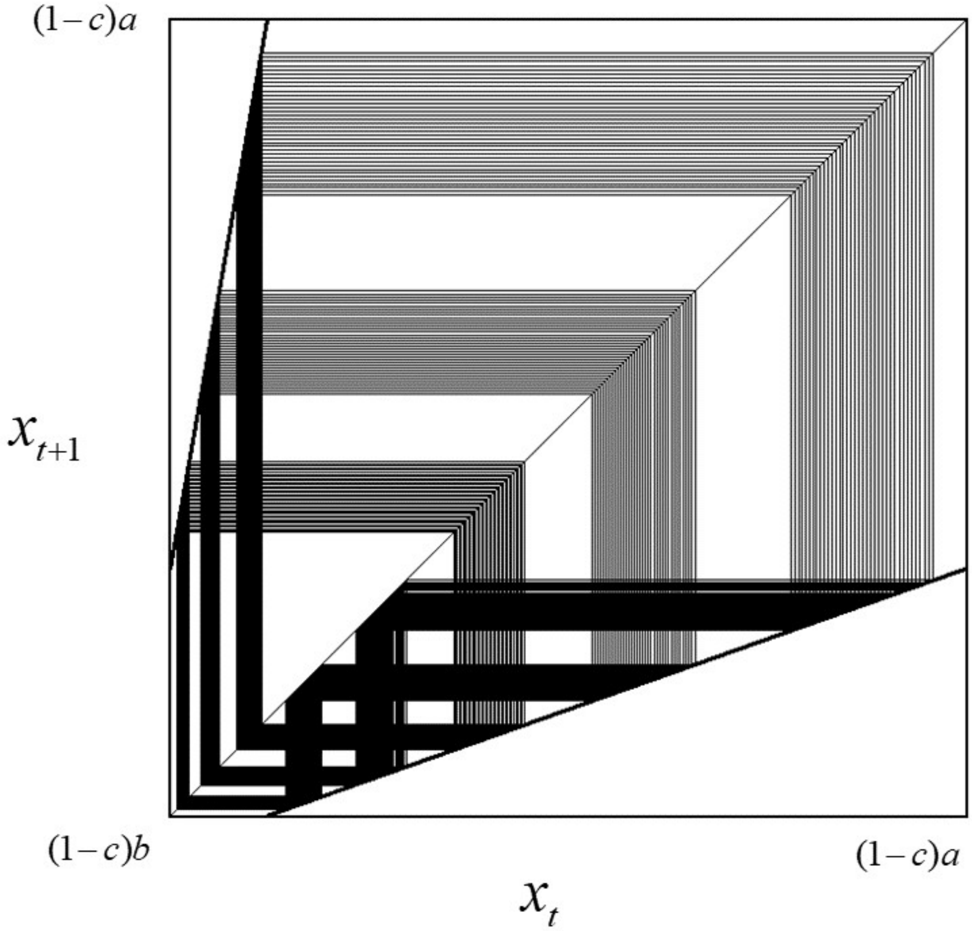

Figure 9: Quasi-periodic cycle with ρ that is irrational but close to 3/8. $a \approx 4\sqrt{2} + 0.01, b = \sqrt{2}/4$, and $c = 0.4$. 300 iterations.

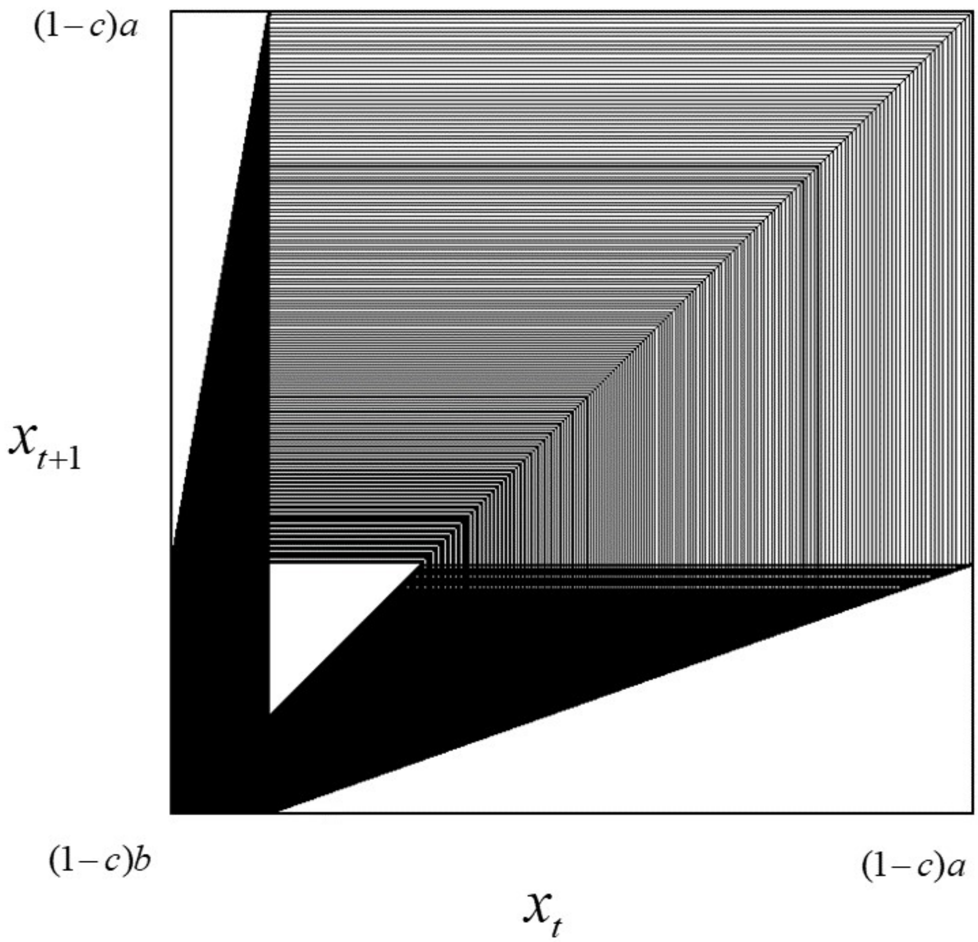

Figure 10: Quasi-periodic cycle with ρ that is irrational but close to 3/8. $a \approx 4\sqrt{2}+0.01, b=\sqrt{2}/4$, and $c=0.4$. 600 iterations.

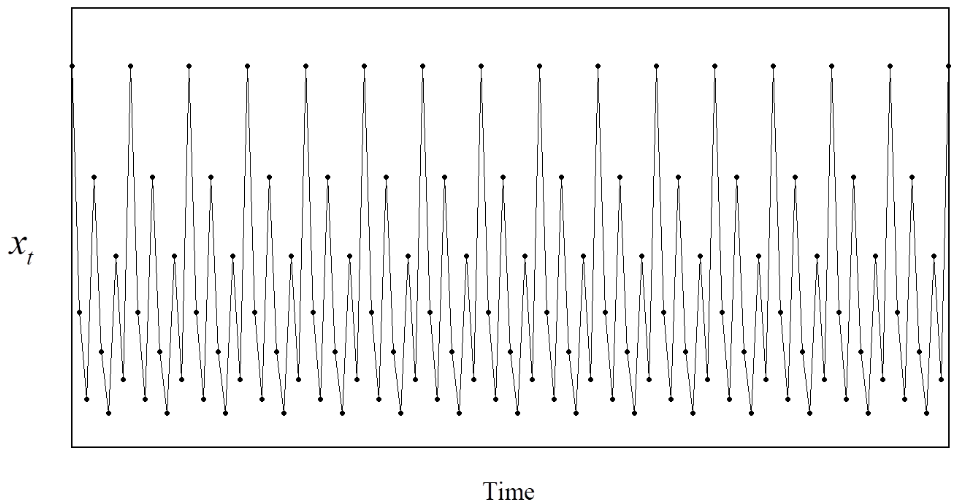

Figure 11: Time series of a period-eight cycle with $\rho = 3/8$.

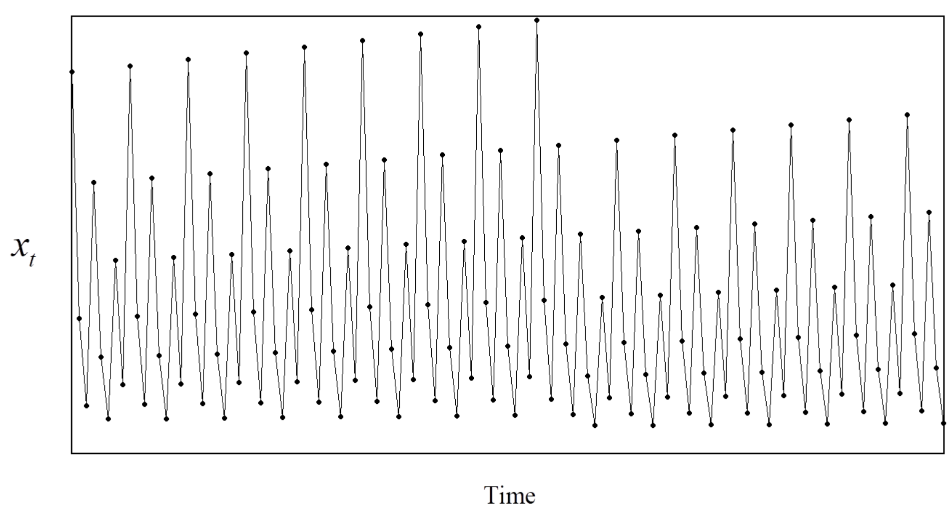

Figure 12: Time series of a quasi-periodic cycle with ρ near 3/8.

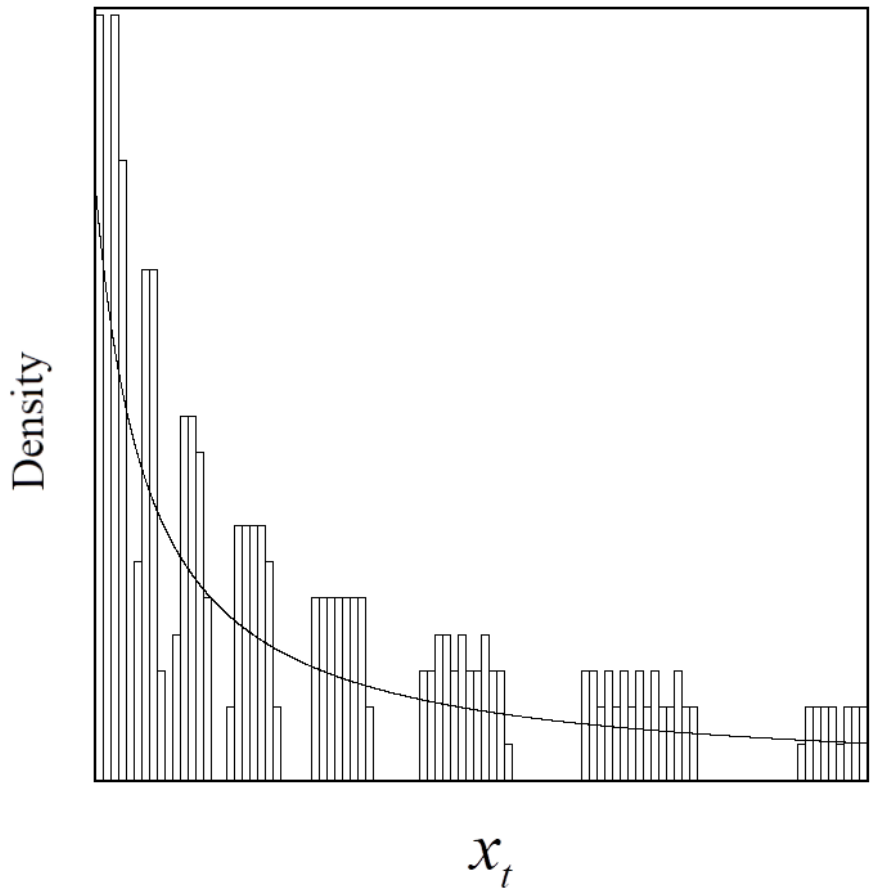

Figure 13: A computer-generated histogram and the theoretical density curve on $I_{a,b,c}$. The parameters are the same as in Figure 8. The number of iterations is 300.

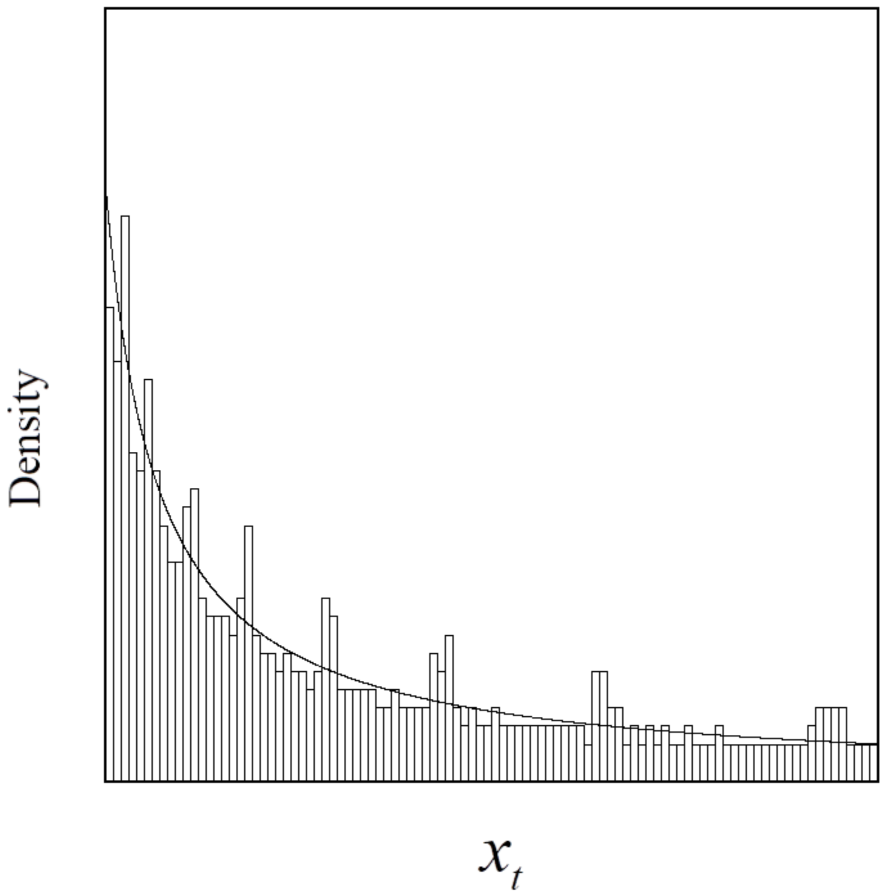

Figure 14: A computer-generated histogram and the theoretical density curve on $I_{a,b,c}$. The parameters are the same as in Figure 8. The number of iterations is 600.

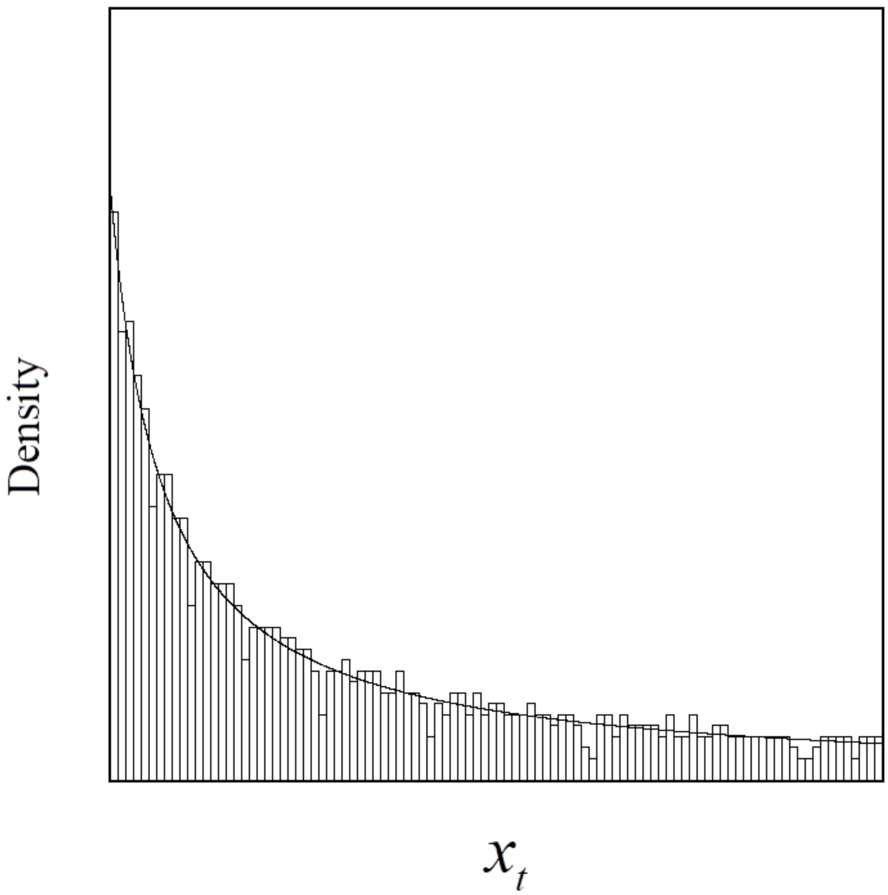

Figure 15: A computer-generated histogram and the theoretical density curve on $I_{a,b,c}$. The parameters are the same as in Figure 8. The number of iterations is 1,000.

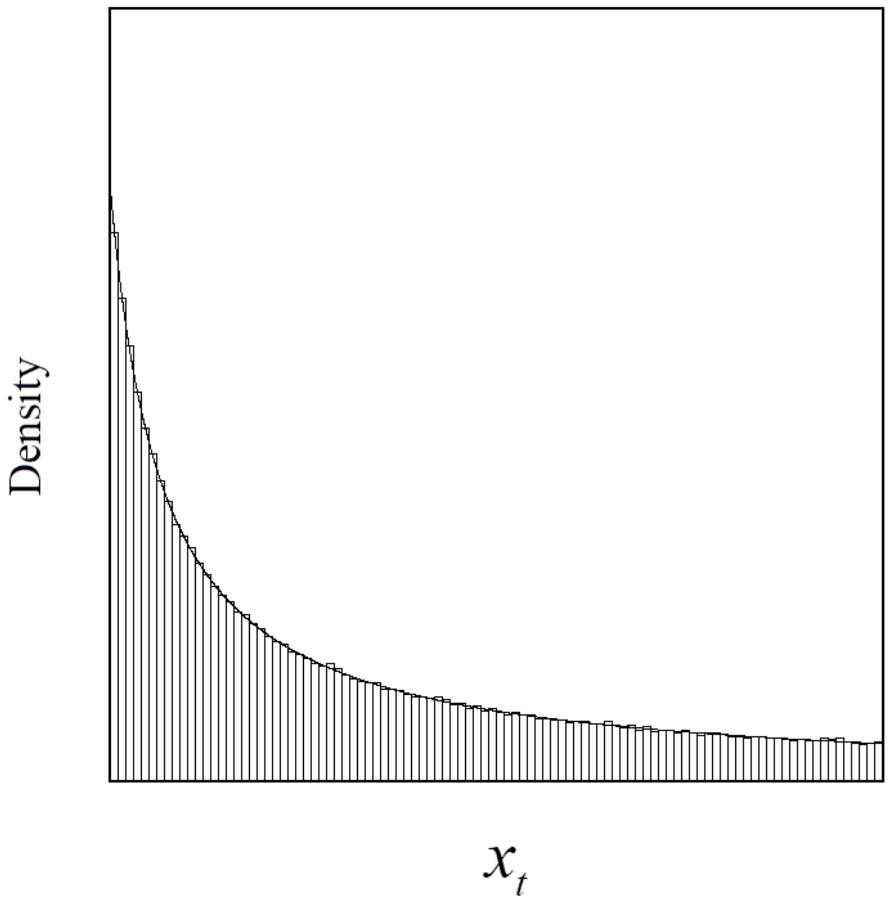

Figure 16: A computer-generated histogram and the theoretical density curve on $I_{a,b,c}$. The parameters are the same as in Figure 8. The number of iterations is 10,000.