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"Indeterminacy and expectation-driven fluctuations with non-separable preferences"

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Indeterminacy and expectation-driven fluctuations with non-separable preferences*

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Abstract: *We consider a continuous-time two-sector infinite-horizon model with sector specific externalities, endogenous labor and a concave homogeneous non-separable utility function. We show that local indeterminacy arises with a low elasticity of intertemporal substitution in consumption provided the wage elasticity of the labor supply and the elasticity of substitution between consumption and leisure are low enough. Such a result cannot hold with additively-separable preferences for which local indeterminacy requires a large enough elasticity of intertemporal substitution in consumption.*

Keywords: *Sector-specific externalities, endogenous labor, non-separable concave homogeneous utility functions, intertemporal substitution in consumption, local indeterminacy.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

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1 Introduction

Is it possible to get local indeterminacy of equilibria in a growth model with a low elasticity of intertemporal substitution in consumption ? This is the central question raised in this paper. Considering a two-sector infinite-horizon model with sector-specific externalities, we have proved in Garnier, Nishimura and Venditti [13, 14] that, when additively separable preferences are assumed, the answer is necessarily negative. In this paper, we consider instead non-separable preferences, and we provide a positive answer giving some sufficient conditions based on the elasticity of substitution between consumption and leisure.

One may wonder why such a question has some importance. A widespread perception among economists is that macroeconomic business-cycle fluctuations can be driven by changes in expectations about fundamentals. A major strand of the literature focussing on fluctuations derived from agents' beliefs is based on the concept of sunspot equilibria that dates back to the early works of Shell [35], Azariadis [1] and Cass and Shell [12]. As shown by Woodford [37], the existence of sunspot equilibria is closely related to the indeterminacy of perfect foresight equilibrium, i.e. the existence of a continuum of equilibrium paths converging toward one steady state from the same initial value of the state variable.

During the last decade a variety of economic models that incorporate some degree of market imperfections have been shown to exhibit multiple equilibria and local indeterminacy.[1] Following the seminal contribution of Benhabib and Farmer [4], infinite horizon Ramsey-type models augmented to include external effects in production have been widely studied. Within aggregate formulations with economy-wide externalities and increasing returns at the social level, the results fundamentaly depend on the specifications of preferences, being additively separable or non separable, and technology, being with a unitary or non-unitary elasticity of capital-labor substitution. However, for all these different formulations, a large enough elasticity of intertemporal substitution in consumption, a close to infinity wage elasticity of the labor supply and some strictly positive but not too large income effects are clearly central conditions.[2]

[1]See Benhabib and Farmer [6] for an extensive bibliography.

[2]See Guo and Lansing [18], Meng and Yip [26], Nishimura, Nourry and Venditti [29], Pintus [33] for conclusions based on additively separable preferences, and Bennett and Farmer [9], Hintermaier [20, 21], Jaimovich [22], Lloyd-Braga, Nourry and Venditti [25], Meng and Yip [26], Nishimura, Nourry and Venditti [29], Pintus [34] for conclusions based

Within two-sector formulations with sector-specific externalities, the literature has only focused on additively separable preferences,[3] and has splitted between models with increasing returns at the social level derived from Benhabib and Farmer [5], and models with constant returns at the social social level derived from Benhabib and Nishimura [8]. Although a large enough elasticity of intertemporal substitution in consumption is still a central condition for the existence of indeterminacy for all these formulations, the restrictions on the labor supply are quite different depending on the returns to scale at the social level:[4] a close to infinity wage elasticity of labor is usually assumed under increasing social returns,[5] but a close to zero wage elasticity appears to be a necessary condition under constant social returns.[6]

When we look at empirical evidence, on the one hand, the elasticity of the aggregate labor supply is shown to be significantly lower than one,[7] but on the other hand, while the elasticity of intertemporal substitution in consumption is usually shown to be low, recent estimates provide divergent views. In a first set of contributions, Campbell [11] and Kocherlakota [24] suggest the interval $(0.2, 0.8)$. More recently, Vissing-Jorgensen [36] partially confirmed such findings by showing that the estimates of this elasticity are around $0.3 - 0.4$ for stockholders and around $0.8 - 1$ for bondholders, and are higher for households with larger asset holdings within these two groups. On the contrary, in a second set of contributions, Mulligan [28] repeatedly obtained estimates above unity, i.e. in the range $1.1 - 2.1$, using different estimation methods. The upper part of this interval has been recently confirmed by Gruber [16] who provides robust estimates of the elasticity around 2.[8]

While two-sector models with constant social returns provide restrictions

on non-separable preferences.

[3]One notable exception if provided by Mino [27] who considers a non-separable utility function within a two-sector Lucas-type endogenous growth model with human capital aggregate externalities. He also shows that local indeterminacy occurs when the elasticity of intertemporal substitution is large enough.

[4]Garnier, Nishimura and Venditti [14] show that such a difference precisely relies on the amount of social returns to scale in the investment good sector.

[5]See Benhabib and Farmer [5], Harrison [19]

[6]See Garnier, Nishimura and Venditti [13], Nishimura and Venditti [32].

[7]Most econometric analysis available in the literature conclude that the wage elasticity of labor belongs to $(0, 0.3)$ for men and to $(0.5, 1)$ for women (see Blundell and MaCurdy [10]).

[8]Barro [2] uses the Gruber's estimates to evaluate the welfare costs of rare disasters in a representative-consumer model.

on the labor supply that fit these evidence, this is not the case for the conditions on the elasticity of intertemporal substitution which is usually required to be close to infinity. Our aim in this paper is then to provide conditions for local indeterminacy which are consistent with these two sets of empirical facts by focussing on non-separable utility functions. We will show that the existence of multiple equilibria can be compatible with a low elasticity of intertemporal substitution in consumption provided the elasticity of substitution between consumption and leisure is adequately chosen.

Our strategy is the following. In order to get local indeterminacy without requiring a large intertemporal substitutability of consumption, we look for a mechanism which is based on the demand for leisure. Indeed, if the reaction of leisure following some modification of consumption is such that the current level of utility remains almost constant, there is no need to require any additional restriction on the intertemporal substitution properties of the utility function. This crucially depends however on the relationship between the elasticity of intertemporal substitution in consumption and the wage elasticity of the labor supply (or leisure). When additively separable preferences are considered, these two elasticities are not connected and the previous mechanism cannot occur. We will show however that when non-separable preferences are considered, this is no longer true and the above-mentioned mechanism can be obtained from the explicit consideration of the elasticity of substitution between consumption and leisure.

We consider a continuous-time two-sector infinite-horizon model with sector-specific externalities. The production side is defined on the basis of Cobb-Douglas production functions.[9] The preference side is defined on the basis of a concave homogeneous non-separable utility function for which the marginal rate of substitution between consumption and leisure depends on their ratio. We build our analysis on a capital intensity reversal between the private and the social levels, the consumption good sector being capital intensive at the private level but labor intensive at the social level.[10] We show that local indeterminacy arises for any given low elasticity of intertemporal substitution in consumption provided the wage elasticity of the labor supply and the elasticity of substitution between consumption and leisure are low enough.

This paper is organized as follows: the next section sets up the basic

[9] CES technologies can be equivalently considered but at the cost of heavier notations (see Nishimura and Venditti [31] and Garnier, Nishimura and Venditti [13]).

[10] See Benhabib and Nishimura [8] and Garnier, Nishimura and Venditti [13] in which these conditions are shown to be necessary for local indeterminacy.

model, defines the intertemporal equilibrium and proves the existence of a steady state. In section 3 we state our main results and we provide economic intuitions in section 4. Section 5 finally contains concluding comments and all the proofs are in a final Appendix.

2 The model

We consider the same basic framework of a two-sector infinite-horizon model with sector-specific externalities as in Garnier, Nishimura and Venditti [13, 14]. But instead of assuming additively separable preferences, we will consider a non-separable utility function.

2.1 Technologies

We consider an economy producing a consumption good y_0 and a capital good y_1. Each good is produced by capital x_{1j} and labor x_{0j}, $j = 0, 1$, through a Cobb-Douglas production function which contains sector specific-externalities. The representative firm in each industry indeed faces the following technology, called *private production function*:

$$y_j = x_{0j}^{\beta_{0j}} x_{1j}^{\beta_{1j}} e_j(X_{0j}, X_{1j}), \quad j = 0, 1 \tag{1}$$

with $\beta_{ij} > 0$. The positive externalities are equal to

$$e_j(X_{0j}, X_{1j}) = X_{0j}^{b_{0j}} X_{1j}^{b_{1j}}$$

with $b_{ij} \geq 0$ and X_{ij} denoting the average use of input i in sector j. We assume that these economy-wide averages are taken as given by each individual firm. At the equilibrium, since all firms of sector j are identical, we have $X_{ij} = x_{ij}$ and we define the *social production functions* as follows

$$y_j = x_{0j}^{\hat{\beta}_{0j}} x_{1j}^{\hat{\beta}_{1j}} \tag{2}$$

with $\hat{\beta}_{ij} = \beta_{ij} + b_{ij}$ and $\hat{\beta}_{0j} + \hat{\beta}_{1j} = 1$, $j = 0, 1$.

Choosing the consumption good as the numeraire, i.e. $p_0 = 1$, a firm in each industry maximizes its profit given the price of the investment (capital) good p_1, the rental rate of capital w_1 and the wage rate w_0. The first order conditions subject to the private technologies (1) give

$$x_{ij}/y_j = p_j \beta_{ij}/w_i \equiv a_{ij}(w_i, p_j), \quad i, j = 0, 1 \tag{3}$$

We call a_{ij} the input coefficients from the *private* viewpoint.

Considering that total labor is given by $\ell = x_{00} + x_{01}$, and the total stock of capital is given by $x_1 = x_{10} + x_{11}$, the factor market clearing equation is directly obtained from the private input coefficients as defined by (3). Denoting $x = (\ell, x_1)'$, $y = (y_0, y_1)'$ and $A(w, p) = [a_{ij}(w_i, p_j)]$, we get

$$A(w, p)y = x \tag{4}$$

Let us now define

$$\hat{a}_{ij}(w_i, p_j) \equiv (\hat{\beta}_{ij}/\beta_{ij})a_{ij}(w_i, p_j) \tag{5}$$

From (3) we get $x_{ij} = (p_j\beta_{ij}/w_i)\, y_j$. Substituting this expression into the social production functions (2) and solving with respect to p_j gives the factor-price frontier, which provides a relationship between input prices and output prices. Denoting $p = (1, p_1)'$, $w = (w_0, w_1)'$ and $\hat{A}(w, p) = [\hat{a}_{ij}(w_i, p_j)]$, we get

$$p = \hat{A}'(w, p)w \tag{6}$$

We call $\hat{a}_{ij}$ the input coefficients from the *social* viewpoint.[11]

Note from (4) and (6) that at the equilibrium, the wage rate and the rental rate are functions of the price of the capital good only, i.e. $w_i = w_i(p_1)$, $i = 0, 1$, while outputs are functions of the capital stock, total labor and the price of the capital good, $y_j = \tilde{y}_j(x_1, \ell, p_1)$, $j = 0, 1$.

Considering the external effects (e_0, e_1) as given, profit maximization in both sectors gives demands for capital and labor as functions of the capital stock, the output of the investment good, total labor and the external effects, namely $\tilde{x}_{ij} = x_{ij}(x_1, y_1, \ell, e_0, e_1)$, $i, j = 0, 1$. The production frontier is then

$$y_0 = T(x_1, y_1, \ell, e_0, e_1) = \tilde{x}_{00}^{\hat{\beta}_{00}} \tilde{x}_{10}^{\hat{\beta}_{10}} \tag{7}$$

From the envelope theorem we easily get $w_1 = T_1(x_1, y_1, \ell, e_0, e_1)$, $p_1 = -T_2(x_1, y_1, \ell, e_0, e_1)$ and $w_0 = T_3(x_1, y_1, \ell, e_0, e_1)$.

2.2 Preferences

The economy is populated by a large number of identical infinitely-lived agents. We assume without loss of generality that the total population is constant and normalized to one. At each period a representative agent supplies elastically an amount of labor $\ell \in (0, \bar{\ell})$, with $\bar{\ell} > 0$ his endowment of labor. He derives utility from consumption c and leisure $\mathcal{L} = \bar{\ell} - \ell$ according to a non-separable function $U(c, \mathcal{L})$ which satisfies:

Assumption 1. *$U(c, \mathcal{L})$ is $\mathbf{C}^r$ over $\mathbb{R}_+ \times [0, \ell]$ for $r \geq 2$, increasing in each argument, concave, homogeneous of degree $\gamma \in (0, 1]$, and for all $(c, \mathcal{L}) \in \mathbb{R}^2_+$, $\lim_{c/\mathcal{L} \to 0} U_2(c, \mathcal{L})/U_1(c, \mathcal{L}) = 0$ and $\lim_{c/\mathcal{L} \to +\infty} U_2(c, \mathcal{L})/U_1(c, \mathcal{L}) = +\infty$.*

[11] If the agents take account of externalities as endogenous variables in profit maximization, the first order conditions subject to the social technologies (2) give the same input coefficients from the *social* viewpoint.

Consumption and leisure are then normal goods. Building on the homogeneity of degree $\gamma \in (0,1]$, we introduce the share of consumption within total utility $\alpha(c,\mathcal{L}) \in (0,\gamma)$ defined as follows:

$$\alpha(c,\mathcal{L}) = \tfrac{U_1(c,\mathcal{L})c}{U(c,\mathcal{L})} \tag{8}$$

The share of leisure within total utility is similarly defined as $\gamma - \alpha(c,\mathcal{L}) \in (0,\gamma)$.[12]

Considering that consumption at time t is given by the output of the consumption good sector, i.e. $c_t = y_{0t}$ as defined by (7), the intertemporal optimization problem of the representative agent is given by:

$$\begin{array}{ll} \max\limits_{\{x_1(t),y_1(t),\ell(t)\}} & \int_0^{+\infty} U(T(x_1(t),y_1(t),\ell(t),e_0(t),e_1(t)),\bar{\ell}-\ell(t))e^{-\delta t}dt \\ s.t. & \dot{x}_1(t) = y_1(t) - gx_1(t) \\ & x_1(0) \ given \\ & \{e_j(t)\}_{t\geq 0},\ j=0,1,\ given \end{array} \tag{9}$$

where $\delta \geq 0$ is the discount rate and $g > 0$ is the depreciation rate of the capital stock.

2.3 Intertemporal equilibrium and steady state

The modified Hamiltonian in current value is given by:

$$\mathcal{H} = U(T(x_1(t),y_1(t),\ell(t),e_0(t),e_1(t)),\bar{\ell}-\ell(t)) + q_1(t)\left(y_1(t) - gx_1(t)\right)$$

with $q_1(t)$ the co-state variable which corresponds to the utility price of the capital good in current value. The first order conditions of problem (9) are given by the following equations:

$$\begin{aligned} q_1(t) &= p_1(t)U_1(c(t),\bar{\ell}-\ell(t)) && (10)\\ U_2(c(t),\bar{\ell}-\ell(t)) &= w_0U_1(c(t),\bar{\ell}-\ell(t)) && (11)\\ \dot{x}_1(t) &= y_1(t) - gx_1(t) && (12)\\ \dot{q}_1(t) &= (\delta+g)q_1(t) - w_1(t)U_1(c(t),\bar{\ell}-\ell(t)) && (13) \end{aligned}$$

As shown in Section 2.1, we have $w_i = w_i(p_1)$, $i=0,1$, $y_1 = \tilde{y}_1(x_1,\ell,p_1)$ and $c = \tilde{y}_0(x_1,\ell,p_1) = T(x_1,\tilde{y}_1(x_1,\ell,p_1),\ell,e_0(x_1,\ell,p_1),e_1(x_1,\ell,p_1))$. Therefore, solving equation (11) describing the labor-leisure trade-off at the equilibrium, we express the labor supply as a function of the capital stock and the

[12] This result is derived from the standard Euler equality for homogeneous functions, namely $\gamma U(c,\mathcal{L}) = U_1(c,\mathcal{L})c + U_2(c,\mathcal{L})\mathcal{L}$.

output price, $\ell = \ell(x_1, p_1)$. Then, we get $y_0 = c(x_1, p_1) \equiv \tilde{y}_0(x_1, \ell(x_1, p_1), p_1)$ and $y_1 = y_1(x_1, p_1) \equiv \tilde{y}_1(x_1, \ell(x_1, p_1), p_1)$.

Let us introduce the following elasticities:

$$\epsilon_{cc} = -\frac{U_1(c,\mathcal{L})}{U_{11}(c,\mathcal{L})c}, \quad \epsilon_{\mathcal{L}c} = -\frac{U_2(c,\mathcal{L})}{U_{21}(c,\mathcal{L})c}, \quad \epsilon_{c\mathcal{L}} = -\frac{U_1(c,\mathcal{L})}{U_{12}(c,\mathcal{L})\mathcal{L}}, \quad \epsilon_{\mathcal{L}\mathcal{L}} = -\frac{U_2(c,\mathcal{L})}{U_{22}(c,\mathcal{L})\mathcal{L}},$$

As it will be more convenient to write the linearized dynamical system in terms of elasticities with respect to labor, let $\tilde{U}(c, \ell) \equiv U(c, \bar{\ell} - \ell)$. We get $\tilde{U}_2(c, \ell) = -U_2(c, \mathcal{L})$, $\tilde{U}_{12}(c, \ell) = -U_{12}(c, \mathcal{L})$, $\tilde{U}_{22}(c, \ell) = U_{22}(c, \mathcal{L})$ and thus:

$$\epsilon_{\ell c} = -\frac{\tilde{U}_2}{\tilde{U}_{21}c} = \epsilon_{\mathcal{L}c}, \quad \epsilon_{c\ell} = -\frac{\tilde{U}_1}{\tilde{U}_{12}\ell} = -\epsilon_{c\mathcal{L}}\frac{\bar{\ell}-\ell}{\ell}, \quad \epsilon_{\ell\ell} = -\frac{\tilde{U}_2}{\tilde{U}_{22}\ell} = -\epsilon_{\mathcal{L}\mathcal{L}}\frac{\bar{\ell}-\ell}{\ell} \tag{14}$$

Note that since $\tilde{U}(c, \ell)$ is decreasing and concave with respect to ℓ, the elasticity $\epsilon_{\ell\ell}$ is negative.

Considering (10)-(13), the equations of motion are finally derived as

$$\begin{aligned} \dot{x}_1 &= y_1(x_1, p_1) - gx_1 \\ \dot{p}_1 &= \frac{(\delta+g)p_1 - w_1(p_1) + \left[\frac{p_1}{\epsilon_{cc}c}\frac{\partial c}{\partial x_1} + \frac{p_1}{\epsilon_{c\ell}\ell}\frac{\partial \ell}{\partial x_1}\right](y_1(x_1,p_1) - gx_1)}{E(x_1,p_1)} \end{aligned} \tag{15}$$

with

$$E(x_1, p_1) = 1 - \left(\frac{p_1}{\epsilon_{cc}c}\frac{\partial c}{\partial p_1} + \frac{p_1}{\epsilon_{c\ell}\ell}\frac{\partial \ell}{\partial p_1}\right) \tag{16}$$

Note that $E(x_1, p_1)$ is equal to 1 minus the sum of the ratio of the elasticity of the consumption good's output with respect to the price of the investment good over the elasticity of intertemporal substitution in consumption, and the ratio of the elasticity of the aggregate labor supply with respect to the price of the investment good over the elasticity of the consumption marginal utility with respect to labor.

Any solution $\{x_1(t), p_1(t)\}_{t\geq 0}$ that satisfies the transversality condition

$$\lim_{t\to+\infty} e^{-\delta t} U_1(c(t), \bar{\ell} - \ell(t)) p_1(t) x_1(t) = 0 \tag{17}$$

is called an equilibrium path.

A steady state is defined by a pair (x_1^*, p_1^*) solution of

$$y_1(x_1, p_1) = gx_1, \quad w_1(p_1) = (\delta + g)p_1 \tag{18}$$

Existence and uniqueness easily follows:

Proposition 1. *Under Assumption 1 there exists a unique steady state* $(x_1^*, p_1^*) > 0$ *with* $\ell^* = \ell(x_1^*, p_1^*) \in (0, \bar{\ell})$.

In order to simplify the analysis, we will consider a normalization of the steady state and choose a particular value for the stationary labor supply. Considering the share of consumption within total utility as defined by (8), the first order condition (11) evaluated at the steady state becomes

$$\frac{\ell^*}{\bar{\ell}-\ell^*} = \frac{\alpha}{(\gamma-\alpha)\Phi} \tag{19}$$

with $\Phi = \chi^*/w_0$ and χ^*, w_0 as given in Appendix 6.1. Hence, choosing a particular value for $\ell^* \in (0, \bar{\ell})$ implies to consider a particular value for $\alpha \in (0, \gamma)$.

2.4 Characteristic polynomial

Linearizing the dynamical system (15) around (x_1^*, p_1^*) gives:

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} - g & \frac{\partial y_1}{\partial p_1} \\ \frac{\left(\frac{p_1}{\epsilon_{cc}c}\frac{\partial c}{\partial x_1} + \frac{p_1}{\epsilon_{c\ell}\ell}\frac{\partial \ell}{\partial x_1}\right)\left(\frac{\partial y_1}{\partial x_1} - g\right)}{E(x_1^*, p_1^*)} & \frac{\delta + g - \frac{\partial w_1}{\partial p_1} + \left(\frac{p_1}{\epsilon_{cc}c}\frac{\partial c}{\partial x_1} + \frac{p_1}{\epsilon_{c\ell}\ell}\frac{\partial \ell}{\partial x_1}\right)\frac{\partial y_1}{\partial p_1}}{E(x_1^*, p_1^*)} \end{pmatrix}$$

All these partial derivatives are functions of ϵ_{cc}, $\epsilon_{c\ell}$, $\epsilon_{\ell c}$ and $\epsilon_{\ell\ell}$. The role of $\epsilon_{\ell c}$ and $\epsilon_{\ell\ell}$ of course occurs through the presence of endogenous labor but remains implicit at that stage mainly because of our methodology to derive the dynamical system (15) from the first order conditions (10)-(13).

Any solution of (15) that converges to the steady state (x_1^*, p_1^*) satisfies the transversality condition (17) and is an equilibrium. Therefore, given $x_1(0)$, if there is more than one initial price $p_1(0)$ in the stable manifold of (x_1^*, p_1^*), the equilibrium path from $x_1(0)$ will not be unique. In particular, if J has two eigenvalues with negative real parts, there will be a continuum of converging paths and thus a continuum of equilibria.

Definition 1. *If the locally stable manifold of the steady state (x_1^*, p_1^*) is two-dimensional, then (x_1^*, p_1^*) is locally indeterminate.*

The eigenvalues of J are given by the roots of the characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} \tag{20}$$

with

$$\begin{aligned} \mathcal{D} &= \frac{\left(\frac{\partial y_1}{\partial x_1} - g\right)\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right)}{E(x_1^*, p_1^*)} \\ \mathcal{T} &= \frac{\frac{\partial y_1}{\partial x_1} + \delta - \frac{\partial w_1}{\partial p_1} + \left(\frac{p_1}{\epsilon_{cc}c}\frac{\partial c}{\partial x_1} + \frac{p_1}{\epsilon_{c\ell}\ell}\frac{\partial \ell}{\partial x_1}\right)\frac{\partial y_1}{\partial p_1} - \left(\frac{p_1}{\epsilon_{cc}c}\frac{\partial c}{\partial p_1} + \frac{p_1}{\epsilon_{c\ell}\ell}\frac{\partial \ell}{\partial p_1}\right)\left(\frac{\partial y_1}{\partial x_1} - g\right)}{E(x_1^*, p_1^*)} \end{aligned} \tag{21}$$

Local indeterminacy requires therefore that $\mathcal{D} > 0$ and $\mathcal{T} < 0$.

3 Local indeterminacy with low intertemporal substitutability

Our aim is to give conditions for the occurrence of local indeterminacy. As initiated by Benhabib and Nishimura [8], these conditions will be based on capital intensity differences across sectors. Using the input coefficients defined in Section 2.1 allows indeed to give the following characterization:

Definition 2. *The consumption good is capital intensive at the* ***private*** *level if and only if* $a_{11}a_{00} - a_{10}a_{01} \equiv T < 0$*, and capital intensive at the* ***social*** *level if and only if* $\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01} \equiv \hat{T} < 0$*.*

From the first order conditions given in Appendix 6.1, we can conveniently relate these input coefficients to the Cobb-Douglas parameters. At the steady state:

i) the consumption good is capital (labor) intensive at the private level if and only if

$$b \equiv 1 - \frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}} < (>)0$$

ii) the consumption good is capital (labor) intensive at the social level if and only if

$$\hat{b} \equiv 1 - \frac{\hat{\beta}_{10}\hat{\beta}_{01}}{\hat{\beta}_{00}\hat{\beta}_{11}} < (>)0$$

Using the Euler theorem for a homogeneous of degree $\gamma \in (0,1]$ utility function, we get $U_{12} = (\gamma-1)U_1/\mathcal{L} - (c/\mathcal{L})U_{11}$ and $U_{22} = (1-\gamma)(U_1 c - U_2\mathcal{L})/\mathcal{L}^2 + (c/\mathcal{L})^2 U_{11}$, and thus from (8), (14), (19) and (44):

$$\epsilon_{\ell c} = -\frac{(\gamma-\alpha)\epsilon_{cc}}{\alpha[1-\epsilon_{cc}(1-\gamma)]}, \ \epsilon_{c\ell} = \frac{(\gamma-\alpha)\epsilon_{cc}\Phi}{\alpha[1-\epsilon_{cc}(1-\gamma)]}, \ \epsilon_{\ell\ell} = -\frac{(\gamma-\alpha)^2\epsilon_{cc}\Phi}{\alpha^2-(1-\gamma)(2\alpha-\gamma)\alpha\epsilon_{cc}} < 0 \quad (22)$$

It follows from Assumption 1 that

$$\frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}} - \frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}} = -\frac{(1-\gamma)\alpha[\gamma-\alpha\epsilon_{cc}(1-\gamma)]}{(\gamma-\alpha)^2\epsilon_{cc}\Phi} \leq 0 \quad (23)$$

Note that (23) and thus $\epsilon_{\ell\ell} < 0$ in (22) hold if $\epsilon_{cc} \leq \gamma/\alpha(1-\gamma)$.

Following Benhabib and Nishimura [8], we focus on a configuration with a factor intensity reversal between the private and the social levels in which the consumption good is capital intensive at the private level ($b < 0$) but labor intensive at the social level ($\hat{b} > 0$). The next Proposition shows that a continuum of equilibria arises if the utility function is not too concave and the share α of consumption in total utility is large enough.

Proposition 2. *Under Assumption 1, let* $b < 0$ *and* $\hat{b} > 0$*. Then there exist* $\underline{\gamma} \in (0,1)$ *and* $\underline{\alpha} \in (0,\gamma)$ *such that when* $\gamma \in (\underline{\gamma}, 1]$ *and* $\alpha \in (\underline{\alpha}, \gamma)$*, the steady state is locally indeterminate.*

As the expression $1-\gamma$ provides a measure of the degree of concavity of the utility function, we get the usual conclusion that local indeterminacy arises if this degree is close enough to zero, i.e. if γ is close enough to one. Note also that the condition on the share α implies to consider a low elasticity of the labor supply as in the case with additively-separable preferences (see Garnier, Nishimura and Venditti [13]). Indeed, we easily compute from (11), (14) and (22) the wage elasticity of labor as follows:

$$\varepsilon_{\ell w} \equiv \frac{d\ell}{dw_0}\frac{w_0}{\ell} = -\frac{1}{\frac{1}{\varepsilon_{\ell\ell}} - \frac{1}{\varepsilon_{c\ell}}} = \frac{(\gamma-\alpha)^2\epsilon_{cc}\Phi}{\alpha[\gamma-\alpha\epsilon_{cc}(1-\gamma)]} > 0 \tag{24}$$

It follows that when α is close to γ, $\varepsilon_{\ell w}$ is close to zero.

As explained in the introduction, most of the contributions in the literature show that local indeterminacy usually needs a large enough elasticity of intertemporal substitution in consumption ϵ_{cc}. The question is now to know whether for a given low value of ϵ_{cc} that would match empirical evidence, we are able to choose a value of α which is close enough to γ to satisfy the condition of Proposition 2. Actually, the answer is not obvious as ϵ_{cc} and α are linked through the elasticity of substitution between consumption and leisure. Denoting this elasticity as

$$\sigma(c,\mathcal{L}) = \frac{\frac{U_2(c/\mathcal{L},1)/U_1(c/\mathcal{L},1)}{c/\mathcal{L}}}{\frac{\partial(U_2(c/\mathcal{L},1)/U_1(c/\mathcal{L},1))}{\partial(c/\mathcal{L})}} \tag{25}$$

and using (8) and (19), we derive at the steady state

$$\sigma = \frac{\epsilon_{cc}(\gamma-\alpha)}{\gamma-\alpha\epsilon_{cc}(1-\gamma)} \tag{26}$$

Obviously, if the utility function is Cobb-Douglas, i.e. $U(c,\mathcal{L}) = c^\alpha\mathcal{L}^\beta$ with $\alpha+\beta=\gamma$, then $\sigma=1$ and for any $\gamma\in(0,1]$, we get $\epsilon_{cc}=1/(1-\alpha)$. Therefore choosing α close to γ, which is itself close to 1, implies to choose ϵ_{cc} large enough, i.e. larger than 1. In this case local indeterminacy is still obtained under a large elasticity of intertemporal substitution in consumption. On the contrary, when σ is different from 1, for a given low value of ϵ_{cc} and a value of α close enough to γ, we can always choose σ to satisfy equation (26). As documented for instance in Campbell [11] and Kocherlakota [24], plausible values of ϵ_{cc} belong to the interval $(0,1)$. We have then proved:

Corollary 1. *Under Assumption 1, let $b<0$, $\hat{b}>0$ and consider the critical bounds $\underline{\gamma}\in(0,1)$ and $\underline{\alpha}\in(0,\gamma)$ as introduced in Proposition 2. When $\gamma\in(\underline{\gamma},1]$ and $\alpha\in(\bar{\alpha},\gamma)$, for any given $\epsilon_{cc}\in(0,1)$, the steady state is locally indeterminate if $\sigma=\sigma^*\equiv\epsilon_{cc}(\gamma-\alpha)/[\gamma-\alpha\epsilon_{cc}(1-\gamma)]<1$.*

We show that with concave homogeneous preferences, for any given low value of the elasticity of intertemporal substitution in consumption, there always exists a low enough value of the elasticity of substitution between consumption and leisure such that local indeterminacy occurs. Note that the lower ϵ_{cc} is, the lower σ must be as $\partial\sigma^*/\partial\epsilon_{cc} > 0$. Proposition 2 and Corollary 1 then prove that the occurrence of local indeterminacy is compatible with empirically plausible values for the fundamental parameters of preferences. However, it remains now to know which kind of values for the elasticity σ of substitution between consumption and leisure is empirically relevant. Up to our knowledge, there does not exist in the literature any precise evaluation of this parameter. However, we can show that σ is closely linked with the income effects of preferences. Indeed we derive from (11), (14) and (24) that:

$$\frac{d\ell}{dc}\frac{c}{\ell} = -\frac{\varepsilon_{\ell w}}{\sigma} < 0$$

For a given wage elasticity of labor $\varepsilon_{\ell w}$, the lower σ is, the larger are the income effects. Corollary 1 thus proves that local indeterminacy can arise with any low elasticity of intertemporal substitution in consumption provided the income effects are large enough. Such a conclusion is similar to the one obtained in one-sector model where income effects have been shown to be necessary for the occurrence of local indeterminacy.[13]

Remark: Other formulations of preferences
Two other specifications of non-separable preferences are often used in the business cycle and growth literature:

i) A King-Plosser-Rebelo [23] (KPR) formulation such that

$$U(c, \mathcal{L}) = \frac{[cv(\mathcal{L})]^{1-\theta}}{1-\theta} \tag{27}$$

which is compatible with both balanced growth and stationary worked hours. Denoting $h(\mathcal{L}) = v'(\mathcal{L})/v(\mathcal{L})$ and

$$\psi(\mathcal{L}) = \mathcal{L}h(\mathcal{L}), \quad \eta(\mathcal{L}) = \frac{\mathcal{L}h'(\mathcal{L})}{h(\mathcal{L})} \tag{28}$$

Assumption 1 holds if $v(\mathcal{L})$ is a positive increasing function, $\theta \geq 0$, $\eta(\mathcal{L}) \leq -\psi(\mathcal{L})(1-\theta)$ and $\eta(\mathcal{L}) \leq \psi(\mathcal{L})(1-1/\theta)$.[14] Moreover, consumption and leisure are normal goods. Note that $\psi(\mathcal{L}) > 0$ can be interpreted as the elasticity of the utility of leisure and $\eta(\mathcal{L}) < 0$ is linked to the elasticity of the labor supply with respect to the wage rate $\epsilon_{\ell w}$, namely $\eta(\mathcal{L}) = -\mathcal{L}/(\ell\epsilon_{\ell w})$.[15]

[13] See Jaimovich [22], Meng and Yip [26].

[14] See Hintermaier [20, 21] and Pintus [34].

[15] This expression is obtained from the total differenciation of equation (11).

As the marginal rate of substitution is given by $\psi(\mathcal{L})c/\mathcal{L}$, the elasticity of substitution between consumption and leisure is at the NSS:

$$\sigma = -1/\eta = \epsilon_{\ell w}\ell/\mathcal{L} \tag{29}$$

with $\eta < 0$. Note also that the income effects are similarly computed as

$$\frac{d\ell}{dc}\frac{c}{\ell} = -\varepsilon_{\ell w} = -\sigma\mathcal{L}/\ell \tag{30}$$

ii) A Greenwood-Hercovitz-Huffman [15] (GHH) formulation such that

$$U(c, \mathcal{L}) = u(c + G(\mathcal{L}/B)) \tag{31}$$

with $u(.)$ and $G(.)$ some increasing and strictly concave functions, and $B > 0$ a normalization constant. Such a specification is concave, satisfies the normality assumption, and implies that the marginal rate of substitution between consumption and leisure depends on the latter only as

$$\frac{U_2(c,\mathcal{L})}{U_1(c,\mathcal{L})} = G'(\mathcal{L}/B)/B$$

This property obviously implies

$$\frac{d\ell}{dc}\frac{c}{\ell} = 0 \tag{32}$$

and thus that there is no income effect associated with the representative agent's labor supply. Let us define

$$\epsilon^G_{\mathcal{L}\mathcal{L}}(\mathcal{L}/B) = -\frac{G'(\mathcal{L}/B)}{G''(\mathcal{L}/B)(\mathcal{L}/B)} > 0 \tag{33}$$

the elasticity of $G(\mathcal{L}/B)$ which also gives the elasticity of the labor supply with respect to the wage rate as $\epsilon_{\ell w} = \epsilon^G_{\mathcal{L}\mathcal{L}}\mathcal{L}/\ell$. As the marginal rate of substitution is given by $G'(\mathcal{L})$, the elasticity of substitution between consumption and leisure is at the steady state:

$$\sigma = \epsilon^G_{\mathcal{L}\mathcal{L}} = \epsilon_{\ell w}\frac{\ell}{\mathcal{L}} \tag{34}$$

We show in Appendix 6.3 and 6.4 that when a KPR or a GHH utility function is considered, local indeterminacy can also arise.[16] However, even if the elasticity of substitution between consumption and leisure is arbitrarily close to zero, the existence of local indeterminacy still requires a large enough elasticity of intertemporal substitution in consumption. An important question is then to understand why such a difference with respect to

[16]Guo and Harrison [17] have independently proved in a recent paper that local indeterminacy can arise in a two-sector model with sector-specific externalities, increasing social returns and a GHH utility function, provided a low enough wage elasticity of the labor supply is considered. However, they assume increasing returns larger than 30% and they do not provide any information about the required elasticity of intertemporal substitution in consumption.

the concave homogeneous specification occurs. The reason is simple and relies on the link between the elasticity of substitution between consumption and leisure σ, the elasticity of the labor supply $\epsilon_{\ell w}$ and the elasticity of intertemporal substitution in consumption ϵ_{cc}. Considering (29) and (34), we conclude that in both cases σ and $\epsilon_{\ell w}$ remain in general disconnected from ϵ_{cc}. It follows that contrary to the conclusion derived from (26), low values for σ and $\epsilon_{\ell w}$ do not imply a low value for ϵ_{cc}. Put differently, we can also conclude from (30) and (32) that KPR and GHH preferences with a low intertemporal substitutability do not generate enough income effects to allow the existence of local indeterminacy.

A CES example:
As an illustration, consider a CES hmogeneous of degree $\gamma \in (0,1]$ utility function such that $U(c,\mathcal{L}) = [\chi c^{-\rho} + (1-\chi)\mathcal{L}^{-\rho}]^{-\gamma/\rho}$ with $\chi \in (0,1)$, $\rho > -1$ and $\sigma = 1/(1+\rho)$ the elasticity of substitution between consumption and leisure. In order to simplify the exposition, let us assume $\gamma = 1$. From the first order condition (11) we derive $c/\mathcal{L} = [w_0\chi/(1-\chi)]^\sigma$ and the share of consumption within total utility is thus given by

$$\alpha = \frac{\chi}{\chi+(1-\chi)(c/\mathcal{L})^\rho} = \frac{\chi^\sigma}{\chi^\sigma+(1-\chi)^\sigma w_0^{1-\sigma}} \tag{35}$$

Obviously, there exists a bound $\underline{\chi} \in (0,1)$ such that the condition $\alpha \in (\underline{\alpha},1)$ of Proposition 2 is satisfied if $\chi \in (\underline{\chi},1)$. Moreover, the elasticity of intertemporal substitution in consumption is given by

$$\epsilon_{cc} = \frac{\chi+(1-\chi)(c/\mathcal{L})^\rho}{(1+\rho)(1-\chi)(c/\mathcal{L})^\rho} \text{ or equivalently } \epsilon_{cc} = \frac{1}{(1-\alpha)(1+\rho)} \tag{36}$$

It follows that for a given $\alpha \in (\underline{\alpha},1)$, local indeterminacy arises with any $\epsilon_{cc} < 1$ if $\rho = [1-\epsilon_{cc}(1-\alpha)]/\epsilon_{cc}(1-\alpha)$.

Let us then consider parameters' values that match quarterly data: $\delta = 2.5\%$, $\rho = 0.010252$, $\beta_{00} = 0.67$, $\beta_{10} = 0.33$, $\beta_{01} = 0.19$, $\beta_{11} = 0.66$, $b_{01} = 0.15$ and $b_{00} = b_{10} = b_{11} = 0$. We get $b < 0$ and $\hat{b} > 0$ with 15% of externalities in the investment good sector, an amount that belongs to the bounds provided by Basu and Fernald [3]. Considering first an elasticity of intertemporal substitution in consumption $\epsilon_{cc} = 0.5$ in the range provided by Campbell [11] and Kocherlakota [24], we find that the steady state is locally indeterminate for any $\alpha \in (0.6773618, 1)$. When $\alpha = 0.6774$, this conclusion holds with an elasticity of substitution between consumption and leisure $\sigma = 0.1613$ and a wage elasticity of labor $\epsilon_{\ell w} = 0.45\%$.[17] Considering

[17] Note that this value belongs to the lower ranges of estimated wage elasticities for men, namely the interval $(0, 8\%)$ (see Blundell and MaCurdy [10]).

now an elasticity of intertemporal substitution in consumption $\epsilon_{cc} = 2$ as recently suggested by Gruber [16], we find again that the steady state is locally indeterminate for any $\alpha \in (0.6773618, 1)$. When $\alpha = 0.6774$, this conclusion holds with an elasticity of substitution between consumption and leisure $\sigma = 0.6452$ and a wage elasticity of labor $\epsilon_{\ell w} = 1.8\%$.

With other preferences and the same values for the technological parameters, the conclusions are drastically different. In the case of a KPR utility function, if we set the wage elasticity of labor at $\epsilon_{\ell w} = 1.8\%$, we find that local indeterminacy arises if $\epsilon_{cc} > 4.25$ while if $\epsilon_{\ell w} = 0.45\%$ we get $\epsilon_{cc} > 13.95$. The results are even worse with a GHH utility function. Indeed, we find that local indeterminacy arises if $\epsilon_{cc} > 4944$ when $\epsilon_{\ell w} = 1.8\%$, or $\epsilon_{cc} > 4920$ when $\epsilon_{\ell w} = 0.45\%$! To complete the comparisons, we can also consider an additively separable utility function with $1/\epsilon_{c\ell} = 1/\epsilon_{\ell c} = 0$ and constant elasticities of intertemporal substitution in consumption and of the labor supply. We find similarly that local indeterminacy arises if $\epsilon_{cc} > 4935$ when $\epsilon_{\ell w} = 1.8\%$, or $\epsilon_{cc} > 4918$ when $\epsilon_{\ell w} = 0.45\%$.

4 Economic intuitions

To understand the intuition for the existence of local indeterminacy, it is convenient to refer to the consequences of multiple equilibria in terms of business-cycles, namely the occurrence of expectation-driven fluctuations. In two-sector models with sector-specific externalities, constant social returns and additively separable preferences, this intuition is now well established. The basic conditions concern the technological side. Let us start from an arbitrary equilibrium, and assume that agents expect an increase in the rate of investment induced by an instantaneous increase in the relative price of the investment good. On the one hand, a higher investment rate results in higher stocks and, when the investment good is labor intensive at the private level, an increase in the capital stock decreases its output at constant prices through the Rybczynski effect. This mechanism thus generates oscillations of the capital stock.[18] On the other hand, when the investment good is capital intensive at the social level, the initial rise in its price causes through the Stolper Samuelson theorem an increase in its return w_1 and requires a price decline to maintain the overall return to capital equal to the discount rate. This offsets the initial rise in the relative price of the invest-

[18] See Benhabib and Nishimura [7].

ment good and prices also reverse direction toward the steady state. As a result, the transversality condition holds and the expected new equilibrium becomes self-fulfilling.

But the properties of preferences also matter: On the one hand, when we consider constructing an alternative equilibrium with a higher investment rate, we must initially curtail consumption. If there is some curvature on the utility function, the desire to smooth consumption over time can overwhelm the technological effects described above. It follows that the associated fluctuations in consumption along the equilibrium path require a high enough elasticity of intertemporal substitution in consumption in order for the representative agent to compensate current loss of consumption by future gain. On the other hand, a low elasticity of labor supply is necessary to prevent the agent from smoothing the fluctuations in his wage and capital incomes associated with the fluctuations in the capital stock.[19] When the utility function is additively separable, these two elasticities are independent and both restrictions are necessary for the existence of local indeterminacy.[20]

Consider now non-separable preferences. While the technological side of the mechanism is identical to the previous one, local indeterminacy can be obtained under less stringent conditions on the side of preferences when concave homogeneous preferences are considered. Indeed, the level of utility essentially depends on the consumption-leisure ratio $c/\mathcal{L}$. When the elasticity of substitution between consumption and leisure σ is sufficiently less than 1, consumption and leisure are complement. It follows that, when the degree of concavity $1-\gamma$ is close enough to zero, if consumption falls, leisure also falls in a similar proportion and the ratio $c/\mathcal{L}$ remains almost constant. Moreover, considering a share α of consumption in total utility close enough to γ implies a low share $\gamma-\alpha$ of leisure in total utility. As a result, this mechanism implies weak variation of utility, and a decrease of consumption today does not need to be compensated by a large increase tomorrow. Expectation-driven fluctuations then become compatible with a low elasticity of intertemporal substitution in consumption.

[19]When the labor supply is highly elastic, fluctuations in the wage rate and the rental rate of capital can be compensated for by major modifications in the labor supply. The fluctuations in income are thus smoothed and the business-cycles can be eliminated. Conversely, when the labor supply is not very elastic, fluctuations in the capital stock generate fluctuations in incomes and business cycles become persistent.

[20]See Nishimura, Garnier and Venditti [13].

5 Concluding comments

In this paper we have studied a continuous-time two-sector infinite-horizon model with Cobb-Douglas technologies augmented to include sector-specific externalities. We have shown that when a concave homogeneous utility function is considered, local indeterminacy arises with a low elasticity of intertemporal substitution in consumption provided the wage elasticity of the labor supply and the elasticity of substitution between consumption and leisure are low enough. This result proves that contrary to the case with additively-separable preferences, local indeterminacy becomes compatible with intertemporal substitutability properties consistent with empirical estimates.

The analysis provided in this paper could be extended following two different lines of research. First, as homothetic utility functions are compatible with endogenous growth, we could consider a Lucas-type version of our model with human capital aggregate externalities. The same kind of argument could be used to prove that local indeterminacy of balanced-growth paths can arise with a low intertemporal elasticity of substitution.

Second, we could also extend our analysis to the consideration of a two-country general equilibrium model. Indeed, Nishimura and Shimomura [30] have shown that indeterminacy can arise in a simple competitive two-country dynamic model of international trade, free of externalities, imperfect competition, and government intervention when there is no international credit market. They use however a non standard quadratic utility function. Considering a non-separable formulation should allow to prove local indeterminacy under weaker conditions on the elasticity of intertemporal substitution. All this is left for future research.

6 Appendix

6.1 Proof of Proposition 1

Maximizing profit subject to the private technologies (1) gives the first order conditions

$$p_j \beta_{ij} y_j / x_{ij} = w_i, \quad i, j = 0, 1 \tag{37}$$

Considering the steady state with $y_1 = g x_1$ and $w_1 = (\delta + g) p_1$, we get

$$x_{11} = \frac{\beta_{11}}{\delta + g} g x_1 \tag{38}$$

Using the social production function (2) for the investment good we derive

$$x_{01} = \left(\frac{\beta_{11}}{\delta+g}\right)^{-\frac{\hat{\beta}_{11}}{\hat{\beta}_{01}}} g x_1 \text{ and thus } \frac{x_{01}}{x_{11}} = \left(\frac{\beta_{11}}{\delta+g}\right)^{-\frac{1}{\hat{\beta}_{01}}} \quad (39)$$

Finally we obtain from (37):

$$\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}} = \frac{x_{01}x_{10}}{x_{00}x_{11}} \quad \Leftrightarrow \quad \frac{x_{10}}{x_{00}} = \frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}} \left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{\hat{\beta}_{01}}} \quad (40)$$

Considering (39), (40) and $x_{00} + x_{01} = \ell$, $x_1 = x_{10} + x_{11}$, we get

$$x_1^* \;=\; \ell \frac{\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{\hat{\beta}_{01}}}}{1-\frac{\beta_{11}}{\delta+g}g\left[1-\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right]} \equiv \ell\kappa^*$$

Equation (37) with (40) gives

$$w_1 = \beta_{10}\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{-\hat{\beta}_{00}} \left(\frac{\beta_{11}}{\delta+g}\right)^{-\frac{\hat{\beta}_{00}}{\hat{\beta}_{01}}} \text{ and } w_0 = w_1 \frac{\beta_{01}}{\beta_{11}} \left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{\hat{\beta}_{01}}} \quad (41)$$

Considering then the fact that $w_1 = (\delta + g)p_1$ implies

$$p_1^* \;=\; \frac{\beta_{10}}{\delta+g}\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{-\hat{\beta}_{00}} \left(\frac{\beta_{11}}{\delta+g}\right)^{-\frac{\hat{\beta}_{00}}{\hat{\beta}_{01}}}$$

The substitution of (38) and (40) into (2) gives the expression of c^*, namely

$$c^* = \ell\kappa^*\left(1 - \frac{\beta_{11}}{\delta+g}g\right)\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{-\hat{\beta}_{00}} \left(\frac{\beta_{11}}{\delta+g}\right)^{-\frac{\hat{\beta}_{00}}{\hat{\beta}_{01}}} \equiv \ell\chi^* \quad (42)$$

Consider now (11) which can be written as follows

$$\frac{U_2(\ell\chi^*, \bar{\ell}-\ell)}{U_1(\ell\chi^*, \bar{\ell}-\ell)} \equiv g(\ell) = w_0 \quad (43)$$

Under Assumption 1, $\lim_{\ell\to 0} g(\ell) = 0$ and $\lim_{\ell\to\bar{\ell}} g(\ell) = +\infty$ with $g'(\ell) > 0$, and there exists a unique steady state with $x_1^* = \ell^*\kappa^*$ and $\ell^* \in (0, \bar{\ell})$. □

6.2 Proof of Proposition 2

We start by the computation of $\mathcal{D}$ and $\mathcal{T}$ using a general formulation for $U(c, \mathcal{L})$. Let us first introduce a useful relationship between $\epsilon_{\ell c}$ and $\epsilon_{c\ell}$.[21]

Lemma 1. *Let Assumption 1 hold. Then at the steady state*

$$\epsilon_{c\ell} = -\frac{\chi^*}{w_0}\epsilon_{\ell c} \equiv -\Phi\epsilon_{\ell c} \quad (44)$$

with χ^* and w_0 as given by (41) in Appendix 6.1.

[21] A similar relationship has been obtained by Hintermaier [20].

Proof: Using (14) and the first order conditions (11), we get $\epsilon_{c\ell} = -\epsilon_{\ell c}(c/w_0\ell)$. At the steady state we have $c^* = \ell^*\chi^*$ and the result follows. □

Consider the expressions (21). We need to compute the following seven derivatives: $\partial c/\partial x_1$, $\partial c/\partial p_1$, $\partial y_1/\partial x_1$, $\partial y_1/\partial p_1$, $\partial w_1/\partial p_1$, $\partial \ell/\partial x_1$ and $\partial \ell/\partial p_1$. Total differenciation of the factor-price frontier (6) and the factor market clearing equation (4) gives:

$$\begin{aligned}
\frac{dw_1}{dp_1} &= \frac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00}-\hat{a}_{10}\hat{a}_{01}} && (45)\\
\frac{dc}{dx_1} &= -\frac{a_{01}}{a_{11}a_{00}-a_{10}a_{01}} + \frac{a_{11}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dx_1} && (46)\\
\frac{dy_1}{dx_1} &= \frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} - \frac{a_{10}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dx_1} && (47)\\
\frac{dy_1}{dp_1} &= \frac{a_{00}\ell^*(\mathcal{Z}_1+\mathcal{Z}_2)}{(a_{11}a_{00}-a_{10}a_{01})p_1^*} - \frac{g\ell^*\kappa^*}{p_1^*} - \frac{a_{10}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dp_1} && (48)\\
\frac{dc}{dp_1} &= \frac{-a_{01}\ell^*(\mathcal{Z}_1+\mathcal{Z}_2)}{(a_{11}a_{00}-a_{10}a_{01})p_1^*} + \frac{a_{11}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dp_1} && (49)
\end{aligned}$$

with

$$\mathcal{Z}_1 = \frac{\kappa^*}{\hat{b}\hat{\beta}_{11}}, \quad \mathcal{Z}_2 = \frac{a_{11}}{a_{01}\mathcal{A}}, \quad \mathcal{A} = \frac{\hat{b}\hat{\beta}_{01}}{1-\hat{b}} \tag{50}$$

As $\hat{b} < 1$, the sign of $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{A}$ is given by the sign of $\hat{b}$. We have finally to compute $d\ell/dx_1$ and $d\ell/dp_1$. Total differenciation of (11) gives:

$$\frac{d\ell}{\ell}\left(\frac{1}{\epsilon_{c\ell}} - \frac{1}{\epsilon_{\ell\ell}}\right) = dp_1\left[\frac{\partial c}{\partial p_1}\frac{1}{c}\left(\frac{1}{\epsilon_{\ell c}} - \frac{1}{\epsilon_{cc}}\right) + \frac{1}{w_0}\frac{dw_0}{dp_1}\right] + dx_1\frac{\partial c}{\partial x_1}\frac{1}{c}\left(\frac{1}{\epsilon_{\ell c}} - \frac{1}{\epsilon_{cc}}\right)$$

When $dp_1 = 0$, using (46) with (42) in the proof of Proposition 1, we derive:

$$\frac{d\ell}{dx_1} = \frac{\frac{a_{01}}{(a_{11}a_{00}-a_{10}a_{01})\chi^*}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{\ell c}}\right)}{\frac{1}{\epsilon_{c\ell}} - \frac{1}{\epsilon_{\ell\ell}} + \frac{a_{11}}{(a_{11}a_{00}-a_{10}a_{01})\chi^*}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{\ell c}}\right)} \tag{51}$$

When $dx_1 = 0$, consider the factor-price frontier (6). Solving for w_0 gives:

$$w_0 = \frac{\hat{a}_{10}p_1\mathcal{A}}{\hat{a}_{11}\hat{a}_{00}-\hat{a}_{10}\hat{a}_{01}} \tag{52}$$

Equations (45) and (52) then imply

$$\frac{1}{w_0}\frac{dw_0}{dp_1} = \frac{-1}{p_1\mathcal{A}}$$

Therefore, considering (48) with (42) in the proof of Proposition 1, we derive:

$$\frac{d\ell}{dp_1} = \frac{\frac{\ell^*}{p_1^*}\left[\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{(a_{11}a_{00}-a_{10}a_{01})\chi^*}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{\ell c}}\right) - \frac{1}{\mathcal{A}}\right]}{\frac{1}{\epsilon_{c\ell}} - \frac{1}{\epsilon_{\ell\ell}} + \frac{a_{11}}{(a_{11}a_{00}-a_{10}a_{01})\chi^*}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{\ell c}}\right)} \tag{53}$$

Let us denote $T = (a_{11}a_{00} - a_{10}a_{01})$ and $\hat{T} = (\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01})$. Substituting (46), (48), (51) and (53) into (21) finally gives:

$$\mathcal{D} = \frac{\left[\left(\frac{a_{00}}{T}-g\right)\left(\frac{1}{\epsilon_{c\ell}}-\frac{1}{\epsilon_{\ell\ell}}\right)+\frac{(1-ga_{11})}{T\chi^*}\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{\ell c}}\right)\right]\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{\frac{1}{\epsilon_{c\ell}}-\frac{1}{\epsilon_{\ell\ell}}+\frac{a_{11}\left[\frac{1}{\epsilon_{cc}}\left(1+\frac{1}{\mathcal{A}}\right)-\frac{1}{\epsilon_{\ell c}}\right]}{T\chi^*}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)\left(\frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}}-\frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}}\right)}{T\chi^*}+\frac{1}{\mathcal{A}\epsilon_{c\ell}}}$$

$$\begin{aligned}\mathcal{T} &= \frac{\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)\left(\frac{1}{\epsilon_{c\ell}}-\frac{1}{\epsilon_{\ell\ell}}\right)+\frac{a_{11}\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)\left(\frac{1}{\epsilon_{cc}}-\frac{1}{\epsilon_{\ell c}}\right)}{T\chi^*}}{\frac{1}{\epsilon_{c\ell}}-\frac{1}{\epsilon_{\ell\ell}}+\frac{a_{11}\left[\frac{1}{\epsilon_{cc}}\left(1+\frac{1}{\mathcal{A}}\right)-\frac{1}{\epsilon_{\ell c}}\right]}{T\chi^*}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)\left(\frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}}-\frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}}\right)}{T\chi^*}+\frac{1}{\mathcal{A}\epsilon_{c\ell}}} \\ &+ \frac{\frac{(1-ga_{11})\left[\frac{1}{\epsilon_{cc}}\left(1+\frac{1}{\mathcal{A}}\right)-\frac{1}{\epsilon_{\ell c}}\right]}{T\chi^*}-\frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*]\left(\frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}}-\frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}}\right)}{T\chi^*}+\frac{\frac{a_{00}}{T}-g}{\mathcal{A}\epsilon_{c\ell}}}{\frac{1}{\epsilon_{c\ell}}-\frac{1}{\epsilon_{\ell\ell}}+\frac{a_{11}\left[\frac{1}{\epsilon_{cc}}\left(1+\frac{1}{\mathcal{A}}\right)-\frac{1}{\epsilon_{\ell c}}\right]}{T\chi^*}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)\left(\frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}}-\frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}}\right)}{T\chi^*}+\frac{1}{\mathcal{A}\epsilon_{c\ell}}}\end{aligned} \quad (54)$$

with $\mathcal{Z}_1 + \mathcal{Z}_2 > \kappa^*$. Note that concavity and normality of consumption and leisure imply $1/\epsilon_{cc}\epsilon_{\ell\ell} - 1/\epsilon_{\ell c}\epsilon_{c\ell} \leq 0$, $1/\epsilon_{cc} - 1/\epsilon_{\ell c} \geq 0$ and $1/\epsilon_{c\ell} - 1/\epsilon_{\ell\ell} \geq 0$.

Let us now prove Proposition 2 by focusing on a concave homogeneous of degree $\gamma \in (0,1]$ utility function. Using (22) and (23), we derive from (54):

$$\mathcal{D} = \frac{[\gamma-\alpha\epsilon_{cc}(1-\gamma)]\left[\left(\frac{a_{00}}{T}-g\right)+\frac{(\gamma-\alpha)(1-ga_{11})\Phi}{\alpha T\chi^*}\right]\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{\gamma-\alpha\epsilon_{cc}(1-\gamma)+\frac{(\gamma-\alpha)a_{11}\Phi\left[\frac{\gamma-\alpha}{\mathcal{A}}+\gamma-\alpha\epsilon_{cc}(1-\gamma)\right]}{\alpha T\chi^*}+\frac{(\gamma-\alpha)[1-\epsilon_{cc}(1-\gamma)]}{\mathcal{A}}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)(1-\gamma)[\gamma-\alpha\epsilon_{cc}(1-\gamma)]}{T\chi^*}}$$

$$\begin{aligned}\mathcal{T} &= \frac{[\gamma-\alpha\epsilon_{cc}(1-\gamma)]\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)+\frac{(\gamma-\alpha)\Phi\left[a_{11}[\gamma-\alpha\epsilon_{cc}(1-\gamma)]\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)+(1-ga_{11})\left[\frac{\gamma-\alpha}{\mathcal{A}}+\gamma-\alpha\epsilon_{cc}(1-\gamma)\right]\right]}{\alpha T\chi^*}}{\gamma-\alpha\epsilon_{cc}(1-\gamma)+\frac{(\gamma-\alpha)a_{11}\Phi\left[\frac{\gamma-\alpha}{\mathcal{A}}+\gamma-\alpha\epsilon_{cc}(1-\gamma)\right]}{\alpha T\chi^*}+\frac{(\gamma-\alpha)[1-\epsilon_{cc}(1-\gamma)]}{\mathcal{A}}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)(1-\gamma)[\gamma-\alpha\epsilon_{cc}(1-\gamma)]}{T\chi^*}} \\ &- \frac{\frac{ga_{01}(\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*)(1-\gamma)[\gamma-\alpha\epsilon_{cc}(1-\gamma)]}{T\chi^*}-\frac{(\gamma-\alpha)[1-\epsilon_{cc}(1-\gamma)]\left(\frac{a_{00}}{T}-g\right)}{\mathcal{A}}}{\gamma-\alpha\epsilon_{cc}(1-\gamma)+\frac{(\gamma-\alpha)a_{11}\Phi\left[\frac{\gamma-\alpha}{\mathcal{A}}+\gamma-\alpha\epsilon_{cc}(1-\gamma)\right]}{\alpha T\chi^*}+\frac{(\gamma-\alpha)[1-\epsilon_{cc}(1-\gamma)]}{\mathcal{A}}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)(1-\gamma)[\gamma-\alpha\epsilon_{cc}(1-\gamma)]}{T\chi^*}}\end{aligned}$$

with $\Phi = \chi^*/w_0$ and χ^*, w_0 given in Appendix 6.1. From Definition 2, we get the following results:

$$\frac{a_{00}}{T} - g < 0 \;\Leftrightarrow\; b < 0 \quad \text{and} \quad \delta + g - \frac{\hat{a}_{00}}{\hat{T}} < 0 \;\Leftrightarrow\; \hat{b} > 0 \quad (55)$$

Under Assumption 1 and since $\hat{b} < 1$, we also have

$$\mathcal{A} > 0 \text{ and } 1 + \frac{1}{\mathcal{A}} = \frac{1-\hat{b}\hat{\beta}_{11}}{\hat{b}\hat{\beta}_{01}} > 0 \;\Leftrightarrow\; \hat{b} > 0 \quad (56)$$

If $b < 0$ and $\hat{b} > 0$, we then get from the above expressions of $\mathcal{D}$ and $\mathcal{T}$:

$$\lim_{\alpha\to\gamma} \mathcal{D} = \frac{\left(\frac{a_{00}}{T}-g\right)\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{1+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)(1-\gamma)}{T\chi^*}} \equiv \mathcal{D}(\gamma)$$

$$\lim_{\alpha\to\gamma} \mathcal{T} = \frac{\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)+\frac{ga_{01}(\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*)(1-\gamma)}{T\chi^*}}{1+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)(1-\gamma)}{T\chi^*}} \equiv \mathcal{T}(\gamma)$$

with

$$\mathcal{D}(1) = \left(\frac{a_{00}}{T} - g\right)\left(\delta + g - \frac{\hat{a}_{00}}{\hat{T}}\right) > 0 \text{ and } \mathcal{T}(1) = \left(\frac{a_{00}}{T} + \delta - \frac{\hat{a}_{00}}{\hat{T}}\right) < 0$$

while

$$\lim_{\alpha \to 0} \mathcal{D} = \frac{(1-ga_{11})}{a_{11}} \frac{\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{\left(1+\frac{1}{\mathcal{A}}\right)} < 0$$

The result is proved. □

6.3 KPR utility function

We first need to prove the existence of a steady state. We get from (43)

$$g(\ell) = \frac{c^* v'(\bar{\ell}-\ell)}{v(\ell-\ell)} = c^* h(\bar{\ell} - \ell)$$

Using the fat that $c^* = \ell\chi^*$, equation (43) can thus be written as

$$\ell h(\bar{\ell} - \ell) \equiv \tilde{g}(\ell) = w_0/\chi^*$$

If we assume that $\lim_{\mathcal{L}\to 0}(\bar{\ell} - \mathcal{L})h(\mathcal{L}) = +\infty$ and $\lim_{\mathcal{L}\to\bar{\ell}}(\bar{\ell} - \mathcal{L})h(\mathcal{L}) = 0$, then we get $\lim_{\ell\to 0} \tilde{g}(\ell) = 0$, $\lim_{\ell\to\ell} \tilde{g}(\ell) = +\infty$ and $\tilde{g}'(\ell) > 0$. It follows that there exists a unique steady state with $x_1^* = \ell^*\kappa^*$ and $\ell^* \in (0, \bar{\ell})$. Normalyzing the steady state with $(\bar{\ell} - \ell^*)/\ell^* = \nu > 0$ gives $\psi = \nu/\Phi$.

We easily derive that

$$\epsilon_{cc} = \frac{1}{\theta}, \quad \epsilon_{\ell c} = -\frac{1}{1-\theta}, \quad \epsilon_{c\ell} = \frac{\Phi}{(1-\theta)}, \quad \epsilon_{\ell\ell} = \frac{\nu\Phi}{\eta\Phi+(1-\theta)\nu} < 0 \tag{57}$$

so that

$$\frac{1}{\epsilon_{cc}} = \frac{1}{\epsilon_{\ell c}} + 1 \text{ and } \frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}} - \frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}} = \frac{\theta\eta\Phi+(1-\theta)\nu}{\nu\Phi} \leq 0 \tag{58}$$

Proposition 3. *Let $U(c, \mathcal{L}) = [cv(\mathcal{L})]^{1-\theta}/(1-\theta)$ with c and $\mathcal{L}$ some normal goods, $b < 0$ and $\hat{b} > 0$. Then, there exist $\bar{\sigma} = -1/\bar{\eta} > 0$ and $\bar{\theta} \in (0, 1)$ such that when $\sigma \in [0, \bar{\sigma})$, the steady state is locally indeterminate if $\theta \in (0, \bar{\theta})$.*

Proof: Using (57) and (58), we derive from (54):

$$\begin{aligned}
\mathcal{D} &= \frac{\left[-\left(\frac{a_{00}}{T}-g\right)\frac{\eta}{\nu}+\frac{(1-ga_{11})}{T\chi^*}\right]\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{-\frac{\eta}{\nu}+\frac{a_{11}\left(1+\frac{\theta}{\mathcal{A}}\right)}{T\chi^*}-\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)[\eta\theta\Phi+(1-\theta)\nu]}{T\chi^*\nu\Phi}+\frac{1-\theta}{\mathcal{A}\Phi}} \\
\mathcal{T} &= \frac{-\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)\frac{\eta}{\nu}+\frac{a_{11}\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)+(1-ga_{11})\left(1+\frac{\theta}{\mathcal{A}}\right)}{T\chi^*}}{-\frac{\eta}{\nu}+\frac{a_{11}\left(1+\frac{\theta}{\mathcal{A}}\right)}{T\chi^*}-\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)[\eta\theta\Phi+(1-\theta)\nu]}{T\chi^*\nu\Phi}+\frac{1-\theta}{\mathcal{A}\Phi}} \\
&+ \frac{\frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*][\eta\theta\Phi+(1-\theta)\nu]}{T\chi^*\nu\Phi}+\frac{\left(\frac{a_{00}}{T}-g\right)(1-\theta)}{\mathcal{A}\Phi}}{-\frac{\eta}{\nu}+\frac{a_{11}\left(1+\frac{\theta}{\mathcal{A}}\right)}{T\chi^*}-\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)[\eta\theta\Phi+(1-\theta)\nu]}{T\chi^*\nu\Phi}+\frac{1-\theta}{\mathcal{A}\Phi}}
\end{aligned} \tag{59}$$

with $\Phi = \chi^*/w_0$ and χ^*, w_0 given in Appendix 6.1. Assuming $b < 0$ and $\hat{b} > 0$, we first show that for $\epsilon_{cc} \in (0,1]$, i.e. $\theta \geq 1$, local indeterminacy is ruled out when σ is large, i.e. $-\eta$ is close to 0. We get indeed from (59):

$$\lim_{\eta \to 0_-} \mathcal{D} = \frac{(1-ga_{11})\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{a_{11}\left(1+\frac{\theta}{\mathcal{A}}\right)-\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)(1-\theta)}{\Phi}+\frac{(1-\theta)T\chi^*}{\mathcal{A}\Phi}} < 0 \tag{60}$$

On the contrary, when low values for σ are considered, i.e. $-\eta$ is large, we derive from (59):

$$\begin{aligned} \lim_{\eta \to -\infty} \mathcal{D} &= \frac{\left(\frac{a_{00}}{T}-g\right)\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)\frac{1}{\nu}}{\frac{1}{\nu}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)\theta}{T\chi^*\nu}} \equiv \mathcal{D}_0(\theta) \\ \lim_{\eta \to -\infty} \mathcal{T} &= \frac{\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)\frac{1}{\nu}-\frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*]\theta}{T\chi^*\nu}}{\frac{1}{\nu}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)\theta}{T\chi^*\nu}} \equiv \mathcal{T}_0(\theta) \end{aligned} \tag{61}$$

It follows that local indeterminacy requires a large enough value of $\epsilon_{cc} = 1/\theta$. Indeed we conclude from (61) that

$$\mathcal{D}_0(0) = \left(\frac{a_{00}}{T} - g\right)\left(\delta + g - \frac{\hat{a}_{00}}{\hat{T}}\right) > 0, \quad \mathcal{T}_0(\theta) = \left(\frac{a_{00}}{T} + \delta - \frac{\hat{a}_{00}}{\hat{T}}\right) < 0$$

and

$$\lim\nolimits_{\theta \to +\infty} \mathcal{D}_0(\theta) = 0_-$$

The result is proved. □

Remark: Proposition 3 provides a drastically different conclusion than the one obtained within aggregate models. Indeed local indeterminacy is ruled out under standard parameterizations in one-sector models with KPR preferences.[22] This result is easily explained by the fact that the existence of multiple equilibria in aggregate models relies on large values for the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply, but the conditions for concavity prevent these two conditions to hold simultaneously. However, in two-sector models with constant social returns, local indeterminacy requires a weakly elastic labor supply which is compatible with a large intertemporal substitution in consumption.

6.4 GHH utility function

We first need to prove the existence of a steady state. Let $\ell = \bar{l} \in (0, \bar{\ell})$. We get

$$g(\bar{l}) = G'((\bar{\ell} - \bar{l})/B)/B \equiv \tilde{g}(B)$$

[22]See Hintermaier [20, 21], Jaimovich [22], Nishimura, Nourry and Venditti [29], Pintus [34].

If $G'(\mathcal{L}/B) + (\mathcal{L}/B)G''(\mathcal{L}/B) \neq 0$ then $\tilde{g}'(B) \neq 0$. Assume also that $\epsilon^G_{\mathcal{L}\mathcal{L}}(\mathcal{L}/B) > 0$ for any $\mathcal{L}/B > 0$, and

i) $\lim_{x\to 0} G'(x)x = +\infty$ and $\lim_{x\to +\infty} G'(x)x = 0$,

or

ii) $\lim_{x\to 0} G'(x)x = 0$ and $\lim_{x\to +\infty} G'(x)x = +\infty$.[23]

This implies that there exists a unique value B^* of B such that when $B = B^*$, $\bar{l}$ satisfies equation (43). It follows that there exists a unique steady state with $x_1^* = \ell^*\kappa^*$ and $\ell^* = \bar{l} \in (0, \bar{\ell})$.

Using the elasticity of $G(\mathcal{L}/B)$ as defined by (33), we get from (14) that

$$\epsilon_{cc} = \epsilon_{\ell c} \text{ and } \frac{1}{\epsilon_{\ell\ell}} = \frac{1}{\epsilon_{c\ell}} + \frac{1}{\epsilon^G_{\ell\ell}} \text{ with } \epsilon^G_{\ell\ell} = -\epsilon^G_{\mathcal{L}\mathcal{L}}\frac{\bar{\ell}-\bar{l}}{\bar{l}} < 0 \tag{62}$$

and thus

$$\frac{1}{\epsilon_{cc}\epsilon_{\ell\ell}} - \frac{1}{\epsilon_{c\ell}\epsilon_{\ell c}} = \frac{1}{\epsilon_{cc}\epsilon^G_{\ell\ell}} < 0 \tag{63}$$

Let $b < 0$, $\hat{b} > 0$ and consider the following bounds for ϵ_{cc}:

$$\begin{aligned} \epsilon^1_{cc} &\equiv \frac{\epsilon^G_{\ell\ell}}{\mathcal{A}\chi^*}\left[\frac{a_{11}}{T} - w_0\right] - \frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{T\chi^*} > 0 \\ \epsilon^2_{cc} &\equiv \frac{\frac{\epsilon^G_{\ell\ell}}{\mathcal{A}\chi^*}\left[\frac{1-ga_{11}}{T} - \left(\frac{a_{00}}{T}-g\right)w_0\right] + \frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*]}{T\chi^*}}{\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}} \end{aligned} \tag{64}$$

Note that while ϵ^1_{cc} is clearly positive, ϵ^2_{cc} can be positive or negative.

Proposition 4. *Let $U(c, \mathcal{L}) = u(c+G(\mathcal{L}/B))$ with $B = B^*$, $\hat{b} > 0$ and $\underline{\epsilon}_{cc} = \max\{\epsilon^1_{cc}, \epsilon^2_{cc}\}$. Then, when $\epsilon_{cc} > \underline{\epsilon}_{cc}$, the steady state is locally indeterminate.*

Proof: Using (62) and (63), we conclude from (54) and Lemma 1:

$$\begin{aligned} \mathcal{D} &= \frac{\left(\frac{a_{00}}{T}-g\right)\left(\delta+g-\frac{\hat{a}_{00}}{\hat{T}}\right)}{1-\frac{\epsilon^G_{\ell\ell}a_{11}}{\epsilon_{cc}\mathcal{A}T\chi^*}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\epsilon_{cc}T\chi^*}+\frac{\epsilon^G_{\ell\ell}}{\epsilon_{cc}\mathcal{A}\Phi}} \\ \mathcal{T} &= \frac{\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)-\frac{\epsilon^G_{\ell\ell}(1-ga_{11})}{\epsilon_{cc}\mathcal{A}T\chi^*}-\frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*]}{\epsilon_{cc}T\chi^*}+\frac{\epsilon^G_{\ell\ell}\left(\frac{a_{00}}{T}-g\right)}{\epsilon_{cc}\mathcal{A}\Phi}}{1-\frac{\epsilon^G_{\ell\ell}a_{11}}{\epsilon_{cc}\mathcal{A}T\chi^*}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\epsilon_{cc}T\chi^*}+\frac{\epsilon^G_{\ell\ell}}{\epsilon_{cc}\mathcal{A}\Phi}} \end{aligned} \tag{65}$$

with $\Phi = \chi^*/w_0$ and χ^*, w_0 given in Appendix 6.1. Let $b < 0$ and $\hat{b} > 0$. Using (55) and (56), we derive from (65) that $\mathcal{D} > 0$ if

$$\epsilon_{cc} > \frac{\epsilon^G_{\ell\ell}}{\mathcal{A}\chi^*}\left[\frac{a_{11}}{T} - w_0\right] - \frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{T\chi^*} \equiv \epsilon^1_{cc} > 0 \tag{66}$$

Moreover, when (66) is satisfied, $\mathcal{T} < 0$ if

[23] If $G(x) = x^{1-\varsigma}/(1-\varsigma)$ with $\varsigma = 1/\epsilon^G_{\mathcal{L}\mathcal{L}} > 0$, i) is satisfied when $\varsigma > 1$ while ii) holds if $\varsigma \in [0, 1)$.

$$\epsilon_{cc} > \frac{\frac{\epsilon_{\ell\ell}^G}{\mathcal{A}\chi^*}\left[\frac{1-ga_{11}}{T} - \left(\frac{a_{00}}{T} - g\right)w_0\right] + \frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*]}{T\chi^*}}{\frac{a_{00}}{T} + \delta - \frac{\hat{a}_{00}}{\hat{T}}} \equiv \epsilon_{cc}^2 \tag{67}$$

Note that while ϵ_{cc}^1 is clearly positive, ϵ_{cc}^2 can be positive or negative. The result is proved taking $\underline{\epsilon}_{cc} = \max\{\epsilon_{cc}^1, \epsilon_{cc}^2\} > 0$. □

Note that the bound $\underline{\epsilon}_{cc}$ is an increasing function of the elasticity $-\epsilon_{\ell\ell}^G$. Recalling that $\epsilon_{\ell w} = -\epsilon_{\ell\ell}^G$, the lowest value of $\underline{\epsilon}_{cc}$ is obtained when the labor supply is inelastic, i.e. $\epsilon_{\ell w} = 0$. In this case indeed we have

$$\underline{\epsilon}_{cc}\Big|_{\epsilon_{\ell\ell}^G=0} = \max\left\{-\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{T\chi^*}, \frac{ga_{01}[\mathcal{Z}_1+\mathcal{Z}_2-\kappa^*]}{T\chi^*\left(\frac{a_{00}}{T}+\delta-\frac{\hat{a}_{00}}{\hat{T}}\right)}\right\} > 0 \tag{68}$$

Remark: Proposition 4 also provides a drastically different conclusion than the one obtained within aggregate models. Local indeterminacy is indeed completely ruled out in one-sector models with GHH preferences.[24] In this case, the absence of income effect implies that an expected increase in the marginal rate of return on capital does not generate a strong enough variation of labor supply to be compatible with the Euler equation, and thus the expectation of a new equilibrium cannot be self-fulfilling.[25] However, in a two-sector model, an expected increase in the marginal rate of return on capital generates movements of productive resources across sectors that modify the marginal product of capital and labor and the relative price of the investment good. As a result, the Euler equation can be satisfied even with variations of the labor supply that remain small, and the expectation of a new equilibrium can be self- fulfilling.

References

[1] Azariadis, C. (1981): "Self Fulfilling Prophecies", *Journal of Economic Theory*, 25, 380-396.

[2] Barro, R. (2009): "Rare Disasters, Asset Prices and Welfare Costs," *American Economic Review*, 99, 243-264.

[24] See Jaimovich [22], Meng and Yip [26], Nishimura, Nourry and Venditti [29].

[25] Considering preferences that nest as special cases the KPR and GHH utility functions, Jaimovich [22] shows that local indeterminacy can arise under mild increasing returns provided the magnitude of income effects admits intermediary values, i.e. larger than GHH preferences but lower than KPR preferences.

[3] Basu, S., and J. Fernald (1997): "Returns to Scale in US Production: Estimates and Implications," *Journal of Political Economy*, 105, 249-283.

[4] Benhabib, J., and R. Farmer (1994): "Indeterminacy and Increasing Returns," *Journal of Economic Theory*, 63, 19-41.

[5] Benhabib, J., and R. Farmer (1996): "Indeterminacy and Sector Specific Externalities," *Journal of Monetary Economics*, 37, 397-419.

[6] Benhabib, J., and R. Farmer (1999): "Indeterminacy and Sunspots in Macroeconomics," in J.B. Taylor and M. Woodford (eds.), *Handbook of Macroeconomics*, North-Holland, Amsterdam, 387-448.

[7] Benhabib, J., and K. Nishimura (1985): "Competitive Equilibrium Cycles", *Journal of Economic Theory*, 35, 284-306.

[8] Benhabib, J., and K. Nishimura (1998): "Indeterminacy and Sunspots with Constant Returns," *Journal of Economic Theory*, 81, 58-96.

[9] Bennett, R, and R. Farmer (2000): "Indeterminacy with non-separable utility," *Journal of Economic Theory* 93, 118-143.

[10] Blundell, R., and T. MaCurdy (1999): "Labour Supply: a Review of Alternative Approaches," in O. Ashenfelter and D. Card (eds.), *Handbook of Labor Economics*, North-Holland, Amsterdam, 1559-1695.

[11] Campbell, J. (1999): "Asset Prices, Consumption and the Business Cycle," in Taylor J.B. and Woodford M. (eds.), *Handbook of Macroeconomics*, North-Holland, Amsterdam.

[12] Cass, D., and K. Shell (1983): "Do Sunspots Matter ?", *Journal of Political Economy*, 91, 193-227.

[13] Garnier, J.P., K. Nishimura and A. Venditti (2007): "Intertemporal Substitution in Consumption, Labor Supply Elasticity and Sunspot Fluctuations in Continuous-Time Models," *International Journal of Economic Theory*, 3, 235-259.

[14] Garnier, J.P., K. Nishimura and A. Venditti (2007): "Local Indeterminacy in Continuous-Time Models: The Role of Returns to Scale," GREQAM Working Paper.

[15] Greenwood, J., Z. Hercovitz, and G. Huffman (1988): "Investment, Capacity Utilization and the Real Business Cycle," *American Economic Review*, 78, 402-417.

[16] Gruber, J. (2006): "A Tax-Based Estimates of the Elasticity of Intertemporal Substitution," *NBER Working Paper* 11945.

[17] Guo, J.T., and S. Harrison (2010): "Indeterminacy with No-Income Effect Preferences and Sector-Specific Externalities," *Journal of Economic Theory*, 145, 287-300.

[18] Guo, J.T., and K. Lansing (2009): "Capital-Labor Substitution and Equilibrium Indeterminacy," *Journal of Economic Dynamics and Control*, 12, 1991-2000.

[19] Harrison, S. (2001): "Indeterminacy in a Model with Sector-Specific Externalities," *Journal of Economic Dynamics and Control*, 25, 747-764.

[20] Hintermaier, T. (2001): "Lower Bounds on Externalities in Sunspot Models," *Working Paper EUI.*

[21] Hintermaier, T. (2003): "On the Minimum Degree of Returns to Scale in Sunspot Models of Business Cycles," *Journal of Economic Theory*, 110, 400-409.

[22] Jaimovich, N. (2008): "Income Effects and Indeterminacy in a Calibrated One-Sector Growth Model," *Journal of Economic Theory*, 143, 610-623.

[23] King, R., C. Plosser, and S. Rebelo (1988): "Production, Growth and Business Cycles," *Journal of Monetary Economics*, 21, 195-232.

[24] Kocherlakota, N. (1996): "The Equity Premium: It's Still a Puzzle," *Journal of Economic Literature*, 36, 42-71.

[25] Lloyd-Braga, T., C. Nourry and A. Venditti (2006): "Indeterminacy with Small Externalities: the Role of Non-Separable Preferences," *International Journal of Economic Theory*, 2, 217-239.

[26] Meng, Q., and C. Yip (2008): "On Indeterminacy in One-Sector Models of the Business Cycle with Factor-Generated Externalities," *Journal of Macroeconomics*, 30, 97-110.

[27] Mino, K. (1999): "Non-Separable Utility Function and Indeterminacy of Equilibrium in a Model with Human Capital," *Economics Letters*, 62, 311-317.

[28] Mulligan, C. (2002): "Capital interest and aggregate intertemporal substitution," *NBER Working Paper* 9373.

[29] Nishimura, K., C. Nourry and A. Venditti (2009): "Indeterminacy in Aggregate Models with Small Externalities: an Interplay Between Preferences and Technology," *Journal of Nonlinear and Convex Analysis*, 10(2), 279-298.

[30] Nishimura, K., and K. Shimomura (2006): "Indeterminacy in a Dynamic Two-Country Model", *Economic Theory*, 29, 307-324.

[31] Nishimura, K., and A. Venditti (2004): "Indeterminacy and the Role of Factor Substitutability," *Macroeconomic Dynamics*, 8, 436-465.

[32] Nishimura, K., and A. Venditti (2007): "Indeterminacy in Discrete-Time Infinite-Horizon Models with Non-Linear Utility and Endogenous Labor," *Journal of Mathematical Economics*, 43, 446-476.

[33] Pintus, P. (2006): "Indeterminacy with Almost Constant Returns to Scale: Capital-Labor Substitution Matters", *Economic Theory*, 28, 633-649.

[34] Pintus, P. (2007): "Local Determinacy with Non-Separable Utility," *Journal of Economic Dynamics and Control*, 31, 669-682.

[35] Shell, K. (1977): "Monnaie et Allocation Intertemporelle", mimeo, Séminaire d'Econométrie Roy-Malinvaud, Centre National de la Recherche Scientifique, Paris.

[36] Vissing-Jorgensen, A. (2002): "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution," *Journal of Political Economy*, 110, 825-853.

[37] Woodford, M. (1986): "Stationary Sunspot Equilibria: the Case of Small Fluctuations Around a Deterministic Steady State", Mimeo, University of Chicago.