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Abstract

In an exchange economy under uncertainty populated by consumers having constant and equal relative risk aversion but heterogeneous probabilistic beliefs, we analyze the nature of the representative consumer's probabilistic belief and discount rates. We prove a formula that implies that the representative consumer's discount rates are raised or lowered by belief heterogeneity depending on whether the constant relative risk aversion is greater or smaller than one. We also show that the representative consumer's discount rates may be a hyperbolic function of time even when the individual consumers' discount rates are equal to one another, as long as their beliefs are heterogeneous.

JEL Classification Codes: D51, D53, D81, D91, G12, G13, Q51, Q54.

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1 Introduction

In dynamic macroeconomics and finance, the use of representative-consumer models is prevalent. As in Mehra and Prescott (1985), the standard (and by now classical) representative-consumer model consists of a single consumer having a utility function U of the form

$$U(c) = E\left(\sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\beta}}{1-\beta}\right) \text{ or } U(c) = E\left(\int_0^{\infty} \exp(-\rho t)\frac{c_t^{1-\beta}}{1-\beta}\right),$$

depending on whether the time span is discrete or continuous, and an initial endowment process $e = (e_t)_t$. Given that there is only one consumer, the equilibrium of such an economy must necessarily be the no-trade equilibrium, in which the consumer is induced to demand his own endowment process $e = (e_t)_t$. The equilibrium state price deflator $\pi = (\pi_t)_t$, which evaluates

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each consumption process $c = (c_t)_t$ via $E\left(\sum_{t=0}^{\infty} \pi_t c_t\right)$ or $E\left(\int_0^{\infty} \pi_t c_t\right)$, can then be written in the simple form of $\delta^t c_t^{-\beta}$ or $\exp(-\rho t) c_t^{-\beta}$. The task of identifying equilibrium asset price process with dividend process $d = (d_t)_t$ can therefore be reduced to one of calculating

$$E_t\left(\sum_{\tau=t}^{\infty} \delta^{t-\tau}\left(\frac{c_\tau}{c_t}\right)^{-\beta} d_\tau\right) \text{ or } E_t\left(\int_t^{\infty} \exp(-\rho(\tau-t))\left(\frac{c_\tau}{c_t}\right)^{-\beta} d_\tau \,\mathrm{d}\tau\right)$$

at each time t.

There are a couple of important assumptions embedded in this specification. First, the representative consumer has an expected utility function, thereby conforming the independence axiom. Second, the discount rate is deterministic, constant, and independent of consumption levels. Third, the representative consumer exhibits constant relative risk aversion.

When we take up any representative-consumer model, we are not really interested in the analysis of an economy consisting of a single consumer per se. Rather, we regard the representative consumer economy as a reduced form of a more complicated economy consisting of multiple, heterogeneous consumers. Then a question arises: if we explicitly model an economy of multiple, heterogeneous consumers and derive the utility function for the representative consumer by aggregating their utility functions, are we likely to obtain an expected utility function, with the discount rate constant, deterministic, and independent of consumption levels and the relative risk aversion constant?

There are already some papers that answer these questions, and the answers are, on the whole, negative. Wilson (1968) and Amershi and Stroeckenius (1983) showed that the representative consumer's utility function need not have the expected-utility form. Gollier and Zeckhauser (2005) and Hara (2008, 2009) showed that the representative consumer's discount rates tend to decrease over time. Calvet, Grandmont, and Lemaire (1999) and Hara, Huang, and Kuzmics (2007) showed that the representative consumer tends to exhibit decreasing relative risk aversion. Some of these papers and others, such as Franke, Stapleton, and Subrahmanyam (1999), Huang (2003), and Jouini and Napp (2007), explored implications of heterogeneous consumers on asset pricing.

This paper adds yet another negative answer to this literature. We take up a continuous-time model in which all individual consumers are assumed to have constant and equal relative risk aversion. Although this assumption is stringent, we allow the individual consumers' probabilistic beliefs and discount rates to be quite arbitrary. In such a model, we prove a formula that relate the representative consumer's probabilistic belief and discount rates to the individual consumers' counterparts. While his probabilistic belief is, in an appropriate sense given via Girsanov's Theorem, a weighted average of their counterparts, his discount rates depend on both the weighted average of their counterparts and the weighted variance of their probabilistic beliefs. The formula, a generalization of Proposition 4 of Jouini and Napp (2007), shows that the belief heterogeneity raises or lowers the discount rates depending on whether the relative risk aversion is greater or smaller than one.

The second, more important, result is obtained in a more special setting. As before, all individual consumers are assumed to have constant and equal relative risk aversion. Moreover, their discount rates are deterministic, constant, and equal, and the biases in their probabilistic beliefs (to be defined via Girsanov's Theorem) are constant and state-independent. Furthermore, these biases are normally distributed across individual consumers. Under these assumptions, we show that the representative consumer's discount rates are a hyperbolic function of time. The novelty of this result lies in the fact that hyperbolic discounting may emerge even when all individual consumers have constant and equal discount rates, as long as their beliefs are heterogeneous. This result should therefore be contrasted with those of Weitzman (2001), Gollier and Zeckhauser (2005), and Hara (2008), who showed that hyperbolic discounting emerge if the individual consumers' (constant) discount rates are distributed according to gamma distributions.

The results of this paper are most relevant to finance, general equilibrium theory, and dynamic macroeconomics. The technique employed here draws much on the techniques developed in mathematical finance. Yet, the message of the paper is relevant to environmental economics, where a cost-benefit analysis is often executed for long-term projects, such as measures to mitigate climate change. The conclusion of any cost-benefit analysis depends inevitably on the discount rate used, and the results of this paper tell us that in an heterogeneous economy under uncertainty, the appropriate discount rate should be determined not only by the average of the individual consumers' discount rates but also by the variance of their probabilistic beliefs. The question of what the appropriate discount rate is under uncertainty has been considered, most notably, by Stern (2007, Section 2 and the Appendix to Chapter 2) and one of its supporting documents, Hepburn (2006, Section 4.2). Neither of them, however, incorporated heterogeneous beliefs. On the other hand, Weitzman (2007), which is a review article on Stern (2007), argued that the uncertainties on the mechanics of climate change and its impact on economic growth are so ambiguous and interrelated that any reduced-form probability distribution of possible growth rates would have a fat left tail. Although he did not provide any formal framework to support his argument, the point we will make towards the end of Section 4, that the heterogeneity in the individual consumers' probabilistic beliefs induces the representative consumer's probabilistic belief to have fat tails, seems to be underlain by the same basic principle as his argument. Also, unlike Stern (2007), Weitzman (2007) argued for the (real) option value of waiting to gather information on the likelihood of environmental disasters. The model of this paper seems suitable for the analysis of the option value, as it is based on stochastic calculus and, as such, can accommodate gradual information revelation.

This paper is organized as follows. Section 2 spells out our model and review some elementary and well known results. Section 3 gives a theorem on how the heterogeneity in the individual consumers' probabilistic beliefs may affect the representative consumer's discount rates. Section 4 gives a theorem on the representative consumer's hyperbolic discounting arising from the heterogeneity of the individual consumers' probabilistic beliefs. Section 5 summarizes these results and suggests directions of future research.

2 Setup

The uncertainty surrounding the economy by a probability space $(\Omega, \mathscr{F}, P)$. The time span, along which the consumption and asset trading take place, is $[0,1]$. The uncertainty and the gradual information revelation are given by a one-dimensional standard Brownian motion $B = (B_t)_{t\in[0,1]}$. That is, each element of the state space Ω is identified with the full specification of realized values of the standard Brownian motion over the entire time span $[0,1]$ and the sub-σ-field $\mathscr{F}_t$ represents the information obtained by observing the standard Brownian motion up to time $t \in [0,1]$.

To allow for the case where there are infinitely many consumers, we let $(I, \mathscr{I}, \nu)$ be a probability measure space representing (the names of) the individual consumers in the economy. They are assumed to have constant and equal relative risk aversion, but their probabilistic beliefs and subjective time discount rates may be quite arbitrary. Specifically, we let $\beta \in \boldsymbol{R}_{++}$ and $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ satisfy $u'(x) = x^{-\beta}$ for every $x \in \boldsymbol{R}_{++}$. Consumer i's discount rate is given by a progressively measurable process $\rho^i = (\rho^i_t)_{t\in[0,1]}$ and his subjective probability measure is given by a probability measure P_i that is absolutely continuous with respect to P. His utility function U_i is defined by

$$U_i(c^i) = E^{P_i}\left(\int_0^1 \exp\left(-\int_0^t \rho^i_s \,\mathrm{d}s\right) u(c^i_t)\,\mathrm{d}t\right). \tag{1}$$

As can be seen from this expression, the standard case of exponential discounting corresponds to the case where ρ^i takes a constant value across time and states. We assume that there exists an adapted process $\gamma^i = (\gamma^i_t)_{t\in[0,1]}$ such that

$$E_t\left(\frac{\mathrm{d}P_i}{\mathrm{d}P}\right) = \exp\left(-\int_0^t \frac{\left(\gamma^i_s\right)^2}{2}\,\mathrm{d}s - \int_0^t \gamma^i_s\,\mathrm{d}B_s\right). \tag{2}$$

Equivalently, if we let $\xi^i_t = E_t\left(\mathrm{d}P_i/\mathrm{d}P\right)$, then $\mathrm{d}\xi^i_t = -\xi^i_t\gamma^i_t\,\mathrm{d}B_t$. By Girsanov's theorem, the process $B^i = \left(B^i_t\right)_{t\in[0,1]}$ defined by $B^i_t = B_t + \int_0^t \gamma^i_s\,\mathrm{d}s$ (that is, $\mathrm{d}B^i_t = \mathrm{d}B_t + \gamma^i_t$) is a standard Brownian motion under P_i. Then the utility function (1) can be written as

$$U_i(c^i) = E\left(\int_0^1 \exp\left(-\int_0^t \left(\rho^i_s + \frac{\left(\gamma^i_s\right)^2}{2}\right)\mathrm{d}s - \int_0^t \gamma^i_s\,\mathrm{d}B_s\right) u(c^i_t)\,\mathrm{d}t\right). \tag{3}$$

Define an Ito process $\varphi^i = (\varphi^i_t)_{t\in[0,1]}$ by

$$\varphi^i_t = \int_0^t \left(\rho^i_s + \frac{\left(\gamma^i_s\right)^2}{2}\right)\mathrm{d}s + \int_0^t \gamma^i_s\,\mathrm{d}B_s, \tag{4}$$

then (3) can be more succinctly written as

$$U_i(c^i) = E\left(\int_0^1 \exp\left(-\varphi^i_t\right) u(c^i_t)\,\mathrm{d}t\right). \tag{5}$$

To find a Pareto efficient allocation of a given aggregate consumption process $c = (c_t)_{t\in[0,1]}$ and its supporting (decentralizing) state-price deflator, it is sufficient to choose positive numbers $\lambda : I \to \boldsymbol{R}_{++}$ and consider the following maximization problem:

$$\begin{aligned} \max_{(c_i)_{i\in I}} \quad & \int_I \lambda(i) U_i(c^i)\,\mathrm{d}\nu(i) \\ \text{subject to} \quad & \int_I c_i\,\mathrm{d}\nu(i) = c. \end{aligned} \tag{6}$$

By (5) and Fubini's Theorem, this problem can be rewritten as

$$\begin{aligned} \int_I \lambda(i) U_i(c_i)\,\mathrm{d}\nu(i) &= E\left(\int_0^1 \int_I \lambda(i)\exp(-\varphi_t^i) u(c_t^i)\,\mathrm{d}\nu(i)\,\mathrm{d}t\right) \\ &= \int_{\Omega\times[0,1]} \left(\int_I \lambda(i)\exp(-\varphi_t^i(\omega)) u(c_t^i(\omega))\,\mathrm{d}\nu(i)\right)\,\mathrm{d}(P\otimes\Lambda)(\omega,t), \end{aligned}$$

where Λ is the Lebesgue measure on $[0,1]$. Hence, to solve the original maximization problem (6), it suffices to solve the simplified maximization problem

$$\begin{aligned} \max_{(x^i)_{i\in I}} \quad & \int_I \lambda(i)\exp(-z^i) u(x^i)\,\mathrm{d}\nu(i) \\ \text{subject to} \quad & \int_I x^i\,\mathrm{d}\nu(i) = x. \end{aligned} \tag{7}$$

for each pair of realization $x \in \boldsymbol{R}_{++}$ of c and a profile $z = (z^i)_{i\in I} \in \boldsymbol{R}^I$ of realizations $z^i \in \boldsymbol{R}$ of φ_t^i. It can be easily proved that for each (x, z), there is a unique solution,[1] which we denote by $(f_i(x, z))_{i\in I}$. It follows from the first-order condition to the solution to (7) that

$$f_i(x,z) = \frac{\big(\lambda(i)\exp(-z^i)\big)^{1/\beta}}{\displaystyle\int_I \big(\lambda(j)\exp(-z^j)\big)^{1/\beta}\,\mathrm{d}\nu(j)} \tag{8}$$

Thus the value function of this problem, $v : \boldsymbol{R}_{++} \times \boldsymbol{R}^I \to \boldsymbol{R}$, satisfies[2]

$$v(x,z) = \int_i \lambda(i) u\left(f_i(x,z)\right)\,\mathrm{d}\nu(i) = \left(\int_I \big(\lambda(i)\exp(-z^i)\big)^{1/\beta}\,\mathrm{d}\nu(j)\right)^{\beta} u(x).$$

The solution to the original maximization problem is given by $(c^i)_{i\in I}$, where, for each $i \in I$ and $t \in [0,1]$, $c_t^i = f_i\left(c, \left(\varphi_t^i\right)_{i\in I}\right)$. The representative consumer's utility function is defined by

$$U(c) = E\left(\int_0^1 v(c_t, (\varphi_t^i)_{i\in I})\,\mathrm{d}t\right) = E\left(\int_0^1 \left(\int_I \big(\lambda(i)\exp(-\varphi_t^i)\big)^{1/\beta}\,\mathrm{d}\nu(i)\right)^{\beta} u(c_t)\,\mathrm{d}t\right). \tag{9}$$

[1]In fact, to guarantee the existence of a solution, we need to impose some measurability and integrability conditions on the function $i \mapsto z^i$, though we shall not explicitly state them here as they do not affect the formulas obtained at the end of our analysis.

[2]To be exact, the value function may be different from the right-hand side by a constant term. But this difference is irrelevant to the subsequent analysis.

In the next section, we show that U can be written in the same form as (1).

By multiplying a positive constant if necessary, we can assume that $\int_I (\lambda(i))^{1/\beta} \, \mathrm{d}\nu(i) = 1$. If all the individual consumers have a common belief and a common discount rate, then the φ^i_t are equal at all times and we can concentrate on the case where the z^i are equal. According to (8), then, $f_i(x,z)/x = (\lambda(i))^{1/\beta}/(\int \lambda^{1/\beta} \, \mathrm{d}\nu)$. Thus, if $\int (\lambda(i))^{1/\beta} \, \mathrm{d}\nu(i) = 1$, then $(\lambda(i))^{1/\beta}$ is the ratio of consumer i's consumption to the average consumption, and hence, his wealth to the average wealth in the economy. Although it is no longer equal to the wealth share when the φ^i are not equal, Lemma 4.1 of Jouini and Napp (2007) shows that it does indeed approximate the wealth share.

The representative consumer is, of course, not an "actual" consumer, who would trade on financial markets. Rather, he is a theoretical construct, whom we can use to identify asset prices.[3] Specifically, if v is the value function to the maximization problem and c is the aggregate consumption process, then his marginal utility process evaluated at the aggregate consumption, $\partial v \left(c, \left(\varphi^i_t \right)_{i \in I} \right) / \partial x$, is a state price density. This means that the price at time $t \in [0,1]$ of an asset with dividend (rate) process $d = (d_\tau)_{\tau \in [0,1]}$ is equal to

$$E_t \left(\frac{1}{\pi_t} \int_t^1 \pi_\tau d_\tau \, \mathrm{d}\tau \right),$$

where $\pi_\tau = \partial v \left(c, \left(\varphi^i_t \right)_{i \in I} \right) / \partial x$.

Although we analyze the Pareto efficient allocations and their supporting (decentralizing) prices, if the asset markets are complete, then our analysis is applicable to the equilibrium allocations and asset prices. This is because the first welfare theorem holds in complete markets, so that the equilibrium allocations are Pareto efficient and the equilibrium asset prices are given by the corresponding support prices. Since u is concave, the second welfare theorem also holds, so that every Pareto efficient allocation is an equilibrium allocation for some distribution of initial endowments. Hence an analysis of Pareto efficient allocations is also an analysis of equilibrium allocations.

3 Belief and discount rates

In this section, we identify the representative consumer's discount rates and the probabilistic belief. They are embedded in the factor

$$\left(\int_I \left(\lambda(i) \exp(-\varphi^i_t) \right)^{1/\beta} \, \mathrm{d}\nu(i) \right)^\beta \tag{10}$$

of (9), our task is to rewrite it in a more understandable manner. The following theorem, though somewhat lengthy, is the main result of this section. It generalizes Proposition 4 of Jouini and Napp (2007) to the case with stochastic and heterogeneous subjective discount rates.

[3] For this reason, the dynamic inconsistency of the representative consumer (arising from, say, hyperbolic discounting) does not imply that individual consumers' choices are dynamically inconsistent.

Theorem 1 *There are two progressively measurable processes $\rho = (\rho_t)_{t\in[0,1]}$ and $\gamma = (\gamma_t)_{t\in[0,1]}$ such that if an equivalent probability measure P_0 is defined by*

$$E_t\left(\frac{\mathrm{d}P_0}{\mathrm{d}P}\right) = \exp\left(-\int_0^t \frac{(\gamma_s)^2}{2}\,\mathrm{d}s - \int_0^t \gamma_s\,\mathrm{d}B_s\right), \tag{11}$$

then

$$U(c) = E^{P_0}\left(\int_0^1 \exp\left(-\int_0^t \rho_s\,\mathrm{d}s\right) u(c_t)\,\mathrm{d}t\right).$$

Moreover, if we define an Ito process $\varphi = (\varphi_t)_{t\in[0,1]}$ by

$$\varphi_t = \int_0^t \left(\rho_s + \frac{(\gamma_s)^2}{2}\right)\mathrm{d}s + \int_0^t \gamma_s\,\mathrm{d}B_s, \tag{12}$$

and, for each i, an Ito process $\theta^i = \left(\theta^i_t\right)_{t\in[0,1]}$ by

$$\theta^i_t = \frac{1}{\beta}\left(\kappa(i) - \left(\varphi^i_t - \varphi_t\right)\right),$$

where $\kappa(i) = \ln\lambda(i)$, then

$$1 = \int_I \exp\theta^i_t\,\mathrm{d}\nu(i), \tag{13}$$

$$\gamma_t = \int_I \gamma^i_t \exp\theta^t_i\,\mathrm{d}\nu(i), \tag{14}$$

$$\rho_t = \int_I \rho^i_t \exp\theta^t_i\,\mathrm{d}\nu(i) + \frac{1}{2}\left(1-\frac{1}{\beta}\right)\int_I \left(\gamma^i_t - \gamma_t\right)^2 \exp\theta^i_t\,\mathrm{d}\nu(i). \tag{15}$$

This theorem says that the factor (10) is can be written as $\exp(-\varphi_t)$ with φ_t defined by (12). Like Proposition 4 of Jouini and Napp (2007), it claims that even if the individual consumers' subjective discount rates are homogeneous, the representative consumer's discount rates may well be stochastic and time-varying. To see this point, note first that because of (13), for each t, the $\exp\theta^i_t$ can be considered as weights across consumers.[4] Then (14) says that the diffusion term γ_t defining the representative consumer's probabilistic belief is simply the weighted average of the individual consumers' counterparts. According to (15), however, the representative consumer's discount rate ρ_t is not the weighted average of the individual consumers' counterparts. Rather, it is equal to the sum of the weighted average and the weighted variance of the heterogeneous beliefs γ^i_t multiplied by $(1/2)(1-1/\beta)$. Since this is positive if $\beta > 1$ and negative if $\beta < 1$, the representative consumer's discount rate is higher than the weighted average of the individual consumers' counterparts if their coefficient of constant relative risk aversion is greater than one, while the former is lower than the latter if it is less than one. It can be stochastic and time-varying even when the ρ^i are constant, deterministic, and equal to one another, as long as the γ^i are different, because, then, the weights $\exp\theta^i_t$ are stochastic and time-varying. The

[4]Indeed, it can be shown that $c^i_t/c_t = \exp\theta^i_t$. This means that the $\exp\theta^i_t$ are consumption weights.

next section will deal with such a situation.

Since the representative consumer has constant relative risk aversion equal to β, this theorem implies that a state-price deflator $\pi = (\pi_t)_{t\in[0,1]}$ is given by

$$\pi_t = \exp(-\varphi_t)c_t^{-\beta}.$$

That is, the price at time t of an asset with dividend (rate) process is $d = (d_t)_{t\in[0,1]}$ is equal to

$$E_t\left(\int_t^1 \exp\left(-(\varphi_\tau - \varphi_t)\right)\left(\frac{c_\tau}{c_t}\right)^\beta d_\tau \, \mathrm{d}\tau\right).$$

Here, of course, the expectation is taken with respect to the objective probability measure P. If we were to use the representative consumer's probabilistic belief P_0, then a state price deflator would be given by

$$\exp(-\rho_t)c_t^{-\beta}.$$

and the price at time t of the asset can be expressed as

$$E_t^{P_0}\left(\int_t^1 \exp\left(-(\rho_\tau - \rho_t)\right)\left(\frac{c_\tau}{c_t}\right)^\beta d_\tau \, \mathrm{d}\tau\right).$$

Proof of Theorem 1 Then define a process $\varphi = (\varphi_t)_{t\in[0,1]}$ by

$$\varphi_t = -\beta \ln\left(\int_I (\lambda(i))^{1/\beta} \exp\left(-\frac{\varphi_t^i}{\beta}\right) \mathrm{d}\nu(i)\right).$$

Then $\varphi_0 = 0$ and

$$\exp(-\varphi_t) = \left(\int_I (\lambda(i))^{1/\beta} \exp\left(-\frac{\varphi_t^i}{\beta}\right) \mathrm{d}\nu(i)\right)^\beta. \tag{16}$$

Hence

$$U(c) = E\left(\int_0^1 \exp\left(-\varphi_t\right) \frac{c_t^{1-\beta}}{1-\beta} \mathrm{d}t\right).$$

By Ito's lemma, φ is an Ito process, with its drift and diffusion terms to be found as follows. First, define the processes $\Phi^i = (\Phi_t^i)_{t\in[0,1]}$ and $\Phi = (\Phi_t)_{t\in[0,1]}$ by

$$\Phi_t^i = \exp\left(-\frac{\varphi_t^i}{\beta}\right), \tag{17}$$

$$\Phi_t = \exp\left(-\frac{\varphi_t}{\beta}\right). \tag{18}$$

Then, by (16),

$$\Phi_t = \int_I (\lambda(i))^{1/\beta} \Phi_t^i \, \mathrm{d}\nu(i). \tag{19}$$

Write $\mu_t^i = \rho_t^i + (\gamma_t^i)^2/2$, then, by (17) and Ito's lemma,

$$-\frac{\mathrm{d}\Phi_t^i}{\Phi_t^i} = \left(\frac{\mu_t^i}{\beta} - \frac{1}{2}\left(\frac{\gamma_t^i}{\beta}\right)^2\right)\mathrm{d}t + \frac{\gamma_t^i}{\beta}\,\mathrm{d}B_t.$$

Thus, by (19), $\mathrm{d}\Phi_t = \mu_t^\Phi\,\mathrm{d}t + \sigma_t^\Phi\,\mathrm{d}B_t$, where

$$\begin{aligned}
\mu_t^\Phi &= -\frac{1}{\beta}\int_I \lambda(i)^{1/\beta}\Phi_t^i\left(\mu_t^i - \frac{\left(\gamma_t^i\right)^2}{2\beta}\right)\mathrm{d}\nu(i),\\
\sigma_t^\Phi &= -\frac{1}{\beta}\int_I \lambda(i)^{1/\beta}\Phi_t^i\gamma_t^i\,\mathrm{d}\nu(i).
\end{aligned}$$

By (18), $\varphi_t = -\beta\ln\Phi_t$ and hence, by Ito's lemma, $\mathrm{d}\varphi_t = \mu_t^\varphi\,\mathrm{d}t + \gamma_t\,\mathrm{d}B_t$, where

$$\begin{aligned}
\gamma_t &= \int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\gamma_t^i\,\mathrm{d}\nu(i),\\
\mu_t^\varphi &= \left(-\frac{\beta}{\Phi_t}\right)\left(\mu_t^\Phi - \frac{\left(\sigma_t^\Phi\right)^2}{2\Phi_t}\right)\\
&= \int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\left(\mu_t^i - \frac{\left(\gamma_t^i\right)^2}{2\beta}\right)\mathrm{d}\nu(i) + \frac{1}{2\beta}\left(\int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\gamma_t^i\,\mathrm{d}\nu(i)\right)^2\\
&= \int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\left(\rho_t^i + \left(1-\frac{1}{\beta}\right)\frac{\left(\gamma_t^i\right)^2}{2}\right)\mathrm{d}\nu(i) + \frac{1}{2\beta}\left(\gamma_t\right)^2.
\end{aligned}$$

Define a progressively measurable process $\rho = (\rho_t)_{t\in[0,1]}$ by $\rho_t = \mu_t^\varphi - (\gamma_t)^2/2$, then

$$\varphi_t = \int_0^t \left(\rho_s + \frac{(\gamma_s)^2}{2}\right)\mathrm{d}s + \int_0^t \gamma_s\,\mathrm{d}B_s$$

and

$$\begin{aligned}
\rho_t &= \int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\left(\rho_t^i + \left(1-\frac{1}{\beta}\right)\frac{\left(\gamma_t^i\right)^2}{2}\right)\mathrm{d}\nu(i) + \left(\frac{1}{2\beta}-\frac{1}{2}\right)(\gamma_t)^2\\
&= \int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\rho_t^i\,\mathrm{d}\nu(i) + \frac{1}{2}\left(1-\frac{1}{\beta}\right)\left(\int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\left(\gamma_t^i\right)^2\mathrm{d}\nu(i) - (\gamma_t)^2\right)\\
&= \int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\rho_t^i\,\mathrm{d}\nu(i) + \frac{1}{2}\left(1-\frac{1}{\beta}\right)\int_I \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}\left(\gamma_t^i - \gamma_t\right)^2\mathrm{d}\nu(i).
\end{aligned}$$

For each i, define a process $\theta^i = \left(\theta_t^i\right)_{t\in[0,1]}$ by

$$\theta_t^i = \frac{1}{\beta}\left(\kappa(i) - \left(\varphi_t^i - \varphi_t\right)\right),$$

then

$$\exp\theta_i^t = \frac{\lambda(i)^{1/\beta}\Phi_t^i}{\Phi_t}$$

and hence (13), (14), and (15) are obtained. ///

4 Hyperbolic discounting

In the following, we impose additional restrictions on the ρ^i and γ^i to show that if the subjective beliefs are heterogeneous, then the representative consumer's discount rates may change over time even when the individual consumers have a common, constant discount rate. Specifically, we assume that all ρ^i are deterministic, constant, and equal to one another, which we denote by $\bar{\rho}$, and that each γ^i is deterministic and constant, which we denote by $\bar{\gamma}(i)$. We shall prove that if the $\bar{\gamma}(i)$ are distributed in a way to be articulated below, then the representative consumer has hyperbolic discounting.

The following facts should be helpful to understand the theorem. First, if the process γ^i is always equal to a constant $\bar{\gamma}(i)$, then, for every $t \in [0,1]$, $B_\tau - B_t$ follows a normal distribution with mean $-\bar{\gamma}(i)(\tau - t)$ and variance $\tau - t$ with respect to the subjective probabilistic belief P_0 under the common information $\mathscr{F}_t$ whenever $\tau > t$. Thus, the higher the $\bar{\gamma}(i)$, the more pessimistic the consumer is. Second, his subjective probabilistic belief P_i and the objective probability P differ only in terms of the mean of the Brownian motion B, and they agree on its variance. Third, regarding $\bar{\gamma}$ as a function from I to $\boldsymbol{R}$, we can define a measure ν^* on $\boldsymbol{R}$ by letting

$$\nu^*(B) = \int_{\bar{\gamma}^{-1}(B)} (\lambda(i))^{1/\beta} \, \mathrm{d}\nu(i) \tag{20}$$

for every $B \in \mathscr{B}(\boldsymbol{R})$. Since $\int_I (\lambda(i))^{1/\beta} \, \mathrm{d}\nu(i) = 1$, ν^* is, in fact, a probability measure. Since, as stated in Section 2, $(\lambda(i))^{1/\beta}$ approximates the proportion of consumer i's wealth relative to the average wealth of the economy, the probability measure ν^* approximates the distribution of the $\bar{\gamma}(i)$ on $\boldsymbol{R}$ in terms of wealth shares in the economy.

The following proposition is the second main result of this paper.

Theorem 2 *Suppose that for every i, the discount rate process ρ^i is always equal to a common constant $\bar{\rho}$ and the progressively measurable process γ^i is always equal to a constant $\bar{\gamma}(i)$. Suppose that the probability measure ν^* on $\boldsymbol{R}$ defined by (20) is a normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$. Define a progressively measurable process $\gamma = (\gamma_t)_{t \in [0,1]}$ by*

$$\gamma_t = -\frac{B_t - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}. \tag{21}$$

and an equivalent probability measure P_0 on Ω by (11). Then

$$U(c) = E^{P_0}\left(\int_0^1 \exp\left(-\bar{\rho}t - \frac{\beta - 1}{2} \frac{1}{t + \beta/\hat{\sigma}^2} \right) u(c_t) \, \mathrm{d}t \right). \tag{22}$$

According to (22), the representative consumer's discount rate is equal to

$$\bar{\rho}t + \frac{\beta - 1}{2} \frac{1}{t + \beta/\hat{\sigma}^2}.$$

The first term, $\bar{\rho}$ is simply the common discount rate of the individual consumers. The second term is a hyperbolic function of t and arises from the heterogeneity in the individual consumers' beliefs. What makes this fact interesting is that in an economy under uncertainty, hyperbolic discounting may emerge even when all individual consumers' subjective discount rates are equal. This result is consistent with Proposition 4 of Jouini and Napp (2007): The hyperbolic part is a decreasing function of time (a special case of a supermartingale) if $\beta > 1$ and an increasing function (a special case of a submartingale) if $\beta < 1$. We should also note that the the hyperbolic factor of Rohde (2008), which is a measure of time consistency, of the second factor $(t+\beta/\hat{\sigma}^2)^{-1}$ of the second term is equal to $\hat{\sigma}^2/\beta$. Since this is an increasing function of the variance $\hat{\sigma}$ of the biases of the individual consumers' beliefs, the theorem implies that the more dispersed the beliefs are, the more hyperbolic the representative consumer's discount rates are.

Proof of Theorem 2 Under the assumptions of this theorem,

$$\exp\left(-\int_0^t \left(\rho_s^i + \frac{\left(\gamma_s^i\right)^2}{2}\right) \mathrm{d}s - \int_0^t \gamma_s^i \, \mathrm{d}B_s\right) = \exp\left(-\bar{\rho}t\right) \exp\left(-\bar{\gamma}(i)B_t - \frac{\left(\bar{\gamma}(i)\right)^2}{2} t\right)$$

for every i. Define $p_i : \boldsymbol{R} \times [0,1] \to \boldsymbol{R}_{++}$ by

$$p_i(z,t) = \exp\left(-\bar{\gamma}_i z - \frac{\left(\bar{\gamma}(i)\right)^2}{2} t\right)$$

Then, by (3),

$$U_i(c^i) = E\left(\int_0^1 \exp\left(-\bar{\rho}t\right) p_i(B_t,t) u(c_t^i) \, \mathrm{d}t\right). \tag{23}$$

Since the discount factor $\exp\left(-\bar{\rho}t\right)$ is common across consumers, we can write the simplified maximization problem (7) as

$$\begin{array}{ll} \max\limits_{(x^i)_{i \in I}} & \displaystyle\int_I \lambda(i) p_i(B_t,t) u(x^i) \, \mathrm{d}\nu(i) \\ \text{subject to} & \displaystyle\int_I x^i \, \mathrm{d}\nu(i) = x. \end{array} \tag{24}$$

It follows from the first-order condition of this maximization problem that if we define $p : \boldsymbol{R} \times [0,1] \to \boldsymbol{R}_{++}$ by

$$p(z,t) = \left(\int_I \left(\lambda(i) p_i(z,t)\right)^{1/\beta} \, \mathrm{d}\nu(i)\right)^{\beta},$$

then

$$U(c) = E\left(\int_0^1 \exp(-\bar{\rho}t) p(B_t,t) u(c_t) \, \mathrm{d}t\right).$$

Write $h(z,t) = \int_I \left(\lambda(i) p_i(z,t)\right)^{1/\beta} \, \mathrm{d}\nu(i)$. Then

$$h(z,t) = \int_I (\lambda(i))^{1/\beta} \exp\left(-\frac{1}{\beta}\left(\bar{\gamma}(i)z + \frac{(\bar{\gamma}(i))^2}{2} t\right)\right) \, \mathrm{d}\nu(i) = \int_{\boldsymbol{R}} \exp\left(-\frac{1}{\beta}\left(qz + \frac{q^2}{2} t\right)\right) \, \mathrm{d}\nu^*(q)$$

by the change-of-variable formula. Since ν^* is a normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$,

$$h(z,t) = \frac{1}{(2\pi)^{1/2}\hat{\sigma}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{\beta}\left(qz + \frac{q^2}{2}t\right)\right) \exp\left(-\frac{(q-\hat{\mu})^2}{2\hat{\sigma}^2}\right) \mathrm{d}q.$$

Since the integrand is equal to

$$\exp\left(-\frac{1}{2\beta}\left(t + \frac{\beta}{\hat{\sigma}^2}\right)\left(q + \frac{z - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}\right)^2\right) \exp\left(\frac{1}{2\beta}\frac{(z - \beta\hat{\mu}/\hat{\sigma}^2)^2}{t + \beta/\hat{\sigma}^2}\right) \exp\left(-\frac{\hat{\mu}^2}{2\hat{\sigma}^2}\right),$$

the definition of density functions for normal distributions implies that

$$h(z,t) = \exp\left(-\frac{\hat{\mu}^2}{2\hat{\sigma}^2}\right)\left(\frac{\beta}{\hat{\sigma}^2 t + \beta}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2\beta}\frac{(z - \beta\hat{\mu}/\hat{\sigma}^2)^2}{t + \beta/\hat{\sigma}^2}\right)$$

and hence

$$p(z,t) = \exp\left(-\frac{\beta\hat{\mu}^2}{2\hat{\sigma}^2}\right)\left(1 + \frac{\hat{\sigma}^2}{\beta}t\right)^{-\frac{\beta}{2}} \exp\left(\frac{1}{2}\frac{(z - \beta\hat{\mu}/\hat{\sigma}^2)^2}{t + \beta/\hat{\sigma}^2}\right).$$

By the definition of φ, $\exp(-\varphi_t) = \exp(-\bar{\rho}t)\, p(B_t, t)$. Hence $\varphi_t = \bar{\rho}t - \ln p(B_t, t)$. Thus, if we define $g : \boldsymbol{R} \times [0,1] \to \boldsymbol{R}$ by

$$g(z,t) = \bar{\rho}t - \ln p(z,t) = \bar{\rho}t + \frac{\beta\hat{\mu}^2}{2\hat{\sigma}^2} + \frac{\beta}{2}\ln\left(1 + \frac{\hat{\sigma}^2}{\beta}t\right) - \frac{1}{2}\frac{(z - \beta\hat{\mu}/\hat{\sigma}^2)^2}{t + \beta/\hat{\sigma}^2},$$

then $\varphi_t = g(B_t, t)$. Moreover,

$$\begin{aligned}
\frac{\partial g}{\partial t}(z,t) &= \bar{\rho} + \frac{1}{2}\frac{\beta}{t + \beta/\hat{\sigma}^2} + \frac{1}{2}\left(\frac{z - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}\right)^2, \\
\frac{\partial g}{\partial z}(z,t) &= -\frac{z - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}, \\
\frac{\partial^2 g}{\partial z^2}(z,t) &= -\frac{1}{t + \beta/\hat{\sigma}^2}.
\end{aligned}$$

By Ito's Lemma, the diffusion term of φ, γ_t, is equal to

$$-\frac{B_t - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}.$$

The drift term of φ, $\rho_t + (\gamma_t)^2/2$, is equal to

$$\bar{\rho} + \frac{1}{2}\frac{\beta}{t + \beta/\hat{\sigma}^2} + \frac{1}{2}\left(\frac{B_t - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}\right)^2 - \frac{1}{2}\frac{1}{t + \beta/\hat{\sigma}^2} = \bar{\rho} + \frac{\beta - 1}{2}\frac{1}{t + \beta/\hat{\sigma}^2} + \frac{1}{2}\left(\frac{B_t - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}\right)^2$$

Thus,

$$\rho_t = \bar{\rho} + \frac{\beta - 1}{2}\frac{1}{t + \beta/\hat{\sigma}^2} + \frac{1}{2}\left(\frac{B_t - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}\right)^2 - \frac{1}{2}\left(-\frac{B_t - \beta\hat{\mu}/\hat{\sigma}^2}{t + \beta/\hat{\sigma}^2}\right)^2 = \bar{\rho} + \frac{\beta - 1}{2}\frac{1}{t + \beta/\hat{\sigma}^2}.$$

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The theorem states that the representative consumer's belief P_0 can be represented by the process γ defined by (21), but it does not intuitively give the idea on what his belief is like. In the following, we show that although B_t is normally distributed with mean 0 and variance t with respect to the objective measure P, it is normally distributed with mean $-\hat{\mu}t$ and variance $t\left(1+\left(\hat{\sigma}^2/\beta\right)t\right)$ with respect to P_0. Since $t\left(1+\left(\hat{\sigma}^2/\beta\right)t\right) > t$, we can say that an important consequence of the individual consumers' heterogeneous beliefs is that the representative consumer's belief is more dispersed than the objective probability, and also more dispersed than any individual consumer's belief.

To prove that B_t is normally distributed with mean $-\hat{\mu}$ and variance $t\left(1+\left(\hat{\sigma}^2/\beta\right)t\right)$ with respect to P_0, note that since $\exp(-\varphi_t) = \exp(-\rho_t)p(B_t,t)$, $p(B_t,t)$ consists of the hyperbolic part of the representative consumer's discount factor and his state-price density. Since the hyperbolic part of the discount rate is equal to

$$\frac{\beta-1}{2}\frac{1}{t+\beta/\hat{\sigma}^2},$$

the corresponding discount factor is equal to

$$\left(1+\frac{\hat{\sigma}^2}{\beta}t\right)^{\frac{1-\beta}{2}}.$$

Thus, we can infer that the density process $\left(E_t\left(\mathrm{d}P_0/\mathrm{d}P\right)\right)_{t\in[0,1]}$ is equal to

$$\frac{p(B_t,t)}{\left(1+\frac{\hat{\sigma}^2}{\beta}t\right)^{\frac{1-\beta}{2}}} = \exp\left(-\frac{\beta\hat{\mu}^2}{2\hat{\sigma}^2}\right)\left(1+\frac{\hat{\sigma}^2}{\beta}t\right)^{-\frac{1}{2}}\exp\left(\frac{1}{2}\frac{(B_t-\beta\hat{\mu}/\hat{\sigma}^2)^2}{t+\beta/\hat{\sigma}^2}\right). \tag{25}$$

Writing this process by $\xi = (\xi_t)_{t\in[0,1]}$ and applying Ito's Lemma, we can show that the drift term of ξ_t is equal to zero and the diffusion term of ξ_t is equal to $-\gamma_t\xi_t$. Thus, the density process $\left(E_t\left(\mathrm{d}P_0/\mathrm{d}P\right)\right)_{t\in[0,1]}$ is, in fact, equal to ξ defined by (25).

Note that ξ_t depends only on B_t and t. Hence, the density function (25), which is the Radon-Nikodym derivative of P_0 with respect to P, is also the Radon-Nikodym derivative of the (marginal) distribution of B_t induced from P_0 with respect to the (marginal) distribution of B_t induced from P. Formally,

$$\frac{\mathrm{d}(P_0\circ(B_t)^{-1})}{\mathrm{d}(P\circ(B_t)^{-1})}(z) = \exp\left(-\frac{\beta\hat{\mu}^2}{2\hat{\sigma}^2}\right)\left(1+\frac{\hat{\sigma}^2}{\beta}t\right)^{-\frac{1}{2}}\exp\left(\frac{1}{2}\frac{(z-\beta\hat{\mu}/\hat{\sigma}^2)^2}{t+\beta/\hat{\sigma}^2}\right).$$

Let Λ be the Lebesgue measure on $\boldsymbol{R}$, then, of course,

$$\frac{\mathrm{d}(P\circ(B_t)^{-1})}{\mathrm{d}\Lambda}(z) = \frac{1}{(2\pi t)^{\frac{1}{2}}}\exp\left(-\frac{z^2}{2t}\right).$$

Therefore,

$$\frac{\mathrm{d}(P_0 \circ (B_t)^{-1})}{\mathrm{d}\Lambda}(z) = \frac{\mathrm{d}(P_0 \circ (B_t)^{-1})}{\mathrm{d}(P \circ (B_t)^{-1})}(z)\frac{\mathrm{d}(P \circ (B_t)^{-1})}{\mathrm{d}\Lambda}(z)$$
$$= \frac{1}{\left(2\pi t\left(1+\frac{\hat{\sigma}^2}{\beta}t\right)\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{(z+\hat{\mu}t)^2}{t\left(1+\frac{\hat{\sigma}^2}{\beta}t\right)}\right).$$

This means that B_t is normally distributed with mean $-\hat{\mu}$ and variance $t\left(1+\left(\hat{\sigma}^2/\beta\right)t\right)$ with respect to P_0.

5 Conclusion

We have shown how the heterogeneity in probabilistic beliefs affect the representative consumer's discount rates. When the biases of the individual consumers' probabilistic beliefs are normally distributed, in a sense that was made precise via Girsanov's Theorem, the representative consumer has hyperbolic discounting, even when the individual consumers share the same discount rate.

There are many interesting directions of future research. Among them, the most important one is perhaps to attempt to dispense with the assumption that all individual consumers have constant and equal relative risk aversion. As shown by Wilson (1968) and Amershi and Stroeckenius (1983), then, the representative consumer's utility function need not have the expected-utility form and his belief need not be well defined, implying that we cannot meaningfully talk about any impact of heterogeneous beliefs on the representative consumer's belief. However, what ultimately matters to asset pricing is not the representative consumer's utility function but the state-price deflator derived from his marginal utilities, which can be decomposed into the short-rate process and the market-price-of-risk process. This fact suggests that we look for an expected utility function for the representative consumer that leads to the same short-rate process and the market-price-of-risk process as the "true" non-expected utility function for the representative consumer. This is an interesting direction of future research.

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