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"Effectively Complete Asset Markets
with Multiple Goods and over Multiple Periods"

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Effectively Complete Asset Markets with Multiple Goods and over Multiple Periods

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Abstract

Following LeRoy and Werner (2001), we propose a definition of effectively complete asset markets in a model with multiple goods and multiple periods, and establish the first and second welfare theorems in such markets. As applications of the first welfare theorem, we derive the sunspot irrelevance theorem of Mas-Colell (1992), and extend the no-retrade theorem of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003) to the case where the asset prices need not be time-invariant Markov processes.

JEL Classification Codes: D51, D52, D53, D61, D91, G11, G12.

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1 Introduction

Asset markets are said to be *complete* if any pattern of transfers of purchasing power across time and states can be attained by trading assets. In the case of two consumption periods, with no uncertainty on the first period and S possible states of the world on the second, asset markets are complete if and only if there are S non-redundant assets. The consequence of market completeness is that the equilibrium allocations are Pareto-efficient.

In complete asset markets, consumers are guaranteed to be able to attain their optimal patterns of transfers of purchasing power across time and states, regardless of their utility function and initial endowments. If we impose some restrictions utility functions and initial endowments, then we may narrow down a class of candidates for equilibrium asset prices and hence that for optimal patterns of transfers. It might even be the case that for some appropriately chosen collection of fewer than S assets, all consumers can attain the patterns

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of transfers that we would find optimal should the markets be complete. The equilibrium allocations would then be Pareto-efficient, as in the case of complete markets.

LeRoy and Werner (2001, Section 16.3) made this observation precise by giving a definition of *effectively complete* asset markets in a model of a single consumption good and two consumption periods. They defined asset markets as being effectively complete if every Pareto-efficient allocation can be attained through some trades of assets, and proved (Theorem 16.4.1) that the equilibrium allocations are Pareto-efficient in effectively complete asset markets. They then provided three examples, including one of the so-called two-fund separation, for which asset markets are effectively complete, and the equilibrium allocations are Pareto-efficient and can be easily characterized. These examples shows that the notion of effectively complete asset market, restrictive as it may seem, deserves special attention thanks to its applicability to many important economic issues.

In this paper, we extend LeRoy and Werner's definition of effectively complete asset markets to the case of multiple goods and over multiple periods. Although the extension is straightforward and the class of economies with effectively complete asset markets is small, it admits a couple of important applications. We then prove that, as in the case of the original definition of LeRoy and Werner (2001), if asset markets are effectively complete, then every equilibrium allocation is Pareto-efficient. This is the first welfare theorem in effectively complete asset markets. We also establish the second welfare theorem in effectively complete asset markets.

The first application of the first welfare theorem in effectively complete asset markets is the sunspot irrelevance theorem in sunspot economies by Mas-Colell (1992). The second application is the no-retrade theorem in Markov economies of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003). In fact, we extend their theorem to the case where the asset prices need not be time-invariant Markov processes. Since the first welfare theorem in effectively complete asset markets is the driving force behind these results, and since it owes much to Mas-Colell (1992) and LeRoy and Werner (2001), the contribution of this paper lies in showing that the technique by Mas-Colell (1992) and LeRoy and Werner (2001) can used to extend the no-retrade theorem to the case where the asset prices need not be time-invariant Markov processes.

This paper is organized as follows. Section 2 describes the setup for our analysis. Section 3 gives the definition of effectively complete asset markets and establishes the first and second welfare theorems. Section 4 provides the first application of effectively complete asset markets and proves the sunspot irrelevance theorem. Section 5 provides the second application of effectively complete asset markets and proves the no-retrade theorem. Section 6 sums up our analysis and suggests a direction of future research.

2 Setup

There are $1+T$ periods, $t = 0, 1, \dots, T$. There are S possible states of the world, $s = 1, 2, \dots, S$ over the entire time span $\{0, 1, \dots, T\}$. The gradual information revelation concerning the true state of the world is given by the filtration $(\mathscr{F}_0, \mathscr{F}_1, \dots, \mathscr{F}_T)$. We assume that $\mathscr{F}_0 = \{\varnothing, \{1, 2, \dots, S\}\}$ and $\mathscr{F}_T$ coincides with the power set of $\{1, 2, \dots, S\}$. Denote by $\mathscr{G}_t$ the partition of $\{1, 2, \dots, S\}$ that generates $\mathscr{F}_t$. For each positive integer n, we denote by X^n the set of all processes of n dimensional vectors over the time span $\{0, 1, \dots, T\}$ that are adapted to the filtration $\{\mathscr{F}_0, F_1, \dots, F_T\}$. This is a vector space of dimension $n \sum_{t=0}^{T} |\mathscr{G}_t|$.

There are L types of physically distinguished perishable goods, $\ell = 1, 2, \dots, L$ on each period and state. The number of (types of) contingent commodities is equal to $L \sum_{t=0}^{T} |\mathscr{G}_t|$.

There are I consumers, $i = 1, 2, \dots, I$. Their consumption sets are the non-negative orthant X_+^L of X^L, utility functions are $U_i : X_+^L \to \boldsymbol{R}$, and initial endowment vectors are $e^i = (e_0^i, e_t^i, \dots, e_T^i) \in X^L$. We assume that the U_i are continuous and strongly monotone. We say that an allocation $(x^1, x^2, \dots, x^I)$ of contingent commodities is *feasible* if $x^i \in X_+^L$ for every i and $\sum_i x^i = \sum_i e^i$.

Definition 1 A feasible contingent-commodity allocation $(x_*^1, x_*^2, \dots, x_*^I)$ is *Pareto-efficient* if there is no other feasible allocation $(x^1, x^2, \dots, x^I)$ such that $U_i(x^i) \geq U_i(x_*^i)$ for every i and $U_i(x^i) > U_i(x_*^i)$ for some i.

There are J assets, $j = 1, 2, \dots, J$. Each asset j is characterized by its dividend process $d^j = (d_0^i, d_t^i, \dots, d_T^i) \in X^L$. An *asset price process* is an element of X^J that represents the transition, expected by all consumers, of asset prices under uncertainty and over time. A *spot price process* is an element of X^L that represents the transition, expected by all consumers, of prices for the L goods, for immediate consumption, under uncertainty and over time. A *trading plan* is an element of X^J that represents the transition of portfolios of the J assets under uncertainty and over time.

Suppose consumer i employs a trading plan y^i under the asset price process q and a spot price process p. Define $d^{y^i} = (d_0^{y^i}, d_1^{y^i}, \dots, d_T^{y^i}) \in X^1$ by

$$d_0^{y^i} = -\sum_j q_0^j y_0^{ji},$$

$$d_t^{y^i} = \sum_j y_{t-1}^{ji}(p_t \cdot d_t^j) - \sum_j q_t^j (y_t^{ji} - y_{t-1}^{ji}) \text{ for every } t \geq 1,$$

where $q = (q_0, q_1, \dots, q_T)$ with $q_t = (q_t^1, q_t^2, \dots, q_t^J)$ for each t, $p = (p_0, p_1, \dots, p_T)$, and $y^i = (y_0^i, y_1^i, \dots, y_T^i)$ with $y_t^i = (y_t^{1i}, y_t^{2i}, \dots, y_t^{Ji})$ for each t. Then he can finance any consumption process $x^i \in X_+^L$ that satisfies

$$p_t \cdot \left(x_t^i - e_t^i\right) \leq d_t^{y^i} \tag{1}$$

for every $t \geq 0$, where $x^i = (x_0^i, x_1^i, \dots, x_T^i)$.

An allocation $(y^1, y^2, \dots, y^I)$ of trading plans is *feasible* if $\sum_i y^i = 0$.

Definition 2 The collection of a feasible contingent-commodity allocation $(x^1_*, x^2_*, \dots, x^I_*)$, a feasible allocation $(y^1_*, y^2_*, \dots, y^I_*)$ of trading plans, an asset price process q, and a spot price process p is an *asset market equilibrium* if for every i, $(x^i, y^i) = (x^i_*, y^i_*)$ maximizes $U_i(x^i)$ under the budget constraint (1) for every $t \geq 0$.

Since the U_i are strongly monotone, p_t is always a strictly positive vector and the weak inequality in (1) holds as an equality. If $L = 1$, then, by replacing q_t by $(1/p_t)q_t$, we can assume that $p_t = 1$ for every t. This convention will be used throughout Section 5.

Definition 3 Asset markets are *complete* under the equilibrium asset price process q and spot price process p if for every $z^i = (z^i_0, z^i_1, \dots z^i_T) \in X^1$, there exists a trading plan y^i such that $z^i_t = d^{y^i}_t$ for every $t \geq 1$.

Consider the special case of $T = 1$. We can identify the payoff d^j_1 of asset j in period 1 with a collection $(d^{1j}_1, d^{2j}_1, \dots, d^{Sj}_1)$ of S L-dimensional vectors. Similarly, we can identify the spot price p^1 in period 1 with a collection $(p^1_1, p^2_1, \dots, p^S_1)$ of S L-dimensional vectors. Then asset markets are complete if and only if for every column vector $a = (a^1, a^2, \dots, a^S) \in \boldsymbol{R}^S$, there exists a column vector $b = (b^1, b^2, \dots, b^J) \in \boldsymbol{R}^S$ such that $a = Db$, where

$$D = \begin{pmatrix} p^1_1 \cdot d^{11}_1 & \cdots & p^1_1 \cdot d^{1J}_1 \\ \vdots & \ddots & \vdots \\ p^S_1 \cdot d^{S1}_1 & \cdots & p^S_1 \cdot d^{SJ}_1 \end{pmatrix} \in \boldsymbol{R}^{S \times J}. \tag{2}$$

This is equivalent to saying that $\operatorname{rank} D = S$. It means that if asset markets are complete, then $J \geq S$, that is, there are at least as many assets as states. In the general case of $T \geq 2$, for each $t \leq T - 1$ and $G_{t-1} \in \mathscr{G}_{t-1}$, let $N(t, G_{t-1}) = |\{G_t \in \mathscr{G}_t \mid G_t \supseteq G_{t-1}\}|$. That is, $N(t, G_{t-1})$ is the number of the elements of partition $\mathscr{G}_t$ that include G_{t-1}. If asset markets are complete, then $J \geq N(t, G_{t-1})$ for every t and G_{t-1}. That is, a necessary condition for market completeness is that

$$J \geq \max_{(t, G_{t-1})} N(t, G_{t-1}),$$

where the maximum is taken over all $t \leq T - 1$ and $G_{t-1} \in \mathscr{G}_{t-1}$.

The following theorem is the well known first welfare theorem for the case of complete asset markets.

Theorem 1 *If the collection of a contingent-commodity allocation $(x^1_*, x^2_*, \dots, x^I_*)$, an allocation $(y^1_*, y^2_*, \dots, y^I_*)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium, and if asset markets are complete under q and p, then $(x^1_*, x^2_*, \dots, x^I_*)$ is Pareto-efficient.*

3 Effective completeness and welfare theorems

We now give the definition of effectively complete asset markets.

Definition 4 Asset markets are *effectively complete* if for every Pareto-efficient allocation $(x^1, x^2, \ldots, x^I)$ of contingent commodities, there exists a feasible allocation $(y^1, y^2, \ldots, y^I)$ of trading plans such that $x_t^i - e_t^i = \sum_j y_{t-1}^{ji} d_t^j$ for every $t \geq 1$ and $y_0^i = y_1^i = \cdots = y_{T-1}^i$ for every i.

According to this definition, asset markets are effectively complete if every Pareto-efficient allocation can be attained by trading goods and assets on period 0 but not trading on asset or spot markets from period 1 onwards. Unlike the notion of completeness, the notion of effective completeness is independent of asset and spot price processes. As in LeRoy and Werner (2001, Chapter 16), if $T = L = 1$, and if asset markets are complete, then they are effectively complete. For, if asset markets are complete, then

$$\operatorname{rank} \begin{pmatrix} d_1^{11} & \cdots & d_1^{1J} \\ \vdots & \ddots & \vdots \\ d_1^{S1} & \cdots & d_1^{SJ} \end{pmatrix} = \operatorname{rank} D = S$$

and, hence, every feasible allocation of contingent commodities, Pareto-efficient or not, can be attained by some feasible allocation of assets. Otherwise, then neither completeness nor effective completeness implies the other.

Elul (1999) took up a problem related to effectively complete asset markets. He clarified when a contingent-commodity allocation at equilibrium in incomplete markets that happens to be Pareto-efficient can be attained at equilibrium in complete markets. His analysis is different from ours in that he first assumed that the contingent-commodity allocation of an incomplete-market equilibrium is Pareto-efficient and then asked whether it can be attained at some complete-market equilibrium, while we ask under what conditions the contingent-commodity allocation of an incomplete-market equilibrium is guaranteed to be Pareto-efficient. His analysis is, therefore, applicable to every equilibrium in asset markets that are effectively complete.

Below the first welfare theorem in effectively complete asset markets. It is because of the validity of this theorem that we have defined the property stated in Definition 4 as effective completeness.

Theorem 2 *If the collection of a contingent-commodity allocation $(x_*^1, x_*^2, \ldots, x_*^I)$, an allocation $(y_*^1, y_*^2, \ldots, y_*^I)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium, and if asset markets are effectively complete, then $(x_*^1, x_*^2, \ldots, x_*^I)$ is Pareto-efficient.*

Proof of Theorem 2 Suppose that the collection of a contingent-commodity allocation $(x_*^1, x_*^2, \ldots, x_*^I)$, an allocation $(y_*^1, y_*^2, \ldots, y_*^I)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium, and that a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-superior to $(x_*^1, x_*^2, \ldots, x_*^I)$.

As shown in LeRoy and Werner (2001, Proposition 16.3.2), since the consumption sets are closed and bounded from below and the utility functions are continuous, there is a Pareto-efficient allocation that is Pareto-superior to $(x_*^1, x_*^2, \ldots, x_*^I)$. Without loss of generality, therefore, we can assume that $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient. By effective completeness, there exists a feasible allocation $(y^1, y^2, \ldots, y^I)$ of trading plans such that for every i, $y_0^i = y_1^i = \cdots = y_{T-1}^i$ and $x_t^i - e_t^i = \sum_j y_t^{ji} d_t^j$ for every $t \geq 1$. Since $y_t^i - y_{t-1}^i = 0$ for every $t \geq 1$, this means that (1) is satisfied on each period $t \geq 1$. Thus, for every i with $U_i(x^i) > U_i(x_*^i)$, (1) fails to hold on period 0, that is,

$$p_0 \cdot \left(x_0^i - e_0^i\right) > -\sum_j q_0^j y_0^{ji}. \tag{3}$$

For every i with $U_i(x^i) = U_i(x_*^i)$, since U_i is strongly monotone,

$$p_0 \cdot \left(x_0^i - e_0^i\right) \geq -\sum_j q_0^j y_0^{ji}. \tag{4}$$

Summing up (3) and (4) over i and using the feasibility constraints, we obtain

$$0 > -\sum_j q_0^j \left(\sum_i y_0^{ji}\right) = 0,$$

which is a contradiction. Thus $(x_*^1, x_*^2, \ldots, x_*^I)$ is Pareto-efficient. ///

Although it will not be used in our applications, it is worth mentioning that the second welfare theorem also holds in effectively complete markets.[1]

Theorem 3 *Suppose that U_i is quasi-concave for every i and that asset markets are effectively complete. Suppose also that a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient. Then there exist a feasible contingent-commodity allocation $(\hat{e}^1, \hat{e}^2, \ldots, \hat{e}^I)$ satisfying $\hat{e}_t^i = e_t^i$ for every $t \geq 1$ and i, a feasible allocation $(y^1, y^2, \ldots, y^I)$ of trading plans, an asset price process q, and a spot price process p such that the collection of $(x^1, x^2, \ldots, x^I)$, $(y^1, y^2, \ldots, y^I)$, q, and p is an asset market equilibrium when the initial endowment allocation is $(\hat{e}^1, \hat{e}^2, \ldots, \hat{e}^I)$.*

This theorem states that if asset markets are effectively complete, then every Pareto-efficient allocation can be attained at equilibrium by some lump-sum transfers of the L contingent commodities available on period 0, not involving any contingent commodity available from period 1 onwards.[2] Note, indeed, that is since $(x^1, x^2, \ldots, x^I)$ is a feasible allocation of $(\hat{e}^1, \hat{e}^2, \ldots, \hat{e}^I)$, $\sum_i \hat{e}_0^i = \sum_i x_0^i = \sum_i e_0^i$. To prove this theorem, it is convenient to use the concept of a contingent-commodity market equilibrium. Before giving the formal definition of

[1] I am grateful to Midori Hirokawa and Atsushi Kajii for suggesting that I check the validity of the second welfare theorem in effectively complete asset markets.

[2] In fact, it suffices to reallocate any one of the L contingent commodities available on period 0.

the concept, note that because of adaptedness, each $z = (z_0, z_1, \dots, z_T) \in X^n$ can be identified with a mapping $(t, G_t) \mapsto z_t^{G_t}$ of $\bigcup_{\tau=0}^{T} (\{\tau\} \times \mathscr{G}_\tau)$ to $\boldsymbol{R}^n$, where $z_t^{G_t} = z_t^s$ for any $s \in G_t$.

Definition 5 The pair of a feasible contingent-commodity allocation $(x_*^1, x_*^2, \dots, x_*^I)$ and a contingent-commodity price process p is a *contingent-commodity market equilibrium* if for every i, $x^i = x_*^i$ maximizes $U_i(x^i)$ under the budget constraint

$$\sum_{t \geq 0} \sum_{G_t \in \mathscr{G}_t} p_t^{G_t} \cdot x_t^{G_t i} \leq \sum_{t \geq 0} \sum_{G_t \in \mathscr{G}_t} p_t^{G_t} \cdot e_t^{G_t i},$$

where $x^i = \left(x_t^{G_t i}\right)_{(t, G_t) \in \bigcup_{\tau=0}^{T} (\{\tau\} \times \mathscr{G}_\tau)}$, and analogously for e^i and p.

Proof of Theorem 3 Let $(x^1, x^2, \dots, x^I)$ be a Pareto-efficient allocation. Since the U_i are continuous, strongly monotone, and quasi-concave, the standard second welfare theorem implies that there exists a contingent-commodity price process p such that the pair of $(x^1, x^2, \dots, x^I)$ and p is a contingent-commodity market equilibrium when the initial endowment allocation is $(x^1, x^2, \dots, x^I)$.[3] Define an asset price process $q = (q^1, q^2, \dots, q^J) \in X^J$, with $q^j = \left(q_t^{G_t j}\right)_{(t, G_t)}$ for each j, by letting

$$q_t^{G_t j} = \sum_{\tau \geq t+1} \sum_{\{G_\tau \in \mathscr{G}_\tau | G_\tau \subseteq G_t\}} p_\tau^{G_\tau} \cdot d_\tau^{G_\tau j} \tag{5}$$

for every $t \geq 0$ and j. By effective completeness, there exists a feasible allocation $(y^1, y^2, \dots, y^I)$ of trading plans such that $x_t^i - e_t^i = \sum_j y_{t-1}^{ji} d_t^j$ for every $t \geq 1$ and i. Define a feasible contingent-commodity allocation $(\hat{e}^1, \hat{e}^2, \dots, \hat{e}^I)$ by letting, for every i, $\hat{e}_t^i = e_t^i$ for every $t \geq 1$, and

$$p_0 \cdot \hat{e}_0^i = p_0 \cdot x_0^i + \sum_j q_0^j y_0^{ji}$$

and $\sum_i \hat{e}^i = \sum_i e^i$. In the following, we show that the collection of $(x^1, x^2, \dots, x^I)$, $(y^1, y^2, \dots, y^I)$, q, and p is an asset market equilibrium when the initial endowment allocation is $(\hat{e}^1, \hat{e}^2, \dots, \hat{e}^I)$. Since the feasibility constraints and the budget constraints are clearly satisfied, it remains to prove that the utility maximization condition is satisfied. For this, let $(\hat{x}^i, \hat{y}^i)$ satisfy the budget constraint for consumer i. Then

$$\begin{aligned}
&\sum_{t \geq 0} \sum_{G_t \in \mathscr{G}_t} p_t^{G_t} \cdot \left(\hat{x}_t^{G_t i} - \hat{e}_t^{G_t i}\right) \\
=& p_0^{\{1,2,\dots,S\}} \cdot \left(\hat{x}_0^{\{1,2,\dots,S\}i} - \hat{e}_0^{\{1,2,\dots,S\}i}\right) + \sum_{t \geq 1} \sum_{G_t \in \mathscr{G}_t} p_t^{G_t} \cdot \left(\hat{x}_t^{G_t i} - e_t^{G_t i}\right) \\
=& p_0^{\{1,2,\dots,S\}} \cdot \left(\hat{x}_0^{\{1,2,\dots,S\}i} - \hat{e}_0^{\{1,2,\dots,S\}i}\right) + \sum_{t \geq 1} \sum_{G_t \in \mathscr{G}_t} d_t^{G_t \hat{y}^i},
\end{aligned} \tag{6}$$

[3] The standard second welfare theorem only claims that the pair of $(x^1, x^2, \dots, x^I)$ and p is a *quasi-equilibrium.* However, since the consumption set is the non-negative orthant X_+^L and the U_i are strongly monotone, every quasi-equilibrium is an equilibrium.

where $d^{\hat{y}^j} = \left(d_t^{G_t \hat{y}^j}\right)_{(t,G_t)}$. Applying the method explained in Duffie (2001, Section 2.C) to (5), we can show that

$$\sum_{t \geq 1} \sum_{G_t \in \mathscr{G}_t} d_t^{G_t \hat{y}^i} = \sum_j q_0^j \hat{y}_0^{ji}. \tag{7}$$

Since $p_0 \cdot \hat{e}_0^i \geq p_0 \cdot \hat{x}_0^i + \sum_j q_0^j \hat{y}_0^{ji}$, (6) and (7) together imply that $\sum_{t \geq 0} \sum_{G_t \in \mathscr{G}_t} p_t^{G_t} \cdot \left(\hat{x}_t^{G_t i} - e_t^{G_t i}\right) \leq 0$. Hence $\hat{x}^i$ satisfies the budget constraint in the contingent-commodity market equilibrium. Since x^i is a solution to the utility maximization problem in the commodity market equilibrium, $U_i(x^i) \geq Ui(\hat{x}^i)$. For every i, therefore, the utility maximization condition is satisfied by (x^i, y^i) at the proposed asset market equilibrium. ///

4 Sunspot irrelevance

In this section, we give our first application of Theorem 2, the first welfare theorem in effectively complete markets. It is the sunspot-irrelevance theorem of Mas-Colell (1992). The following analysis is really a recap of his own, as Theorem 2 owes much to Mas-Colell (1992) (and also LeRoy and Werner (2001)).

Assume that $T = 1$ and that all consumers hold the same probability measure P. Assume that $P(\{s\}) > 0$ for every s and all consumers have expected utility functions

$$U_i(x^i) = E(u_i(x_0^i, x_1^i)) = \sum_s P(\{s\}) u_i(x_0^{si}, x_1^{si}),$$

where $x^i = (x_0^i, x_1^i)$ with $x_t^i = (x_t^{1i}, x_t^{2i}, \ldots, x_t^{Si})$ for each t,[4] and $u_i : \boldsymbol{R}_+^L \times \boldsymbol{R}_+^L \to \boldsymbol{R}$ is strictly concave in the second coordinate. Note that consumption is possible on period 0, as well as on period 1, unlike the model of Mas-Colell (1992). We also assume that e_1^i takes a constant value, that is,

$$e_1^{1i} = e_1^{2i} = \cdots = e_1^{Si}, \tag{8}$$

for every i. Thus the states are irrelevant to utility functions and initial endowments, and thus called *sunspot* states. Of course, under standard assumptions, there is an asset market equilibrium of which the contingent-commodity allocation is sunspot-free. Mas-Colell (1992) showed that if there are not sufficiently many assets available for trade, then there may be an asset market equilibrium of which the contingent-commodity allocation depends on sunspots and some of its realizations are different from any of the sunspot-free equilibrium allocations.

We start with characterizing the Pareto-efficient allocations in this economy. The following lemma shows that they are sunspot-free.

Lemma 1 *If a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient, then x_1^i takes a constant value, that is, $x_1^{1i} = x_1^{2i} = \cdots = x_1^{Si}$ for every i.*

[4] Since x^i is adapted, $x_0^{1i} = x_0^{2i} = \cdots = x_0^{Si}$.

Proof of Lemma 1 For any feasible allocation $(x^1, x^2, \ldots, x^I)$, define another allocation $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^I)$ by $\hat{x}_0^i = x_0^i$ and $\hat{x}_1^{si} = \bar{x}^i$, where $\bar{x}_1^i = \sum_s P(\{s\}) x_1^{si} \in \boldsymbol{R}_+^L$, for every i and s. Then $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^I)$ is feasible because

$$\sum_i \hat{x}_1^{si} = \sum_i \sum_{s'} P(\{s'\}) x_1^{s'i} = \sum_{s'} P(\{s'\}) \sum_i x_1^{s'i} = \sum_{s'} P(\{s'\}) \sum_i e_1^{s'i} = \sum_i e^{si}$$

by (8). Moreover, $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^I)$ is Pareto-superior to $(x^1, x^2, \ldots, x^I)$ unless $x^i = \hat{x}^i$, that is, $x^{1i} = x^{2i} = \cdots = x^{Si}$ for every i, by the strict concavity of the u_i in the second coordinate. Therefore, if $(x^1, x^2, \ldots, x^I)$ is Pareto efficient, then $x^{1i} = x^{2i} = \cdots = x^{Si}$ for every i. ///

Next, we give a sufficient condition for asset markets to be effectively complete.

Lemma 2 *Asset markets are effectively complete if for every good ℓ there is an asset j such that for every s,*

$$d^{sj} = (0, \ldots, 0, \underbrace{1}_{\ell-th}, 0, \ldots, 0) = (\ell\text{-}th \text{ unit vector}) \in \boldsymbol{R}^L.$$

This lemma means that asset markets are effectively complete if it is possible to guarantee receipt of any fixed amount of any good. Thus, asset markets may be effectively complete with fewer than S assets if $L < S$.

Proof of Lemma 2 If $(x^1, x^2, \ldots, x^I)$ is an efficient allocation, then, by Lemma 1, for each i there exists a $z^i \in \boldsymbol{R}^L$ such that $x_1^{si} - e_1^{si} = z^i$ for every s. By assumption, there is a portfolio $\bar{y}^i = (\bar{y}^{1i}, \bar{y}^{2i}, \ldots, \bar{y}^{Ji}) \in \boldsymbol{R}^J$ such that $z^i = \sum_j \bar{y}^{ji} d^{sj}$ for every s. For each $i \geq 2$, define a trading plan $y^i = (y_0^i, y_1^i)$ by letting $y_t^i = \bar{y}^i$ for each t. Define $y^1 = -\sum_{i \geq 2} y^i$. Then $(y^1, y^2, \ldots, y^I)$ is a feasible allocation of trading plans. For every $i \geq 2$ and s,

$$\sum_j y_0^{ji} d_1^{sj} = \sum_j \bar{y}^j d_1^{sj} = z^i = x_1^{si} - e_1^{si}.$$

As for $i = 1$,

$$\sum_j y_0^{j1} d_1^{sj} = \sum_j \left(-\sum_{i \geq 2} y_0^{ji} \right) d_1^{sj} = -\sum_{i \geq 2} \sum_j y_0^{ji} d_1^{sj} = -\sum_{i \geq 2} \left(x_1^{si} - e_1^{si} \right) = x_1^{s1} - e_1^{s1}.$$

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Under the assumption of Lemma 1, asset markets are effectively complete and, by Theorem 2, the equilibrium contingent-commodity allocation are Pareto-efficient and, by Lemma 2, sunspot-free. We have thereby proved the following theorem of Mas-Colell (1992) via effective completeness.

Theorem 4 (Sunspot Irrelevance Theorem of Mas-Colell (1992)) *If for every good ℓ there is an asset j such that d_1^{sj} is the ℓ-th unit vector for every s, then every equilibrium allocation is Pareto-efficient and sunspot-free.*

Mas-Colell's theorem tells us that the equilibrium allocations may be Pareto-efficient even when asset markets are incomplete. Indeed, if the number J of assets is less than the number S of sunspot states, then asset markets must necessarily be incomplete but the equilibrium allocations may be Pareto-efficient. It may even be the case that the J assets turn out to be redundant under the equilibrium prices, rendering asset markets incomplete regardless of whether J is larger or smaller than S, and yet the equilibrium allocations are Pareto-efficient.

To see this point more formally, assume that for every $j \leq L$, the j-th asset pays out one unit of good j for sure (that is, for every $j \leq L$ and s, d_1^{sj} coincides with the j-th unit vector), and that the collection of a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$, a feasible allocation $(y^1, y^2, \ldots, y^I)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium. Then $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient and sunspot-free. Define another spot price process $\hat{p} = (\hat{p}_0, \hat{p}_1)$ by letting $\hat{p}_0 = p_0$ and $\hat{p}_1$ coincide with the first L coordinates of q_0. Then $\hat{p}_1$ is sunspot-free. Define another feasible allocation $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$ of trading plans by letting the first L coordinates of $\hat{y}_0^i$ coincide with $x_1^i - e_1^i$ (which is sunspot-free) and the remaining $J - L$ coordinates equal to zero. Then the collection of $(x^1, x^2, \ldots, x^I)$, $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$, q, and $\hat{p}$ is an asset market equilibrium. This is because every sunspot-free consumption plan that can be attained under (q, p) can also be attained under $(q, \hat{p})$, and vice versa. In this latter equilibrium, the contingent-commodity allocation is the same as in the original equilibrium, but the rank of $D \in \boldsymbol{R}^{S \times J}$ defined by (2) with $p_1 = \hat{p}_1$ is equal to one because the d_1^j and $\hat{p}_1$ are sunspot-free. This implies that asset markets are incomplete as long as $S \geq 2$ but the equilibrium allocation is Pareto-efficient.

5 No-retrade theorem

Our second application of effectively complete markets is the no-retrade theorem of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003).

We consider a Markov environment in which there are M states, $m = 1, 2, \ldots, M$, on each period and a single good in each state. Let $\bar{m} \in \{1, 2, \ldots, M\}$ be the state on period 0, then the state space over the entire history is given by $S = \{\bar{m}\} \times M^T$.[5]

Define $\chi : S \times \{0, 1, \ldots, T\} \to M$ by $\chi(s, t) = s_t$, where $s = (s_0, s_1, \ldots, s_T) \in S$. Write χ_t for $\chi(\cdot, t) : S \to M$. Then χ_t maps each entire history to the state that arises on period t along the history. The filtration $(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_T)$ is defined in such a way that for every t, $\mathscr{F}_t$ is generated by the mapping $(\chi_0, \chi_1, \ldots, \chi_t) : S \to M^t$. That is, for every $s = (s_0, s_1, \ldots, s_T) \in S$ and $s' = (s'_0, s'_1, \ldots, s'_T) \in S$, s and s' belong to the same element of the partition $\mathscr{G}_t$ corresponding to $\mathscr{F}_t$ if and only if $s_{t'} = s'_{t'}$ for every $t' \leq t$.

[5] There is a slight abuse of notation, as S is a set in this section, while it used to be a positive integer up to the previous section. Little confusion will arise from this abuse of notation.

Assume that $L = 1$, that is, there is only one good in each state and on each period. We thus let $p_t = 1$ for every t.

Assume that all consumers hold the same probability measure P and subjective discount factor $\delta > 0$, and have state-dependent expected utility functions

$$U_i(x^i) = E\left(\sum_{t=0}^{T} \delta^t u_i(x_t^i, \chi_t)\right) = \sum_{(s,t)\in S\times\{0,1,\ldots,T\}} \delta^t P(\{s\}) u_i(x_t^{si}, s_t),$$

where $u_i : \boldsymbol{R}_+ \times \{1, 2, \ldots, M\} \to \boldsymbol{R}$ and $x^i = (x_0^i, x_1^i, \ldots, x_T^i)$. Assume that $P(\{s\}) > 0$ for every s. This is equivalent to saying that for every t and $(s_0, s_1, \ldots, s_t) \in \{\bar{m}\} \times M^t$, $P\left(\{s' \in S \mid s'_{t'} = s_{t'} \text{ for every } t' \leq t\} \mid \{s' \in S \mid s'_{t'} = s_{t'} \text{ for every } t' \leq t-1\}\right) > 0$.

Assume that, for the initial endowment process $e^i = (e_0^i, e_1^i, \ldots, e_T^i)$ of each consumer i, each e_t^i depends only on s_t (and not on t), that is, there is a $g_i : \{1, 2, \ldots, M\} \to \boldsymbol{R}$ such that $e^i = g_i(\chi)$. Assume also that, for the dividend process $d^j = (d_0^j, d_1^j, \ldots, d_T^j)$ of each asset j, each d_t^j depends only on s_t (and not on t), that is, there is an $h_i : \{1, 2, \ldots, M\} \to \boldsymbol{R}$ such that $e^j = h_j(\chi)$.

This economy is in the Markov environment, as there are M states that recur over time and the utility functions, initial endowments, and dividend payouts depend only on the state on the period but not on the state on any earlier period. But, unlike the model of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003), the probability that state m occurs on period t may depend not only on the state that occurred on period $t-1$ but also on some earlier periods. Note also that we are assuming that there are only finitely many periods, while they assumed that there are infinitely many periods. Finally, all assets in our model are long lived (traded from period 0 onwards and dividends paid out until period T), while some assets in their model may be short-lived (traded just once and dividends paid out only on the next period). We exclude short-lived assets from our model for the sake of simplicity of exposition.

Just as in the previous section, we start the analysis of the model with characterizing the Pareto-efficient allocations.

Lemma 3 *If an allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient, then for every i, there exists an $f_i : \{1, 2, \ldots, M\} \to \boldsymbol{R}$ such that $x^i - e^i = f_i(\chi)$.*

Proof of Lemma 3 Let $(x^1, x^2, \ldots, x^I)$ be a feasible contingent-commodity allocation. For each m, define $r^m = \sum_{(s,t)\in\chi^{-1}(m)} \delta^t P(\{s\})$. Then, for each m and i, define

$$\bar{x}^{mi} = \sum_{(s,t)\in\chi^{-1}(m)} \frac{\delta^t P(\{s\})}{r^m} x_t^{si}.$$

Then define $\hat{x}^i = \left(\hat{x}_0^i, \hat{x}_1^i, \ldots, \hat{x}_T^i\right)$ by letting $\hat{x}_t^{si} = \bar{x}^{\chi(s,t)i}$ for every s and t. Then the allocation

$(\hat{x}^1, \hat{x}^2, \dots, \hat{x}^I)$ is feasible because

$$\begin{aligned}\sum_{i=1}^{I} \hat{x}_t^{si} = \sum_{i=1}^{I} \bar{x}^{\chi(s,t)i} &= \sum_{i=1}^{I} \sum_{(s',t')\in\chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} x_{t'}^{s'i} \\ &= \sum_{(s',t')\in\chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} \sum_{i=1}^{I} x_{t'}^{s'i} \\ &= \sum_{(s',t')\in\chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} \sum_{i=1}^{I} e_{t'}^{s'i} \\ &= \sum_{(s',t')\in\chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} \sum_{i=1}^{I} g_i(\chi(s,t)) \\ &= \sum_{i=1}^{I} g_i(\chi(s,t)) = \sum_{i=1}^{I} e_t^{si}.\end{aligned}$$

Since $u_i(\cdot, m)$ is strictly concave,

$$U_i(x^i) = \sum_{m=1}^{M} r^m \sum_{(s,t)\in\chi^{-1}(m)} \frac{\delta^t P(\{s\})}{r^m} u_i\left(x_t^{si}, m\right) \geq \sum_{m=1}^{M} r^m \sum_{(s,t)\in\chi^{-1}(m)} u_i\left(\hat{x}_t^{si}, m\right) = U_i(x^i),$$

where the weak inequality holds as a strict inequality unless $x^i = \hat{x}^i$, that is, $x_t^{si} = x_{t'}^{s'i}$ whenever $\chi(s,t) = \chi(s',t')$ for every i. Thus $(\hat{x}^1, \hat{x}^2, \dots, \hat{x}^I)$ is Pareto-superior to $(x^1, x^2, \dots, x^I)$ unless $x_t^{si} = x_{t'}^{s'i}$ whenever $\chi(s,t) = \chi(s',t')$ for every i. Therefore, if $(x^1, x^2, \dots, x^I)$ is Pareto-efficient, then $x_t^{si} = x_{t'}^{s'i}$ whenever $\chi(s,t) = \chi(s',t')$ for every i. This means that for every i, there is a $\hat{f}_i : \{1, 2, \dots, M\} \to \boldsymbol{R}$ such that $x^i = k_i(\chi)$. The proof is completed by taking $f_i = \hat{f}_i - g_i$. ///

To state a sufficient condition for effectively complete asset markets, write

$$H = \begin{pmatrix} h_1(1) & \cdots & h_J(1) \\ \vdots & \ddots & \vdots \\ h_1(M) & \cdots & h_J(M) \end{pmatrix} \in \boldsymbol{R}^{M\times J}.$$

Lemma 4 *If* $\operatorname{rank} H = M$*, then asset markets are effectively complete .*

Proof of Lemma 4 Suppose that $(x^1, x^2, \dots, x^I)$ is a Pareto-efficient contingent-commodity allocation. By Lemma 3, for every i, there exists an $f_i : \{1, 2, \dots, M\} \to \boldsymbol{R}$ such that $x^i - e^i = f_i(\chi)$. Write

$$v_i = \begin{pmatrix} f_i(1) \\ \vdots \\ f_i(M) \end{pmatrix} \in \boldsymbol{R}^M,$$

Since $\operatorname{rank} H = M$, there exists a $b_i \in \boldsymbol{R}^J$ such that $v_i = Hb_i$. For each $i \geq 2$, define a trading plan y^i by letting $y^i_t = b_i$ for every t. For $i = 1$, let $y^1 = \sum_{i\geq 2} y^i$. Then $(y^1, y^2, \ldots, y^I)$ is a feasible allocation of trading plans and $y^i_0 = y^i_1 = \cdots = y^i_{T-1}$ for every i. For every $i \geq 2$, $t \geq 1$, and s,

$$x^i_t - e^i_t = f_i(\chi_t) = Hb_i = \sum_j y^{ji}_{t-1} d^j_t.$$

As for $i = 1$,

$$\sum_j y^{j1}_{t-1} d^j_t = \sum_j \left(-\sum_{i\geq 2} y^{ji}_{t-1} \right) d^j_t = -\sum_{i\geq 2}\sum_j y^{ji}_{t-1} d^j_t = -\sum_{i\geq 2}(x^i_t - e^i_t) = x^1_t - e^1_t.$$

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To state the no-retrade theorem, we need the following notation. Let q be an equilibrium asset price process. Since it is adapted to the filtration $(\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_T)$, $q^{sj}_t = q^{s'j}_t$ whenever $s_{t'} = s'_{t'}$ for every $t' \leq t$. Thus there exists a $k_j : \bigcup_{\tau=0}^{T}(\{\bar{m}\} \times M^\tau) \to \boldsymbol{R}$ such that $q^{sj}_t = k_j(\chi_0(s), \chi_1(s), \ldots, \chi_t(s))$. For each t and each $(s_0, s_1, \ldots, s_{t-1})$, define $K(s_0, s_1, \ldots, s_{t-1}) \in \boldsymbol{R}^{M\times J}$ as

$$\begin{pmatrix} h_1(1) + k_1(s_0, s_1, \ldots, s_{t-1}, 1) & \cdots & h_J(1) + k_J(s_0, s_1, \ldots, s_{t-1}, 1) \\ \vdots & \ddots & \vdots \\ h_1(M) + k_1(s_0, s_1, \ldots, s_{t-1}, M) & \cdots & h_J(M) + k_J(s_0, s_1, \ldots, s_{t-1}, M) \end{pmatrix}.$$

While the matrix H represents the dividends of the J assets on the next period, the matrix $K(s_0, s_1, \ldots, s_{t-1})$ represents the total returns to the J assets, inclusive of their prices on the next period. Asset markets are complete if and only if $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = M$ for every t and $(s_0, s_1, \ldots, s_{t-1})$. In the model of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003), since there are infinitely many periods and the transition probabilities between two states are time-invariant, the asset prices are also time-invariant functions of the M states, and $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = \operatorname{rank} H = M$ for every t and $(s_0, s_1, \ldots, s_{t-1})$. That is, if $\operatorname{rank} H = M$, then asset markets are complete. In contrast, since asset prices need not be time-invariant functions of the M states in our model, the condition that $\operatorname{rank} H = M$ does not imply that asset markets are complete. It is for this reason that we need to assume that $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = M$ for every t and $(s_0, s_1, \ldots, s_{t-1})$ in the second part of our no-retrade theorem.

Theorem 5 (No-Retrade Theorem) *Assume that* $\operatorname{rank} H = M$. *If the collection of a feasible contingent-commodity allocation* $(x^1, x^2, \ldots, x^I)$, *a feasible allocation* $(y^1, y^2, \ldots, y^I)$ *of trading plans, and an asset price process* q *is an asset market equilibrium, then there exists a feasible allocation* $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$ *of trading plans such that* $\hat{y}^i_0 = \hat{y}^i_1 = \cdots = \hat{y}^i_{T-1}$ *for every* i, *and the collection of* $(x^1, x^2, \ldots, x^I)$, $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$, *and* q *is an asset market equilibrium. If, in addition,* $J = M$ *and* $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = M$ *for every* t *and every* $(s_0, s_1, \ldots, s_{t-1})$, *then* $y^i = \hat{y}^i$ *and hence* $y^i_0 = y^i_1 = \cdots = y^i_{T-1}$ *for every* i.

This theorem states that if asset markets are effectively complete, then any equilibrium contingent-commodity allocation can be attained by letting all consumers trade assets once and for all on period 0, and that if, in addition, markets are complete and the J assets are not redundant, then all consumers do in fact trade assets once and for all on period 0 at equilibrium. Since the equilibrium asset price processes need not be time-invariant, the proof of Theorem 5, which relies on effective completeness, is different from that of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003), which relies on the stationary dynamic programming technique.

Proof of Theorem 5 Let the collection of a feasible contingent-commodity allocation $(x^1, x^2, \dots, x^I)$, a feasible allocation $(y^1, y^2, \dots, y^I)$ of trading plans, and an asset price process q be an asset market equilibrium. By Lemma 4 and Theorem 2, $(x^1, x^2, \dots, x^I)$ is Pareto-efficient. Thus, there exists a feasible allocation $(\hat{y}^1, \hat{y}^2, \dots, \hat{y}^I)$ of trading plans such that $x^i_t - e^i_t = \sum_j \hat{y}^{ji}_{t-1} d^j_t$ for every $t \geq 1$ and $\hat{y}^i_0 = \hat{y}^i_1 = \cdots = \hat{y}^i_{T-1}$ for every i. To show that the collection of $(x^1, x^2, \dots, x^I)$, $(\hat{y}^1, \hat{y}^2, \dots, \hat{y}^I)$,and q is an asset market equilibrium, it suffices to prove that $x^i_0 - e^i_0 \leq -\sum_j q^j_0 \hat{y}^{ji}_0$ for every i. This is a consequence of the absence of arbitrage opportunities at equilibrium. Indeed, as explained in Duffie (2001, Section 2.C), there exists a process, called a state-price deflator, $\pi = (\pi_0, \pi_1, \dots, \pi_T) \in L^1$ such that

$$q^j_t = \frac{1}{\pi_t} E\left(\sum_{\tau=t+1}^{T} \pi_\tau d^j_\tau \mid \mathscr{F}_t\right)$$

for every t and j. Hence,

$$\sum_j q^j_t y^{ji}_t = \frac{1}{\pi_t} E\left(\sum_{\tau=t+1}^{T} \pi_\tau d^{y^i}_\tau \mid \mathscr{F}_t\right),$$

$$\sum_j q^j_t \hat{y}^{ji}_t = \frac{1}{\pi_t} E\left(\sum_{\tau=t+1}^{T} \pi_\tau d^{\hat{y}^i}_\tau \mid \mathscr{F}_t\right).$$

Since $d^{y^i}_t = x^i_t - e^i_t = d^{\hat{y}^i}_t$ for every $t \geq 1$, $\sum_j q^j_0 y^{ji}_0 = \sum_j q^j_0 \hat{y}^{ji}_0$. Since $x^i_0 - e^i_0 \leq -\sum_j q^j_0 y^{ji}_0$, $x^i_0 - e^i_0 \leq -\sum_j q^j_0 \hat{y}^{ji}_0$. This completes the proof of the first part.

As for the second part, suppose, in addition, that $J = M$ and $\operatorname{rank} K(s_0, s_1, \dots, s_{t-1}) = M$ for every t and every $(s_0, s_1, \dots, s_{t-1})$. We prove that $y^i = \hat{y}^i$ by a backward induction argument. For each t and $s = (s_0, s_1, \dots, s_T)$, write

$$q(s_0, s_1, \dots, s_t) = \begin{pmatrix} q_t^{s1} \\ \vdots \\ q_t^{sJ} \end{pmatrix} \in \boldsymbol{R}^J,$$

$$y_i(s_0, s_1, \dots, s_t) = \begin{pmatrix} y_t^{s1i} \\ \vdots \\ y_t^{sJi} \end{pmatrix} \in \boldsymbol{R}^J,$$

$$r_i(s_0, s_1, \dots, s_{t-1}) = \begin{pmatrix} q(s_0, s_1, \dots, s_{t-1}, 1) \cdot y_i(s_0, s_1, \dots, s_{t-1}, 1) \\ \vdots \\ q(s_0, s_1, \dots, s_{t-1}, M) \cdot y_i(s_0, s_1, \dots, s_{t-1}, M) \end{pmatrix} \in \boldsymbol{R}^M.$$

We define $\hat{y}_i(s_0, s_1, \dots, s_t)$ and $\hat{r}_i(s_0, s_1, \dots, s_t)$ analogously for $\hat{y}^i$.

Since $p_t = 1$ for every t and $q_T^j = 0$ for every j (because, otherwise, there would be an arbitrage opportunity and the maximization problem would have no solution), (1) with $t = T$ can be rewritten as $v_i = H y_{T-1}^i = H \hat{y}_{T-1}^i$. Since $\operatorname{rank} H = M = J$, this means that $y_{T-1}^i = \hat{y}_{T-1}^i$. As an induction hypothesis, let $t \le T - 2$ and suppose that $y_{t+1}^i = \hat{y}_{t+1}^i$. Then (1) can be written as

$$\begin{aligned} v_i &= K(s_0, s_1, \dots, s_{t-1}) y_i(s_0, s_1, \dots, s_{t-1}) - r_i(s_0, s_1, \dots, s_t), \\ v_i &= K(s_0, s_1, \dots, s_{t-1}) \hat{y}_i(s_0, s_1, \dots, s_{t-1}) - \hat{r}_i(s_0, s_1, \dots, s_t), \end{aligned}$$

which is equivalent to

$$\begin{aligned} K(s_0, s_1, \dots, s_{t-1}) y_i(s_0, s_1, \dots, s_{t-1}) &= v_i + r_i(s_0, s_1, \dots, s_t), \\ K(s_0, s_1, \dots, s_{t-1}) \hat{y}_i(s_0, s_1, \dots, s_{t-1}) &= v_i + \hat{r}_i(s_0, s_1, \dots, s_t). \end{aligned}$$

Since $K(s_0, s_1, \dots, s_{t-1})$ is an invertible $M \times M$ matrix and $r_i(s_0, s_1, \dots, s_t) = \hat{r}_i(s_0, s_1, \dots, s_t)$ by the induction hypothesis, $y_i(s_0, s_1, \dots, s_{t-1}) = \hat{y}_i(s_0, s_1, \dots, s_{t-1})$. Thus $y_t^i = \hat{y}_t^i$. ///

Since the theorem holds even when $\operatorname{rank} K(s_0, s_1, \dots, s_{t-1}) < M$ as long as $\operatorname{rank} H = M$, the theorem shows that effective complete asset markets may not be complete.

6 Conclusion

We have proposed a definition of effectively complete asset markets in a model with multiple goods and multiple periods, and established the first and second welfare theorems in such markets. We have then given two applications of the first welfare theorem, the sunspot irrelevance theorem and the no-retrade theorem. The lesson to be learned from this exercise is that the equilibrium allocations may well be Pareto-efficient even in incomplete asset markets, and effective completeness serves as a sufficient condition for this to occur.

The usefulness of the concept of effective completeness hinges on to what extent it is applicable. We now know five distinct examples of effectively complete asset markets, of which three are presented in LeRoy and Werner (2001, Chapter 16) and two in this paper. But the applicability is severely limited by the fact that it requires every Pareto-efficient allocation to be attained after the first round of asset trades, without using asset or spot markets from the second round onwards. We should, therefore, find a weaker notion of effective completeness with respect to which the first (and, preferably, the second) welfare theorem retains its validity. This seems to be an important direction of future research.

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