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“Optimal Execution in an Evolutionary Setting”

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Optimal Execution in an Evolutionary Setting*

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Abstract

We consider the dynamic trading strategies that minimize the expected cost of trading a large block of securities over a fixed finite number of periods. We obtain the result in which the institutional investor sells more stocks in early stages when we introduce the conjectures about the others' actions off the equilibrium path that is identical to the ones on the equilibrium path, compared to the outcome in the normal setting.

1 Introduction

Big institutional investors frequently have to sell/buy fixed amounts of securities until a certain date. Recently how to trade in such an execution problem has been attracting academic attentions. While many studies exogenously fix the behaviors of small traders, Ishii[17] explicitly modeled it, in addition to the institutional investor in a general equilibrium model.

The former studies give “price impact functions” as reduced form of behavior of small traders. That is, they specify, exogenously, how the price changes according to the order of the institutional investor. Most of the studies (for example, see Bertsimas and Lo [4]) show that the institutional investor executes the same number of securities in each period over the entire time span in this framework.

In contrast, Ishii [17] concludes that the price impact endogenously derived from the behavior of small investors is more complicated than the ones in the former. The orders placed by the institutional investor may be concentrated on the final period over a wide range of parameters in Ishii's [17] model. However, the trade concentration on the final period is not supported by empirical analyses. How can we obtain the outcome in which early trades are active within the framework of Ishii [17]?

This could be due to the assumption that the market participants expect others' actions in all information sets and update their beliefs rationally, which does not make much sense in the actual market. Rather, it is felt that actual market participants behave on the premise of less complex price impacts. Some practitioners insist that it is enough to adopt the impact function in Almgren and Criss [1]. The present situation may be that the market participants expect the others' strategies simply, despite the complex strategies in reality. So, in that case, it could be natural that market participants make decisions only due to the book in the past in similar circumstances.

The relationship between Ishii's [17] model and our model is reminiscent of the relationship between strong-form and semi-strong form in the context of information efficiency. If the price reflects all publicly and private information then the price is strong-form efficient. If it correctly reflects only public information then it is only semi-strong-form efficient. Ishii's [17] model analyzes the situation where the market participants rationally expect others' strategies, and in our model they only take into account the past price flows observable publicly.

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Let us think in perspective of the extensive-form game. Ishii [17] considers the normal sequential equilibrium. On the other hand, we assume the inconsistency between the beliefs and actions of the market participants. It is come across occasionally in the context of evolution of extensive-form game that the players construct some kinds of conjectures on the unobservable others' actions and choose best-responses against the conjectures. (See, for example, Nöldeke and Samuelson [13]). On the pattern of it, we assume that the market participants make their conjecture off the equilibrium path identified as the actions on the equilibrium path.

We show that the institutional investor sells more stocks in early stages in the stable state in the evolutionary dynamic than the equilibrium in the normal setting similar to Ishii [17].

2 Model

There are one institutional investor (which we denote by II in what follows) and many small investors (SI). They can trade at periods $t = 1, 2, \dots, T$. II has to sell W units of security over this time period. Trades occur at $t = 1, 2, \dots, T$, and those who have one unit of securities obtain dividends F_{T+1} at $t = T + 1$.

$$F_t = F_0 + \sum_{u=1}^t \varepsilon_{F_u}. \quad (t = 1, 2, \dots, T + 1),$$

where F_0 is observed by all traders at $t = 0$. F_t is the conditional expectation of the final dividends at the period t , and follows a random walk. That is, each ε_{F_t} ($t = 1, 2, \dots, T$) in the trading hours follows a normal distribution that has a mean 0 and a variance σ_F^2 at the beginning of the period t independently of each other. All traders observe ε_{F_t} and therefore F_t at the beginning of t , and observe $\varepsilon_{F_{T+1}}$ after the trading hours. The variances during market hours are the same as one another.

II places a market order S_t at $t = 1, 2, \dots, T$. We require $\sum_{t=1}^T S_t = W$. II is risk-neutral, and maximizes the expected value of his selling amount

$$\sum_{t=1}^T P_t S_t,$$

where P_t is the security price at t .

There are infinitely many SIs whose population is 1. The measure of each SI is assumed to be zero. These are of two types; irrational one (population μ) and rational one (population $1 - \mu$). SIs have no position at $t = 0$. They can borrow some money or securities and place limit orders at $t = 1, 2, \dots, T$. They face no liquidity constraint. The interest rate is 0 for simplicity. We denote the quantity held by a representative irrational (resp. rational) SI¹ at the end of t as B_t^{ir} (B_t^r), and the order at t is ΔB_t^{ir} (ΔB_t^r).

$$\begin{aligned} \Delta B_t^{ir} &= B_t^{ir} - B_{t-1}^{ir}, \text{ and} \\ \Delta B_t^r &= B_t^r - B_{t-1}^r. \end{aligned}$$

The irrational SIs place their orders “irrationally.” They maximize

$$E_t \left[-\exp \left\{ -\rho \sum_{u=1}^t (F_{T+1} - P_u) \Delta B_u^{ir} \right\} \right].$$

The subscript t means the conditional distribution on the information available until the time t .

The rational SIs place their orders to maximize

$$E_t \left[-\exp \left\{ -\rho \sum_{u=1}^T (P_{u+1} - P_u) B_u^r \right\} \right].$$

¹When all SIs place the identical orders, the aggregate SIs' order is the same as in the case where one trader with the risk aversion ρ places his order. So we can call this virtual trader a “representative SI.”

The price P_t is determined to balance buy and sell orders at every $t = 1, 2, \dots, T$:

$$S_t = \mu \Delta B_t^{ir} + (1 - \mu) \Delta B_t^r. \quad (1)$$

We make a mild assumption for the following analysis. The ex ante risk of the fundamentals is large in some measure²:

$$(T - 1) \sigma_F^2 + \sigma_{F_{T+1}}^2 \geq \frac{\mu}{2}. \quad (2)$$

3 Benchmark

Before considering the evolutionary decision making, we will derive a “one-shot” equilibrium in the benchmark model, where all market participants take into account the opponents’ strategies.

3.1 Decision Making of SIs

First, we consider SIs’ problem. They behave given the price flow, so that we can take the the first order conditions in one lump without considering the order of the induction steps.

We consider a linear equilibrium in the followings. Since a linear combination of variables that follow normal distributions also follows a normal distribution, the price P_t follows a normal distribution. In addition, all observations follow normal distributions, so that we can take it for granted that price P_t follows normal distributions conditional on the events by the beginning of t for all traders.

We consider the equilibrium in which ΔB_t^r and ΔB_t^{ir} depend only on $F_t - P_t$ (not on the individual realizations F_t or P_t), and $F_t - P_t$ is measurable at the period 0. The conditional expectation and the conditional variance of the representative irrational SI’s position at t ($t = 1, 2, \dots, T$) are

$$\begin{aligned} E_t \left[\sum_{u=1}^t (F_{T+1} - P_u) \Delta B_u^{ir} \right] &= \sum_{u=1}^t (F_t - P_u) \Delta B_u^{ir}, \text{ and} \\ Var_t \left[\sum_{u=1}^t (F_{T+1} - P_u) \Delta B_u^{ir} \right] &= \sum_{u=t}^T \sigma_{F_{u+1}}^2 (B_u^{ir})^2. \end{aligned}$$

The first order condition of the representative irrational SI is

$$\begin{aligned} -\rho \left(F_t - P_t - \rho \sum_{u=t}^T \sigma_{F_{u+1}}^2 B_u^{ir} \right) E_t \left[-\exp \{ -\rho (F_{T+1} - P_t) B_t^{ir} \} \right] &= 0 \\ \Rightarrow B_t^{ir} (P_t) &= \frac{F_t - P_t}{\rho \sum_{u=t}^T \sigma_{F_{u+1}}^2}. \end{aligned} \quad (3)$$

where $B_t^{ir}(\cdot)$ is the SI’s holding at the end of the period t as a function of P_t . $B_t^{ir}(P_t)$ (and the realized holding B_t^{ir}) are measurable in terms of the information he has at the beginning of the period t . We obtain the first order condition of the representative rational SI in a similar manner:

$$B_t^r (P_t) = \frac{E_t [P_{t+1}] - P_t}{\rho \sigma_{P_{t+1}}^2}. \quad (4)$$

Substituting (3) and (4) to the market clearing condition (1), we obtain

$$S_t = \mu \frac{F_t - P_t}{\rho \sum_{u=t}^T \sigma_{F_{u+1}}^2} - \mu \frac{F_{t-1} - P_{t-1}}{\rho \sum_{u=t-1}^T \sigma_{F_{u+1}}^2} + (1 - \mu) \frac{E_t [P_{t+1}] - P_t}{\rho \sigma_{P_{t+1}}^2} - (1 - \mu) \frac{E_{t-1} [P_t] - P_{t-1}}{\rho \sigma_{P_t}^2}. \quad (5)$$

²It is inconceivable that the company is liquidated without new information after the cutoff time. In this sense, the assumption that $\sigma_{F_{T+1}}^2$ is large enough seems natural.

3.2 Decision Making of II

We are supposed to solve the II's optimization problem by the backward induction. However such a solving method will make it difficult to derive the outcome analytically. So we focus on a linear equilibrium, replacing the problem with the one that "Before trading, II optimally determines the discrepancies between the actual prices and the fundamentals."³ Specifically, we define

$$FP_t = F_t - P_t \quad (t = 1, 2, \dots, T),$$

and consider the model where II set FP_t in an appropriate manner before trading hours. If we were to view F_t and P_t as separate variables, the outcome would be the same as the case where FP_t were a unified variable. Thus, the definition of FP_t does not affect the solution of this model.

II's decision making problem is:

$$\min_{(FP_t)_{t=1}^T} \sum_{t=1}^T FP_t S_t \quad \text{s.t.} \quad \sum_{t=1}^T S_t = W \quad \text{and (5)}. \quad (6)$$

By the market clearing condition (5), we can see the relationship between FP_t and S_t .

$$S_1 = \left(\frac{\mu}{(T-1)\rho\sigma_F^2 + \rho\sigma_{F_{T+1}}^2} + \frac{1-\mu}{\rho\sigma_F^2} \right) FP_1 - \frac{1-\mu}{\rho\sigma_F^2} FP_2, \quad \text{and} \quad (7)$$

$$S_t = - \left(\frac{\mu}{(T-t+1)\rho\sigma_F^2 + \rho\sigma_{F_{T+1}}^2} + \frac{1-\mu}{\rho\sigma_F^2} \right) FP_{t-1} + \left(\frac{\mu}{(T-t)\rho\sigma_F^2 + \rho\sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\rho\sigma_F^2} \right) FP_t - \frac{1-\mu}{\rho\sigma_F^2} FP_{t+1}, \quad (8)$$

for $t = 2, 3, \dots, T$.

By the first order condition, we obtain the following.

Theorem 1 *In the unique linear equilibrium, the discrepancies between the actual prices and the fundamentals are:*

$$FP_t = \frac{A_{t-1} \prod_{u=t}^{T-1} \left(\frac{\mu}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} \right)}{A_{T-1}} \rho\sigma_{T+1}^2 W,$$

where $A_0 = 1$, $A_1 = 2\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}$, and

$$A_{t+1} = \left(2\frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 4\frac{1-\mu}{\sigma_F^2} \right) A_t - \left(\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} \right)^2 A_{t-1}$$

FP_t is a nondecreasing function with respect to t . In addition to this, (7) and (8) tell us the increasing tendencies of the selling amounts of the security and the discrepancies between the actual prices and the fundamentals over time. The irrational SIs buy the securities by a little amount in early periods. The rational SIs sell by a little amount to the period before the last period, but buy back at a lower price at the last period, and their final positions are long. We see these with the simple calculations here. Eliminating A_t from the above equations, we obtain

$$FP_{t+2} - FP_{t+1} = FP_{t+1} - \frac{\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}}{\frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}} FP_t.$$

³If we were to solve this problem by the backward induction, we would obtain the same linear equilibrium. The author has finished analyses of simple models already. Anyone who wants to know further particulars can e-mail the author.

Note that

$$FP_t \geq 0, \text{ and } 0 < \frac{\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}}{\frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}} < 1,$$

$$\therefore FP_{t+2} - FP_{t+1} \geq FP_{t+1} - FP_t.$$

The speed of discrepancy increasing is increasing over time. As a result,

$$\Delta B_{t+2}^{ir} \geq \Delta B_{t+1}^{ir},$$

which means that the irrational SIs' orders are increasing over time. It also means

$$E_{t+1}[P_{t+2}] - P_{t+1} \leq E_t[P_{t+1}] - P_t,$$

so that

$$B_{t+1}^r \leq B_t^r \Leftrightarrow \Delta B_{t+1}^r \leq 0.$$

Thus, the rational SIs sell at $t = 1, 2, \dots, T-1$ increasingly over time as II.

It may be difficult to imagine the outcome. In the following sections, we see some distinctive examples.

3.3 $\mu = 1$, and $\sigma_F^2 = 0$

All SIs are irrational and there is no fluctuation of fundamentals before T . As in Bertsimas and Lo [4], II's optimal strategy is the "equal selling":

$$S_t = \frac{W}{T} \text{ (for all } t).$$

The discrepancy between the actual price and the fundamental at t is

$$FP_t = \frac{t}{T} \rho \sigma_{F_{T+1}}^2 W.$$

The range of price reduction $FP_{t+1} - FP_t$ are maintained constant. The limit orders that SIs place at t are

$$\begin{aligned} \Delta B_t &= \frac{FP_t}{\rho \sigma_{F_{T+1}}^2} - \frac{t-1}{T} W \\ &= \frac{FP_t}{\rho \sigma_{F_{T+1}}^2} - \sum_{u=1}^{t-1} S_u, \end{aligned}$$

where the market impact (the coefficient that describes the price change in relation to the amount of the II's order) stays constant for all t :

$$\rho \sigma_{F_{T+1}}^2.$$

The cumulative amount of securities executed before t

$$\sum_{u=1}^{t-1} S_u$$

has a permanent effect on the price. This can be interpreted as the inventory effect. SIs are risk-averse, and the price should be low when they have long positions. As buyers of trading before t , they already have large amounts of long positions. If II intends to sell more in this situation, then SIs need lower price to take more risk. As a result, the price at t becomes low in proportion to the cumulative amount of securities executed before t . Trade concentration work out to only a disadvantage after such period. Instead II would like to divide the amounts into many parts, and to sell off in pieces by little and little. II can sell the securities at a comparatively high price in the early stage in this case. These kinds of motives lead to II's evenly selling.

3.4 $\mu = 0$, and $\sigma_F^2 \neq 0$

All SIs are rational.⁴ We obtain the similar result as in Ishii [17]. II concentrates the trade on the final period.

$$\begin{aligned} S_1 = S_2 = \dots = S_{T-1} = 0, \text{ and} \\ S_T = W. \end{aligned}$$

The discrepancies between the actual prices and the fundamentals are constant.

$$FP_t = \rho\sigma_{F_{T+1}}^2 W, \text{ for all } t.$$

This result is attributed to the fact that the rational SIs are greedy. II has executed W until the end of T , and the position of the representative SI at the end of the final period is also W . The risk-averse SIs press to pay the fair share of the risk, and require the low price. The width of the price reduction from the fundamental at T is

$$FP_T = \rho\sigma_{F_{T+1}}^2 W,$$

independently of S_1, S_2, \dots, S_{T-1} . The trade does not take place at $t = 1, 2, \dots, T-1$ in the equilibrium and SIs' positions are 0. In order to maintain the level of 0 position, FP_t has to be the same as FP_{t+1} . Therefore the low price level at the final period sustains over the trading time.

What happens if II sell a part of securities before the final period? Let II sell $S_1 > 0$ at the period 1 and $S_T = W - S_1$ at T . Then, the positions of SIs are S_1 . Since SIs have long positions at the period 1, they expect that the (expected) price at the period 2 is greater than the price at 1. That is,

$$FP_2 - FP_1 = \rho\sigma_F^2 S_1.$$

Furthermore SIs still continue to have the long positions S_1 , so that

$$FP_{t+1} - FP_t = \rho\sigma_F^2 S_1,$$

holds for every $t = 1, 2, \dots, T-2$. On the other hand, the price at the final period is fixed at the level where

$$FP_T = \rho\sigma_{F_{T+1}}^2 W.$$

Thus, even if II brings the execution forward, the price at the final period does not change, he sells S_1 at a bargain price, and the expected revenue becomes small. This can be interpreted as the "inventory" effect in a way apart from the above example.

3.5 $T = 2$

There is the composite population of the irrational and the rational SIs. We consider the 2 period model to avoid complexity. It brings together features of the above two examples.

$$\begin{aligned} S_1 &= \frac{\mu}{2} \frac{\sigma_{F_{T+1}}^2}{\sigma_F^2 + \sigma_{F_3}^2} W, \\ S_2 &= \left(1 - \frac{\mu}{2} \frac{\sigma_{F_{T+1}}^2}{\sigma_F^2 + \sigma_{F_3}^2} \right) W, \\ FP_1 &= \frac{\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + 2\frac{1-\mu}{\sigma_F^2}}{2\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + 2\frac{1-\mu}{\sigma_F^2}} \rho\sigma_{F_3}^2 W, \text{ and} \\ FP_2 &= \rho\sigma_{F_3}^2 W. \end{aligned}$$

⁴The assumption that the fundamentals change in some small measure during the trading period does not have an essential role. We can conduce to the similar outcome in the case of $\sigma_F^2 = 0$. If we assume $\sigma_F^2 = 0$, however, II's any strategy is optimal. In order to avoid such a degenerated outcome, it is natural to assume this condition.

Since

$$\frac{\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + 2\frac{1-\mu}{\sigma_F^2}}{2\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + 2\frac{1-\mu}{\sigma_F^2}} \leq 1,$$

we obtain

$$FP_1 \leq FP_2.$$

The price at the period 2 is undervalued than the price at 1. Because of the assumption (2), this undervaluation $FP_2 - FP_1$ is increasing with respect to μ . The irrational SIs buy at the period 1, and buy at 2 because the price goes down further. The rational SIs have the same position at the end of the period 2, but sell at 1. They see a profit to buy back at the period 2 at a lower price. The irrational SIs work as the buy basis of the selling by II, so that the higher price P_1 than P_2 can be realized. However P_1 is depressed through the rational SIs' arbitrage. FP_2 remains independent of the behavior of II. The way of execution has a tendency of bundle sale on the final period. For every set of parameters,

$$S_1 \leq S_2,$$

holds.

4 Evolutionary Decision Making

In the above section, II rationally expects the actions of SIs and takes into consideration the opponents' strategies. We see the situation where all market participants play the $T + 1$ period trading again and again, and respond optimally from the myopic viewpoints in a sense to be shown later. Let us call the one play of the $T + 1$ period trading a *round*. The orders in the initial round are arbitrarily fixed. Every market participant places his order as one of best responses to realization in the last round. If there are multiple best responses, they choose one randomly with full support.⁵ SIs observe the price flow $(FP_t)_{t=1}^T$ in every round, and place their orders supposing that the price flow in the present round is the same as $(FP_t)_{t=1}^T$ in the last round. SIs' orders at the time t off the equilibrium path (the possible order that SIs would place at the unrealized price) are determined as the best responses to the same price flow as in the last round except the price at the period t . SI's limit orders at the period t are expressed as the function of FP_t , and II observe the sequence of the function.⁶ II best-responses to $\mu B_t^{ir}(P_t - F_t' + F_t) + (1 - \mu) B_t^r(P_t - F_t' + F_t)$ when the fundamentals in the last round are $(F_t')_{t=1}^T$ and when SIs' limit orders in the last round are $(\mu B_t^{ir}(\cdot) + (1 - \mu) B_t^r(\cdot))_{t=1}^T$. Any market participant does not modify his order plan in the middle of a round play.⁷

We can think of this process as a Markov process. A vector that lines up the infinitely many orders of II and SIs corresponds a state. When one state gets decided, (the probability distribution of) the next state gets decided. We interpret the absorbing state in this Markov process as the *stable* state in the evolutionary dynamic.

Theorem 2 *We can express the discrepancies between the actual prices and the fundamentals as a function of S_T , and obtain the unique absorbing state analytically. $(FP_t)_{t=1}^T$ are proportional to ρ , and $(S_t)_{t=1}^T$ are independent of ρ .*

Proof. See Appendix. ■

Since the equations are complex and there is little intuition, so that it may be better to skip the derivation of the explicit solution. In the following we see what happens in the evolutionary dynamic using the same examples as the benchmark models.

⁵Generally we have more than one best responses, but we can obtain unique best response when the orders in the last round can be thought of a pure action profile.

⁶The book II can observe at each period t is only function of FP_t . It does not depend on FP_u ($u \neq t$) because it can not reflect the price at other period than t .

⁷Imagine the situation where one can beforehand send his order to a securities company, but can not monitor conditions of market during market hours (for example, with his business).

4.1 $\mu = 1$, and $\sigma_F^2 = 0$

All SIs are irrational and there is no fluctuation of fundamentals. II sells the securities earlier in the absorbing state:

$$S_t = \frac{2^{T-t}}{2^T - 1} W, \text{ and}$$

$$FP_t = \frac{2^T - 2^{T-t}}{2^T - 1} \rho \sigma_{F_{T+1}}^2 W$$

(for $t = 1, 2, \dots, T$).

It would help the promotion of understanding to consider the following story. The initial state is the equilibrium actions of the above benchmark model. That is, II sells evenly and the prices ratchet down. Observing this outcome, II think about the order flow in the next round “I sold evenly and the price decreased with time gradually. If it will be all the same in the next round, I do not have to have large inventory till late. What would happen if I were to sell in earlier period a part of securities that would be supposed to be sold at the final period. It is true that the price in the earlier period would be depressed, but this loss can be made up for by the advance of the price at the final period through the decrease in the selling amount. More importantly, the price of the securities moved up the schedule for should become higher. I guess that the entire selling plan is well organized.” Realistically there are no rise in price at the final period because of the inventory effect. II who believes that the order flow in the next round will be the same as the one in the last round makes the selling period moved forward. SIs observe II’s selling moved forward, and place the limit orders that comply with it. II’s next order complies with the SIs’ order that he observe... By process like this it converges with the absorbing state above.

4.2 $\mu = 0$, and $\sigma_F^2 = \sigma_{F_{T+1}}^2$

All SIs are rational. We assume here $\sigma_F^2 = \sigma_{F_{T+1}}^2$ for simplicity. II’s earlier selling achieves in the absorbing state as was expected:

$$S_t = \frac{\frac{1+\sqrt{5}}{2} \left(\frac{3+\sqrt{5}}{2}\right)^{t-1} - \frac{1-\sqrt{5}}{2} \left(\frac{3-\sqrt{5}}{2}\right)^{t-1}}{\left(\frac{3+\sqrt{5}}{2}\right)^{T-1} - \left(\frac{3-\sqrt{5}}{2}\right)^{T-1}} W, \text{ and}$$

$$FP_t = \left[\left\{ \left(\frac{3+\sqrt{5}}{2}\right)^T - \left(\frac{3-\sqrt{5}}{2}\right)^T \right\} - (T-t) \left\{ \left(\frac{3+\sqrt{5}}{2}\right)^{t-1} - \left(\frac{3-\sqrt{5}}{2}\right)^{t-1} \right\} \right. \\ \left. + (T-t+1) \left\{ \left(\frac{3+\sqrt{5}}{2}\right)^{t-2} - \left(\frac{3-\sqrt{5}}{2}\right)^{t-2} \right\} \right] \cdot \frac{\rho \sigma_F^2}{\left(\frac{3+\sqrt{5}}{2}\right)^{T-1} - \left(\frac{3-\sqrt{5}}{2}\right)^{T-1}} W$$

(for $t = 1, 2, \dots, T$).

Unlike the above example, the prices increase over time.

4.3 $T = 2$

When (2) holds, II sells earlier:

$$\begin{aligned}
S_1 &= \frac{2\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + \frac{1-\mu}{\sigma_F^2}}{2 + \frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + \frac{1-\mu}{\sigma_F^2}} \sigma_{F_3}^2 W, \\
S_2 &= \left(1 - \frac{2\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + \frac{1-\mu}{\sigma_F^2}}{2 + \frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + \frac{1-\mu}{\sigma_F^2}} \sigma_{F_3}^2 \right) W, \\
FP_1 &= \frac{2 + \frac{\frac{1-\mu}{\sigma_F^2}}{\frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + \frac{1-\mu}{\sigma_F^2}} + \frac{1-\mu}{\sigma_F^2}}{2 + \frac{\mu}{\sigma_F^2 + \sigma_{F_3}^2} + \frac{1-\mu}{\sigma_F^2}} \rho \sigma_{F_{T+1}}^2 W, \text{ and} \\
FP_2 &= \rho \sigma_{F_3}^2 W.
\end{aligned}$$

S_1 is increasing with μ . We can not describe magnitude correlation between FP_1 and FP_2 categorically. However FP_1 is increasing with μ , and if μ is greater than the threshold level, $FP_1 > FP_2$ holds. That is, the greater the proportion of the rational SIs is, the lower the price at the period 1, resulting in the buy orders of both irrational and rational SIs against II's sell order.

5 Conclusion

We have considered a multiperiod model of securities trading in an evolutionary setting. We show that earlier selling in the case of the evolutionary decision making indeed takes place, and the price becomes lower.

6 Appendix

6.1 Proof of Theorem 1

Note that

$$S_T = W - \sum_{t=1}^{T-1} S_t.$$

The first order conditions of (6) with respect to $FP_1, FP_2, \dots, FP_{T-1}$ are:

$$\begin{aligned}
&\left(2\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} \right) FP_1 - \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} \right) FP_2 = 0, \text{ and} \\
&- \left(\frac{\mu}{(T-t+1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} \right) FP_{t-1} + \left(2\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 4\frac{1-\mu}{\sigma_F^2} \right) FP_t \\
&\quad - \left(\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} \right) FP_{t+1} = 0 \quad (t = 2, 3, \dots, T-1).
\end{aligned}$$

When we rewrite these equations using matrix expression, the above first order conditions are

$$\begin{bmatrix} 2\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} & -\left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right) & \cdots & 0 \\ -\left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right) & 2\frac{\mu}{(T-2)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} + 2\frac{1-\mu}{\sigma_F^2} & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 2\frac{\mu}{\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} + 2\frac{1-\mu}{\sigma_F^2} \end{bmatrix} \cdot \begin{bmatrix} F_1 - P_1 \\ F_2 - P_2 \\ \vdots \\ F_{T-1} - P_{T-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \left(\frac{\mu}{\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right)\rho\sigma_{T+1}^2 W \end{bmatrix}. \quad (9)$$

Now we define

$$A_0 = 1, \quad A_1 = 2\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2},$$

$$A_{t+1} = \left(2\frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 4\frac{1-\mu}{\sigma_F^2}\right)A_t - \left(\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right)^2 A_{t-1},$$

and the determinant of the matrix in left side of (9) is A_{T-1} . If $\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2} > 1$, then A_t is increasing with t . In fact,

$$A_1 > \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right)A_0 > A_0,$$

and when we assume

$$A_{t+1} > \left(\frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right)A_t > A_t,$$

then

$$A_{t+2} > \left(\frac{\mu}{(T-t-2)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right)A_{t+2},$$

holds. By Cramer's rule,

$$FP_t = \frac{A_{t-1} \prod_{u=t}^{T-1} \left(\frac{\mu}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 2\frac{1-\mu}{\sigma_F^2}\right)}{A_{T-1}} \rho\sigma_{T+1}^2 W,$$

holds.

6.2 Proof of Theorem 2

Let $(S_t)_{t=1}^T$ be II's strategy in the stable state. We consider II's strategy change into $(S'_t)_{t=1}^T$. The differences between $(S_t)_{t=1}^T$ and $(S'_t)_{t=1}^T$ are only at the period t and u ($t < u$). We denote $S'_t = S_t + \Delta$, and $S'_u = S_u - \Delta$. Now II has to minimize the following under $\Delta = 0$.

$$\begin{aligned} & FP'_t S'_t + FP'_u S'_u - FP_t S_t - FP_u S_u \\ &= \left(FP_t + \frac{\Delta}{\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2}} \right) (S_t + \Delta) + \left(FP_u - \frac{\Delta}{\frac{\mu}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2}} \right) (S_u - \Delta) - FP_t S_t - FP_u S_u \\ &= FP_t \Delta + \frac{\Delta}{\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2}} (S_t + \Delta) - FP_u \Delta - \frac{\Delta}{\frac{\mu}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2}} (S_u - \Delta). \end{aligned}$$

Differentiating it with respect to Δ , and making equal to 0 under $\Delta = 0$, we obtain

$$FP_t + \frac{1}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} S_t - FP_u - \frac{1}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} S_u = 0, \text{ and}$$

$$\frac{1}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} S_t + FP_t = \frac{1}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} S_u + FP_u = \rho\sigma_{F_{T+1}}^2 S_T + FP_T.$$

When we express them as the function of (FP_t) , we obtain

$$2 \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) FP_1 - \frac{1-\mu}{\sigma_F^2} FP_2 = \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) \rho\sigma_{F_{T+1}}^2 (S_T + W), \text{ and}$$

$$- \left(\frac{\mu}{(T-t+1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) FP_{t-1} + \left(2 \frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 3 \frac{1-\mu}{\sigma_F^2} \right) FP_t - \frac{1-\mu}{\sigma_F^2} FP_{t+1}$$

$$= \left(\frac{\mu}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) \rho\sigma_{F_{T+1}}^2 (S_T - W),$$

which can be represented by a matrix form:

$$\begin{bmatrix} 2 \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) & -\frac{1-\mu}{\sigma_F^2} & \cdots & 0 \\ - \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) & 2 \frac{\mu}{(T-2)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 3 \frac{1-\mu}{\sigma_F^2} & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 2 \frac{\mu}{\sigma_F^2 + \sigma_{F_{T+1}}^2} + 3 \frac{1-\mu}{\sigma_F^2} \end{bmatrix} \cdot \begin{bmatrix} FP_1 \\ FP_2 \\ \vdots \\ FP_{T-1} \end{bmatrix} = \begin{bmatrix} \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) \rho\sigma_{F_{T+1}}^2 (S_T + W) \\ \left(\frac{\mu}{(T-2)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) \rho\sigma_{F_{T+1}}^2 (S_T + W) \\ \vdots \\ \left(\frac{\mu}{\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) \rho\sigma_{F_{T+1}}^2 (S_T + W) - \frac{1-\mu}{\sigma_F^2} \rho\sigma_{F_{T+1}}^2 W \end{bmatrix} \rho\sigma_{F_{T+1}}^2.$$

We do the elementary transformation of the matrix with respect to rows in the left side of the equation to obtain the inverse matrix. First, we have to set up:

$$I. \begin{bmatrix} 2 \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) & -\frac{1-\mu}{\sigma_F^2} & \cdots & 0 \\ - \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) & 2 \frac{\mu}{(T-2)\sigma_F^2 + \sigma_{F_{T+1}}^2} + 3 \frac{1-\mu}{\sigma_F^2} & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 2 \frac{\mu}{\sigma_F^2 + \sigma_{F_{T+1}}^2} + 3 \frac{1-\mu}{\sigma_F^2} \end{bmatrix}$$

When we define

$$B_0 = 1, \text{ and}$$

$$B_{t+1} = \frac{\frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2}}{2 \frac{\mu}{(T-t-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + (3 - B_t) \frac{1-\mu}{\sigma_F^2}},$$

$\frac{1}{3} < B_t \leq \frac{1}{2}$ ($t \neq 0$) holds and B_t is decreasing with respect to t . Multiplying the t th row by B_t and adding to the $t + 1$ th row from top to bottom, we obtain

$$\begin{bmatrix} 2 \left(\frac{\mu}{(T-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) & -\frac{1-\mu}{\sigma_F^2} & \cdots & 0 \\ 0 & 2 \frac{\mu}{(T-2)\sigma_F^2 + \sigma_{F_{T+1}}^2} + (3 - B_1) \frac{1-\mu}{\sigma_F^2} & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 2 \frac{\mu}{\sigma_F^2 + \sigma_{F_{T+1}}^2} + (3 - B_{T-2}) \frac{1-\mu}{\sigma_F^2} & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ B_1 & 1 & & \\ \vdots & & \ddots & \vdots \\ \prod_{t=1}^{T-2} B_t & \prod_{t=2}^{T-2} B_t & \cdots & 1 \end{bmatrix}.$$

Next, we multiply the t th row by $\frac{\frac{1-\mu}{\sigma_F^2}}{(T-t)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} B_t$ and add it to the $t - 1$ th row from bottom to top. In this way we obtain the inverse matrix whose i, j element is

$$\frac{\sum_{u=i}^{T-1} \left(\prod_{v=i}^{u-1} \frac{\frac{1-\mu}{\sigma_F^2}}{(T-v-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} B_{v+1} \right) \left(\prod_{v=j}^{u-1} B_v \right)}{\sum_{u=i}^{T-1} \left(\prod_{v=i}^{u-1} \frac{\frac{1-\mu}{\sigma_F^2}}{(T-v-1)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} B_{v+1} \right) \left(2 \frac{\mu}{(T-u)\sigma_F^2 + \sigma_{F_{T+1}}^2} + (3 - B_{u-1}) \frac{1-\mu}{\sigma_F^2} \right)}$$

Therefore

$$FP_t = \rho \sigma_{T+1}^2 \sum_{j=1}^{T-1} \left\{ \left(\frac{\mu}{(T-j)\sigma_F^2 + \sigma_{F_{T+1}}^2} + \frac{1-\mu}{\sigma_F^2} \right) (S_T - W) + \chi_{j=T-1} \frac{1-\mu}{\sigma_F^2} W \right\} \text{ (the } t, j \text{ element of the matrix),}$$

where $1 \leq i, j \leq T - 1$ holds and

$$\chi_{j=T-1} = \begin{cases} 1 & \text{if } j = T - 1 \\ 0 & \text{if } j \neq T - 1 \end{cases},$$

$$\prod_{v=u}^{u-1} \text{(equation of some sort)} = 1, \text{ and}$$

$$\prod_{v=u'}^{u-1} \text{(equation of some sort)} |_{u' > u} = 0.$$

S_t ($t = 1, 2, \dots, T - 1$) can be represented as $(FP_t)_{t=1}^T$, so that we obtain the prices and II's strategy in the absorbing state using the relational expressions

$$\sum_{t=1}^T S_t = W, \text{ and } FP_T = \rho \sigma_{F_{T+1}}^2 W.$$

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