KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE
OF
ECONOMIC RESEARCH

Discussion Paper No. 643

"Reexamination on Updating Choquet Beliefs"

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November 2007

KYOTO UNIVERSITY

KYOTO, JAPAN

Reexamination on Updating Choquet Beliefs

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Version: November 21, 2007

Eichberger, Grant, and Kelsey (2007) characterize the full Bayesian update rule for capacities. This paper shows that a conditional preference relation represented by the Choquet expected utility with respect to the updated capacity through the rule does not satisfy the axiom of Conditional Certainty Equivalence Consistency. A counterexample is provided and it is proved that a relaxation of the axiom maintains their results.

JEL Classification: D81

Keywords: Bayesian update, capacity, conditional preference, Choquet expected utility

1. INTRODUCTION

Recent work by Eichberger, Grant and Kelsey (2007) provides an axiomatic foundation for an updating rule for capacities, which is called the full Bayesian update rule. This rule itself is originally proposed by Dempster (1967) and called the Dempster-Fagin-Halpern rule (Fagin and Halpern, 1991), or the *generalized Bayes rule* (Walley, 1991). On the other hand, the full Bayesian update rule for a set of priors is sometimes called the belief-by-belief updating, which is axiomatized by Pires (2002).

The main axiom for a characterization of Eichberger et al. is called *Conditional Certainty Equivalent Consistency* (CCEC), which assures the existence of a certainty equivalent outcome that connects conditional and unconditional preference relations. The authors prove that the axiom of CCEC together with other axioms are sufficient for conditional capacities to be updated by the DFH update rule.

This paper achieves a necessary improvement in their results. At first, it is shown that, a conditional preference relation represented by the Choquet expected utility with respect to the updated capacity through the DFH rule, does not satisfy the axiom of CCEC. A counterexample is provided. Furthermore, after careful consideration, it is confirmed that they proved the sufficient part only by binary acts (conditional on the realized event). Although the necessary part of the proof was left to the readers, we examine whether the axiom is satisfied or not, and concludes that the relationship in CCEC cannot be satisfied by *all acts*, but *all binary acts*. The main result of this paper proposes a relaxation of the axiom of CCEC to maintain their contributions.

[1]I appreciate Jürgen Eichberger, Simon Grant, and David Kelsey for helpful comments and suggestions, and Atsushi Kajii for many constructive discussions. This research is mainly conducted under the 21th Century COE Research Program, which I thank for the support and encouragement. Remaining errors in this paper are, of course, my own responsibility.

2. BASICS AND AXIOMS

We basically follow the set up introduced by Eichberger, Grant and Kelsey (2007). However, for convenience, we adopt slight simplifications in some definitions.

Let S be a finite set with $|S| = n$ and $\mathcal{E}$ be the set of all subsets of S, that is $\mathcal{E} = 2^S$. A nonempty set $E \in \mathcal{E}$ is called an *event*. E^c indicates the complement of E with respect to S. The set of *outcomes* is denoted by $X = [\underline{x}, \overline{x}]$ with $\underline{x} < \overline{x}$. A function $f : S \to X$ is called an *act*. Let $\mathcal{F}$ be the set of all acts. Every $x \in X$ is considered as a constant act, $f(s) = x$ for all $s \in S$. $f_E g$ is the act which generates $f(s)$ if $s \in E$ and $g(s)$ if $s \in E^c$.

Given an act $f \in \mathcal{F}$, let $R(f) \subset X^n$ be the range of f. Define the set of *binary acts* by $\mathcal{F}^2 = \{f \in \mathcal{F} \mid \dim R(f) \leqq 2\}$. Any binary act in $\mathcal{F}^2$ can be expressed by some b, $w \in X$, $b \geqq w$ and $A \in \mathcal{E}$, denoted by $b_A w$, which is also called a *binary gamble on A*.

A finite set function $\nu : \mathcal{E} \to [0, 1]$ is called a *capacity* on S if it satisfies (i) $\nu(\varnothing) = 0$ and $\nu(S) = 1$, and (ii) for every A and B in $\mathcal{E}$ with $A \subset B$, $\nu(A) \leqq \nu(B)$. A capacity ν is said to be *convex* if for every A and B in $\mathcal{E}$, $\nu(A \cup B) + \nu(A \cap B) \geqq \nu(A) + \nu(B)$. On the other hand, a capacity ν is *additive* if for every for every A and B in $\mathcal{E}$ the previous inequality holds by equality. Let Δ be the set of all additive capacities, that is probability distributions. For future reference, it will be useful to define the *core* of a capacity ν, $C(\nu) \equiv \{p \in \Delta \mid p(A) \geqq \nu(A) \text{ for all } A \in \Sigma\}$.

Given an event $E \in \mathcal{E}$, a *conditional* or *updated capacity* ν_E is a capacity *on* E, i.e. for all A in $\mathcal{E}$ with $A \cap E = E$, $\nu_E(A) = 1$. Note that for any event E in $\mathcal{E}$, ν_E has domain $\mathcal{E}$. When $E = S$, ν_S is interpreted as the *unconditional capacity* and we simply write it as ν. Let $\int f \, d\nu$ denote the Choquet integral of f with respect to ν.

Given each event $E \in \mathcal{E}$, let $\succsim_E$ be a *conditional preference relation on $\mathcal{F}$ given E*. As usual $\succ_E$ and $\sim_E$ represent the asymmetric and symmetric part of $\succsim_E$ respectively. When $E = S$, $\succsim_S$ is considered as the *unconditional preference relation on $\mathcal{F}$*, simply denoted by $\succsim$. Let $\langle \succsim_E \rangle_{E \in \mathcal{E}}$ be a collection of conditional preference relations.

Given an event $E \in \mathcal{E}$ and a conditional preference $\succsim_E$, event $A \in \mathcal{E}$ is called $\succsim_E$*-null* if for all f and g in $\mathcal{F}$, $f_A g \sim_E g$. Let $\mathcal{N}_E$ be the set of $\succsim_E$-null events and write $\mathcal{N}_S$ as $\mathcal{N}$. Given $\succsim_E$, an event $A \in \mathcal{E}$ is said to be *non-null* iff $A \notin \mathcal{N}_E$.

Throughout this work, it is assumed that every $\succsim_E$ of $\langle \succsim_E \rangle_{E \in \mathcal{E}}$ has following representation:

Definition 1. The set of conditional preference relations $\langle \succsim_E \rangle_{E \in \mathcal{E}}$ is said to constitute a *collection of CEU preferences* if for each $\succsim_E$, there exists a capacity ν_E on $\mathcal{E}$ and a continuous non-constant real-valued function u_E on X such that for all f, $g \in \mathcal{F}$

$$f \succsim_E g \Longleftrightarrow \int u_E \circ f \, d\nu_E \geqq \int u_E \circ g \, d\nu_E.$$

When $\succsim_E$ is represented by a Choquet expected utility with respected to u_E and ν_E, we simply say $\succsim_E$ is *represented by* (u_E, ν_E). Since u_E is non-constant, it is compatible to assume that for all $E \in \mathcal{E}$, $\overline{x} \succ_E \underline{x}$. We normalize u_E so that $u_E(\underline{x}) = 0$ and $u_E(\overline{x}) = 1$ for all $E \in \mathcal{E}$ because the Choquet expected utility is unique up to positive linear transformations. We also write u_S as u, that is, $\succsim$ is represented by (u, ν).

In the main result, Eichberger, Grant and Kelsey (2007) prove that the following three axioms are necessary and sufficient for each conditional preference $\succsim_E$ given a non-null event $E \in \mathcal{E}$ represented by (u, ν_E) where

$$\nu_E(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} \quad \text{for every } A \in \mathcal{E}. \tag{1}$$

The result gives a characterization for an update rule for ν, which is called the full Bayesian update rule, or the Dempster-Fagin-Halpern update rule (Dempster, 1967, and Fagin and Halpern, 1991) for capacities.

Their three axioms are formally stated as below:

Axiom 1 (Consequentialism) Fix an event $E \in \mathcal{E}$. The event E^c is $\succsim_E$-null. That is, $f_E g \sim_E f$ for all $f, g \in \mathcal{F}$.

Axiom 2 (State Independence) For any pair of outcomes x, y in X, and any event $E \in \mathcal{E}$, $x \sim y$ if and only if $x \sim_E y$.

Axiom 3 (Conditional Certainty Equivalent Consistency) For any unconditionally non-null event E, any outcome x in X, and any act f in $\mathcal{F}$, $f \sim_E x$ if and only if $f_E x \sim x$.

Even if $\succsim_E$ is represented by (u, ν_E) where ν_E is updated by (1), some acts in $\mathcal{F}$ cannot fulfill the requirement of Axiom 3. Let us find them in the following counterexample.

Let S consist of six states $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The set of outcomes is assumed to be $X = [0, 1]$. Suppose that $\succsim$ is represented by (u, ν) such that:

- $u(x) = x$.
- $\nu(A) = \frac{|A|^2}{36}$ for every $A \subset S$.

This ν is a capacity on $\mathcal{E}$: (i) $\nu(\varnothing) = 0$ and $\nu(S) = 1$ (ii) for every A, $B \in \mathcal{E}$ with $A \subset B$, $\nu(A) \leqq \nu(B)$, since $|A|^2 \leqq |B|^2$. In fact, ν is *convex*, which is verified as follows. For any A, $B \in \mathcal{E}$, $|A \cup B| + |A \cap B| = |A| + |B|$. Then

$$\begin{aligned}
&(|A \cup B| + |A \cap B|)^2 - (|A| + |B|)^2 \\
&= |A \cup B|^2 + |A \cap B|^2 - \left\{|A|^2 + |B|^2 + 2(|A| - |A \cap B|)(|B| - |A \cap B|)\right\} \\
&\geqq |A \cup B|^2 + |A \cap B|^2 - \left\{|A|^2 + |B|^2\right\} \\
&\geqq 0.
\end{aligned}$$

Suppose that an event $E = \{s_1, s_2, s_3\}$ is observed and $\succsim_E$ is represented by (u, ν_E) where ν_E is updated via (1). The updated capacity ν_E is computed as follows: for every $A \subset E$

$$\begin{aligned}
\nu_E(A) &= \frac{\nu(A)}{\nu(A) + 1 - \nu(E^c \cup A)} \\
&= \frac{|A|^2}{|A|^2 + 36 - (|A| + 3)^2} \\
&= \begin{cases} \frac{1}{21} & \text{if } |A| = 1 \\ \frac{4}{15} & \text{if } |A| = 2 \\ 1 & \text{if } A = E \end{cases} .
\end{aligned}$$

Now consider an act $f = 1_{\{s_1\}}\varepsilon_{\{s_2\}}0$ where $\varepsilon > 0$ is sufficiently small. Suppose $x^f \in X$ satisfies $x^f \sim f_E x^f$. Then we have $\int x^f \, d\nu = \int f_E x^f \, d\nu$, which is calculated as

$$\begin{aligned} x^f &= \nu(\{s_1\}) + [\nu(E^c \cup \{s_1\}) - \nu(\{s_1\})] \, x^f \\ &\quad + [\nu(E^c \cup \{s_1, s_2\}) - \nu(E^c \cup \{s_1\})] \, \varepsilon \\ 36x^f &= 1 + 15x^f + 9\varepsilon \\ x^f &= \tfrac{1}{21} + \tfrac{135}{15\times 21}\varepsilon. \end{aligned}$$

Since $\succsim_E$ is represented by (u, ν_E) where ν_E is updated by (1), $\succsim_E$ is expected to satisfy Axiom 3. However, in this case, we have $x^f \succ_E f$, since

$$\begin{aligned} \int f \, d\nu_E &= \tfrac{1}{21} + \left[\tfrac{4}{15} - \tfrac{1}{21}\right] \varepsilon \\ &= \tfrac{1}{21} + \tfrac{69}{15\times 21}\varepsilon. \end{aligned}$$

One may question what the posterior set updated by the belief-by-belief updating, say $P_E \subset \Delta$, is like in this example. Formally, the full Bayes rule (the belief-by-belief update rule) is defined as follows. Given a set of priors $P \subset \Delta$ and an event E such that $p(E) > 0$ for all $p \in P$, the set of posteriors $P_E \subset \Delta$ is

$$P_E = \{p_E \in \Delta \mid p \in P\} \text{ where } p_E(A) = \frac{p(A \cap E)}{p(E)} \text{ for any } A \in \mathcal{E}.$$

For comparison, let $P = C(\nu)$. According to calculations above, $C(\nu_E)$ becomes the convex hull of six vertices. Let us take one vertex, $\widehat{p}_E = \left(\frac{1}{21}, \frac{23}{105}, \frac{11}{15}, 0, 0, 0\right)$. If $\widehat{p}_E \in P_E$, then there exists a $\widehat{p} \in P$ such that $\widehat{p}_E$ is obtained by updating $\widehat{p}$ via the Bayes' rule, where $\widehat{p}(E) = \frac{21}{36}$ and $\widehat{p}(E^c) = \frac{15}{36}$, that is

$$\widehat{p} = \left(\tfrac{1}{36}, \tfrac{4.6}{36}, \tfrac{15.4}{36}, \widehat{p}(\{s_4\}), \widehat{p}(\{s_5\}), \widehat{p}(\{s_6\})\right).$$

However, for $\widehat{p}$ to belong to $C(\nu)$, it has to satisfy for every $i, j = 4, 5, 6$, $i \neq j$

$$\widehat{p}(\{s_1, s_2, s_i, s_j\}) = \tfrac{5.6}{36} + \widehat{p}(\{s_i\}) + \widehat{p}(\{s_j\}) \geqq \tfrac{16}{36},$$

which requires that $\widehat{p}(E^c) \geqq \frac{15.6}{36}$. However, by assumption, $\widehat{p}(E^c) = \frac{15}{36}$, hence $\widehat{p} \notin C(\nu)$. The same argument can be applied to every other vertex, and so P_E includes *none* of them. It is illustrated in Figure 1.

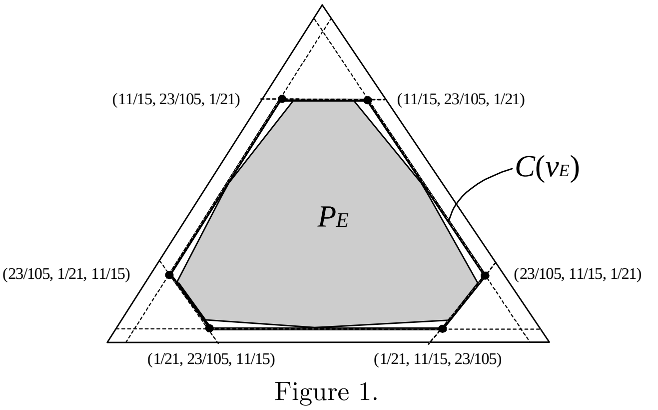

Figure 1.

3. ALTERNATIVE AXIOM AND RESULT

To conform to the fact in the previous section, we propose the following axiom which allows the domain $\mathcal{F}$ in Axiom 3 to be relaxed to the set of binary act $\mathcal{F}^2$.

Axiom 4 (Conditional Certainty Equivalent Consistency for Binary Gambles) For any unconditionally non-null event E, any outcome x in X, and any binary act f in $\mathcal{F}^2$, $f \sim_E x$ if and only if $f_E x \sim x$.

The following theorem proves that the result of Eichberger, Grant and Kelsey (2007) is preserved under Axiom 4.

THEOREM 1. *Suppose that $\langle \succsim_E \rangle_{E \in \mathcal{E}}$ constitutes a collection of CEU preferences represented by (u_E, ν_E) for each $E \in \mathcal{E}$. For any non-null event $E \notin \mathcal{N}$, the following two statements are equivalent:*
(i) $\succsim_E$ satisfies Axiom 1, 2 and 4.
(ii) $\succsim_E$ is represented by (u, ν_E): for all f and g in $\mathcal{F}$

$$f \succsim_E g \;\Leftrightarrow\; \int u \circ f \; d\nu_E \geqq \int u \circ g \; d\nu_E$$

where for every $A \in \mathcal{E}$, $\nu_E(A)$ is well-defined and

$$\nu_E(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)}. \tag{2}$$

Proof. **(i)⇒(ii)** Take any non-null event $E \notin \mathcal{N}$ and assume (i). Since $\succsim_E$ is represented by (u_E, ν_E) and satisfies Axiom 1, ν_E is a capacity on E.

Step 1: (To show that for all $A \in \mathcal{E}$, $\nu(A \cap E) + 1 - \nu(E^c \cup A) > 0$.)

Since ν is a capacity on S, both $\nu(A \cap E)$ and $1 - \nu(E^c \cup A)$ are non-negative. Therefore, it is sufficient to show that for all $A \in \mathcal{E}$, $\nu(A \cap E) > 0$ or $1 - \nu(E^c \cup A) > 0$.

To lead contradiction, assume that there exists an event $A \in \mathcal{E}$ such that $\nu(A \cap E) = 0$ and $\nu(E^c \cup A) = 1$. Consider an act $\overline{x}_{A\cap E}\underline{x}$. Then, $\int u \circ \overline{x}_{A\cap E}\underline{x} \; d\nu = \nu(A \cap E) = 0$ by assumption, hence $\overline{x}_{A\cap E}\underline{x} \sim \underline{x}$. In addition, consider $\overline{x}_{A\cap E}\underline{x}_{A^c\cap E}\overline{x}$. We have $\int u \circ \overline{x}_{A\cap E}\underline{x}_{A^c\cap E}\overline{x} \; d\nu = \nu(E^c \cup A) = 1$, hence $\overline{x}_{A\cap E}\underline{x}_{A^c\cap E}\overline{x} \sim \overline{x}$. However, by Axiom 4 and $\overline{x}_{A\cap E}\underline{x} \in \mathcal{F}^2$, we have $\overline{x}_{A\cap E}\underline{x} \sim_E \underline{x}$ and $\overline{x}_{A\cap E}\underline{x} \sim_E \overline{x}$, which contradicts $\overline{x} \succ_E \underline{x}$.

By Axiom 4, it is also verified that $\nu_E(A \cap E) = 0$ if $\nu(A \cap E) = 0$, and $\nu_E(A \cap E) = 1$ if $\nu(E^c \cup A) = 1$.

Step 2: (To show that $\nu_E(A)$ is equal to (2).)

Take an arbitrary $A \in \mathcal{E}$ and consider an act $\overline{x}_{A\cap E}\underline{x}$. We are interested in an $x \in X$ satisfying $x \sim \overline{x}_{A\cap E}\underline{x}_{A^c\cap E}x$, which is expressed in the following equation:

$$u_E(x) = \nu(A \cap E) + u_E(x)\left[\nu(E^c \cup A) - \nu(A \cap E)\right]. \tag{3}$$

By the argument in Step 1, we have $\nu(A \cap E) + 1 - \nu(E^c \cup A) > 0$, hence for any $A \in \mathcal{E}$, (3) has a solution, say x^A. On the other hand, by Axiom 4, we also have $x^A \sim_E \overline{x}_{A \cap E} \underline{x}$, which comes to $u_E\left(x^A\right) = \nu_E\left(A \cap E\right)$. Then

$$\nu_E\left(A \cap E\right) = \frac{\nu\left(A \cap E\right)}{\nu\left(A \cap E\right) + 1 - \nu\left(E^c \cup A\right)} \quad \text{for every } A \in \mathcal{E}.$$

Furthermore, by Axiom 1, we have $E^c \in \mathcal{N}_E$, $\overline{x}_A \underline{x} \sim_E \left(\overline{x}_A \underline{x}\right)_E x^A$, hence $\nu_E\left(A\right) = \nu_E\left(A \cap E\right)$.

Step 3: (To show that $\succsim_E$ is represented by (u, ν_E).)

By Step 1, $\nu\left(A \cap E\right) + 1 - \nu\left(E^c \cup A\right) > 0$ for every $A \in \mathcal{E}$. Any binary act in $\mathcal{F}^2$ is expressed in the form of a binary gamble on A, $b_A w$, where b, $w \in X$ with $b \geqq w$ and $A \in \mathcal{E}$. By Axiom 4, we have

$$\begin{aligned}
& x \sim (b_A w)_E x \\
& \Leftrightarrow \int u \circ x \; d\nu - \int u \circ b_{A \cap E} w_{A^c \cap E} x \; d\nu = 0 \qquad \text{(by CEU)} \\
& \Leftrightarrow \left[\nu\left(A \cap E\right) + 1 - \nu\left(E^c \cup A\right)\right] \times \\
& \qquad \left\{u\left(x\right) - \tfrac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} u\left(b\right) + \tfrac{1 - \nu(E^c \cup A)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} u\left(w\right)\right\} = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
& x \sim_E b_A w \\
& \Leftrightarrow \int u_E \circ x \; d\nu_E - \int u_E \circ b_A w \; d\nu_E = 0 \qquad \text{(by CEU)} \\
& \Leftrightarrow \left\{u_E\left(x\right) - \tfrac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} u_E\left(b\right) + \tfrac{1 - \nu(E^c \cup A)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} u_E\left(w\right)\right\} = 0.
\end{aligned}$$

Therefore, due to Axiom 4, $\succsim_E$ can be represented by (u, ν_E) where ν_E is defined in (2).

(ii)⇒(i) Assume (ii). If $E \in \mathcal{N}_E$, then $\succsim_E$ satisfies Axiom 1 and 2 directly by definition. Thus assume $E \notin \mathcal{N}_E$.

Axiom 1 is straightforward from the fact that ν_E is a capacity on E.

When $|E| = 1$, $\nu_E\left(E\right)$ defined in (2) is equal to 1. Hence Axiom 2 is also satisfied.

As for Axiom 4, suppose that $\succsim_E$ is represented by (u, ν_E). For any $A \in \mathcal{E}$ with $\nu\left(A\right) + 1 - \nu\left(A \cup E^c\right) > 0$ and any b, $w \in X$ with $b \geqq w$

$$\begin{aligned}
& x \sim (b_A w)_E x \\
& \Leftrightarrow \int u \circ x \; d\nu - \int u \circ b_{A \cap E} w_{A^c \cap E} x \; d\nu = 0 \qquad \text{(by CEU)} \\
& \Leftrightarrow \left[\nu\left(A \cap E\right) + 1 - \nu\left(E^c \cup A\right)\right] \times \\
& \qquad \left\{u\left(x\right) - \tfrac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} u\left(b\right) + \tfrac{1 - \nu(E^c \cup A)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} u\left(w\right)\right\} = 0 \\
& \Leftrightarrow \int u \circ x \; d\nu_E - \int u \circ b_A w \; d\nu_E = 0 \qquad \text{(by (2))} \\
& \Leftrightarrow x \sim_E b_A w. \qquad \text{(by CEU)}
\end{aligned}$$

It follows that we have $x \sim_E f \Leftrightarrow x \sim f_E x$ for any $f \in \mathcal{F}^2$, which completes the proof. ∎

4. CONCLUDING REMARKS

The result presented here is consistently extended to the biseparable preferences (Ghirardato and Marinacci, 2001)[2], since Axiom 4 is imposed only on binary gambles. Since the family of biseparable preferences includes the maxmin expected utility by Gilboa and Schmeidler (1989) and Casadesus-Masanell, et al. (2000), the extension of our result would also characterize the DFH rule and show another way to derive the *lower envelope* of the posterior set updated by the belief-by-belief update rule, but not the rule itself.

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