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Risk-Free Bond Prices in Incomplete Markets with Recursive Utility Functions and Multiple Beliefs

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Risk-Free Bond Prices in Incomplete Markets with Recursive Utility Functions and Multiple Beliefs[1]

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Abstract

We consider an exchange economy under uncertainty, in which agents' utility functions exhibit constant absolute risk aversion, but they may be recursive and the expected utility calculation may be based on multiple subjective beliefs. The risk aversion coefficients, subjective beliefs, subjective time discount factors, initial endowments, and tradeable assets may differ across agents. We prove that the risk-free bond price goes down (and the interest rate goes up) monotonically as the markets become more complete. We find the range of equilibrium bond prices that depends on the primitives of the economy but not on the structures of financial markets.

JEL Classification Codes: D52, D91, E21, E44, G12.

Keywords: The risk-free rate puzzle, constant absolute risk aversion, incomplete markets, general equilibrium, multiple priors.

1 Introduction

1.1 Setup and Results

In this paper, we consider a model of an exchange economy under uncertainty with two consumption periods and one physical good, where consumption smoothing over time and uncertainty is done by asset transactions in financial markets. We are interested in how market incompleteness affects the price for the risk-free (real or indexed) discount bond, which pays one unit of the physical commodity in every state of the second period in this setup.

The setup of the model is very general. Markets may be incomplete, so that not all risks are *hedgeable*, that is, can be hedged through asset transactions. In particular, initial endowments may not be hedgeable. We also allow different agents to have access to different assets markets. For the preferences of agents, we require that the utility from consumption in each period be given by a utility function exhibiting constant absolute risk aversion (CARA for short), but otherwise the dynamic utility function has a general recursive from with possibly multiple priors. Agents' preferences are heterogenous.

The first contribution of the paper is to prove that when the asset markets become more complete, in the sense that every agent can access more markets, then the bond price never goes up (Theorem 5). Moreover, it must go down if some agent's consumption plan changes. In other words, *the interest rate increases as the market incompleteness diminishes.* The result is reported in Section 3.

This result implies that the bond price is highest when the markets are least complete (where the risk-free bond is the only tradeable asset) and it is lowest when the markets are fully complete (where all risk are marketable). Thus we can derive theoretical bounds for the equilibrium bond prices *independent of the market structure* by studying these two extreme hypothetical cases: we do not need to know what kind of risky assets are available for trade in the economy. In Section 4, we first identify a theoretical upper bound for equilibrium bond prices of the economy in question (Proposition 9). For the case of single common prior, with some additional assumptions, we also find a lower bound, and hence a range of theoretical bond prices (Propositions 12 and 13). These results constitute the second contribution of the paper.

1.2 The Preferences

It may first appear that the assumption of a CARA utility function is unduly restrictive. But as the matter of fact, our model admits a very general class of two period utility functions across the agents. First, the utility function may have the recursive form, so that the attitude towards intertemporal substitution can be disentangled from the attitude towards risk taking.

Secondly, the agent may have multiple subjective probability measures over the state space, so the attitude towards subjective uncertainty can be discussed in addition to the risk attitudes. More specifically, an agent's utility level is given as the minimum of the expected utility levels with respect to all such subjective probability measures, so that he behaves in the maximin manner when choosing a consumption plan.

In symbols, we consider the class of utility functions U over two-period consumptions of the form

$$U(x_0, \mathbf{x}) \equiv v(x_0) + \delta v \left(u^{-1} \left(\min_{\pi \in \Pi} \mathbf{E}^{\pi} \left(u(\mathbf{x}) \right) \right) \right), \tag{1}$$

where x_0 denotes the consumption level on period 0 (the first period); $\mathbf{x}$ denotes the random variable representing the consumption on period 1 (the second period); v is the negatively exponential function representing the attitude towards intertemporal substitution; u is the negatively exponential function representing the attitude towards risk taking; δ is the subjective time discount factor; Π is the set of subjective probability measures; and $\mathbf{E}^{\pi}$ is the expectation operator with respect to the probability measure π. Of course, the requirement on the form of v and u above is our CARA assumption.

Not only the individual utility function is very general as above, but also a great deal of heterogeneity in agents' utility functions is allowed in our model. Indeed, all the constituents for the utility function U, which are u, v, δ, and Π, may vary across agents.

1.3 Significance of the Results

First, our results are important contributions to the extensive literature on the *risk-free rate puzzle*, identified by Weil (1989) and later surveyed in, for example, Kocherlakota (1996): in the US financial markets for about a century until the 1980s, the observed risk-free interest rates have been far lower than the rates a representative agent model with reasonable parameter values could explain. As Weil (1992) subsequently pursued, we also ask the question of to what extent the market incompleteness and agent heterogeneity can possibly explain the observed low risk-free interest rates in a general equilibrium model. Our analysis also takes into account the effects from time non-separability of preferences and multiple priors. The results in this paper are therefore very general answers to this question. In particular, the upper bound result (Proposition 9) identifies the limit of explanatory power of time non-separability and multiple priors as well as the market incompleteness.

Secondly, their implications to the financial innovation literature are valuable. How the prices of existing assets are affected by financial innovations, such as introduction of new derivative assets, reduction in transaction fees, and abolition of short sales constraints? Our results answer this question for the risk-free bond. It is known that this question is difficult to answer in general. The issue is complicated because of the pecuniary externalities arising from changes in prices for existing assets. Obviously, these must be taken into account when evaluating the overall welfare consequences of financial innovations.

Although our results constitute only a partial answer since they do not describe the other assets, for certain cases they are enough to identify the pecuniary externalities, and thus they give a complete answer to the welfare issues. Indeed, it is known that in a class of single common prior normal payoffs models with CARA expected utility, the prices for risky assets are not affected by the introduction of new assets.[1] Hence the overall welfare consequence can be identified once the changes in bond prices are known as our results indicate.[2]

Finally, one should not overlook the advantage of this paper over some recent contributions in the so called general equilibrium theory with incomplete asset markets (GEI) with numerical analyses: while this paper is able to look into detailed pricing implications, it does not appeal to any numerical analysis restricted to specific cases of incomplete markets. While we readily admit that the scope of the analysis is still limited, it deepens our understanding far beyond numerical analyses can offer, and therefore it suggests a new line

[1] To be precise, all agents conform the expected utility hypothesis with CARA utility (that is, $u = v$ in (1)), and share the same, single, probabilistic belief over the state space and the same subjective time discount factor, and if the risk factors defining the assets' payoffs and agents' endowments are assumed to be normally distributed. The result is contained in Oh (1996) and his predecessors referred to therein.

[2] To be exact, we need to assume that the payoffs of the risky assets have zero mean for this conclusion, because otherwise the change in the risk-free interest rate would affect relative prices of risky assets according to how large their means are.

of research on GEI.

1.4 Related Literature

Weil (1992, Proposition 1) showed that the bond price is lower in fully complete markets than in the least complete markets. The intermediate cases, where the markets are neither fully nor least complete, are indeed difficult to analyze, due to the general equilibrium effects among multiple asset markets and inefficiency of equilibrium allocations; and it would cast serious doubts on the relevance of his result if the direction of changes in the bond prices could be reversed in some intermediate cases. Yet the monotonicity result of this paper shows that such reversion never occurs. We can thus say that our result confirms the robustness of his result.

Our result generalizes Proposition 1 of Elul (1997), in that we do not use any of the four conditions he needed to derive the same result. The recursive utility functions were introduced in Kreps and Porteus (1978). The case of constant elasticity of substitution was subsequently presented by Epstein and Zin (1989), and recursive utility functions of CARA type were used in Calvet, Gonzalez-Eiras, and Sodini (2001). Gilboa and Schmeidler (1989) considered the problem of when attitudes towards risk and uncertainty can be represented as the minimum of expected utility levels with respect to multiple subjective probabilities over the state space. To the best of our knowledge, this paper is the first one to explore implications on asset prices of *heterogenous* agents with recursive utility functions and multiple subjective probability beliefs, let alone being the first one to predict (correctly!) possible directions of changes in asset prices when more assets become tradeable.

A model of restricted participation was considered by Balasko, Cass, and Siconolfi (1990), where different agents may trade different types of assets. The types of trade restrictions we assume in this paper are more general than theirs. In particular, short-sales constraints and nonlinear constraints can be accommodated. Although the restrictions on asset trades are exogenously given for each agent in our and their models, agents may choose to trade more varieties of risky assets upon paying entry costs in the model of Calvet, Gonzalez-Eiras, and Sodini (2001); and they point to the possibility (Proposition 4) that financial innovations may cause the bond price to go up due to the endogenous market participation.

In a model of a continuous-time stochastic world economy of two countries having the log expected utility functions, Devereux and Saito (1997) considered three structures of international capital markets: Autarchy, in which no trade between the two countries is at all possible; fully complete markets, in which both the short-term risk-free bond and the shares in each country's production technologies can be traded between the two countries; and the bond trade regime, in which only the short-term risk-free bond can be traded. They identified a parameter region (inequality (18) on page 471) over which one country may get worse off as the market structure moves from the bond trade regime to the complete markets. The reduction in this country's welfare can be attributed to the negative pecuniary externality arising from changes in the bond price, which dominates the benefit of enhanced risk-hedging (diversification) opportunities.

2 The Model

2.1 State Space and Commodity Space

There are two consumption periods, 0 and 1, and there is a single perishable good in each period. There is no uncertainty in period 0, when the consumption good and assets are exchanged. At the beginning of period 1, the assets pay off and consumption then takes place. The uncertainty in period 1 is described by a finite state space Ω. We often refer to each function from Ω to $\mathbb{R}$ as a random variable. Denote by $\mathbf{1}$ the function from Ω to $\mathbb{R}$ that takes constant value one. The constant variable $\mathbf{1}$ will be interpreted as the risk-free discount bond. Let X be the set of all random variables. We take the commodity space to be $\mathbb{R} \times X$. A generic element of $\mathbb{R} \times X$ will be denoted by $(x_0, \mathbf{x})$, where x_0 corresponds to consumption in period 0, and $\mathbf{x}$ is a random variable that corresponds to consumption in period 1, and we write $\mathbf{x}(\omega)$ for consumption in state ω.

2.2 Primitives

There are H *agents* in the economy. Each agent, indexed $h \in \{1, \ldots, H\}$, is characterized by:

- Set Π^h of subjective probability measures putting a strictly positive probability on every state. We assume that Π^h is non-empty and compact when regarded as a subset of $\mathbb{R}^{|\Omega|}$. The expectation operator for each $\pi \in \Pi^h$ is denoted by $\mathbf{E}^\pi$; $\mathbf{E}^\pi(\mathbf{x}) :=$

$\sum_{\omega\in\Omega} \mathbf{x}(\omega)\,\pi(\omega)$. The minimum expectation operator $\mathbf{E}^{\Pi^h}$ is defined by $\mathbf{E}^{\Pi^h}(\mathbf{z}) = \min_{\pi\in\Pi^h} \mathbf{E}^{\pi}(\mathbf{z})$ for every $\mathbf{z} \in X$, which is well defined by the compactness.

- Atemporal von Neumann Morgenstern utility function u^h and intertemporal utility function v^h. They are assumed to be CARA functions:[3] for every h, $u^h(w) = -\exp\left(-\alpha^h w\right)$, and $v^h(w) = -\exp\left(-\beta^h w\right)$.
- Subjective time discount factor $\delta^h > 0$.
- Initial endowment vector $(e_0^h, \mathbf{e}^h) \in \mathbb{R} \times X$.

The preference relation of agent h is represented by the utility function $U^h : \mathbb{R} \times X \to \mathbb{R}$ defined by

$$U^h(x_0^h, \mathbf{x}^h) \equiv v^h\left(x_0^h\right) + \delta^h v^h\left(\mathbf{E}_{u^h}^{\Pi^h}\left(\mathbf{x}^h\right)\right), \tag{2}$$

where $\mathbf{E}_{u^h}^{\Pi^h}\left(\mathbf{x}^h\right)$ is the certainty equivalent of $\mathbf{x}^h$ with respect to the minimum expectation operator of Π^h and u^h; formally,

$$\mathbf{E}_{u^h}^{\Pi^h}\left(\mathbf{x}^h\right) \equiv \left(u^h\right)^{-1}\left(\mathbf{E}^{\Pi^h}\left(u^h\left(\mathbf{x}^h\right)\right)\right). \tag{3}$$

This is a combination of a modification of recursive utility functions investigated in Kreps and Porteus (1978) and multiple probability measures of Gilboa and Schmeidler (1989). The class of the expected utility functions is given by letting $u^h = v^h$ and Π^h be a singleton; note that homogeneity of beliefs is not required.[4] Writing the CARA form explicitly in (2), we have,

$$U^h(x_0^h, \mathbf{x}^h) = -\exp(-\beta^h x_0^h) - \delta^h\left(\min_{\pi\in\Pi^h} \mathbf{E}^{\pi}\left(\exp(-\alpha^h \mathbf{x}^h)\right)\right)^{\beta^h/\alpha^h}. \tag{4}$$

The reciprocal $1/\beta^h$ of the absolute risk aversion represents the agent's tolerance for intertemporal substitution and is denoted by γ^h.

[3] A special case of these utility functions has been used in Calvet, Gonzales-Eiraz, and Sodini (2001), in which all consumers have the same atemporal utility function, the same intertemporal utility function, the same unique subjective probability, and the same subjective discount factor, and the risky asset payoffs and initial endowments are jointly normally distributed. We shall not impose any of these additional assumptions.

[4] Thus even when all the Π^h are singletons, our model is one of heterogenous beliefs, extensively investigated by Calvet, Grandmont, and Lemaire (1999).

2.3 Market Structures

To investigate the implications of the structure of markets on the equilibrium prices, we shall consider various market structures. A market structure is determined by a collection of assets and a profile of permissible portfolios for the H agents in the economy, as follows.

Each *asset* is characterized by its payoff in period 1 and hence identified with an element of X. There are finitely many assets, $\mathbf{a}_1, \ldots, \mathbf{a}_J$, where $J \geq 1$ and $\mathbf{a}_j \in X$ for every j, and a generic collection of assets is denoted by $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_J) \in X^J$. It is assumed that $\mathbf{a}_1 = \mathbf{1}$ throughout the paper. This means that the risk-free bond can be traded in all the setups we shall consider. In particular, if $J = 1$, then the risk-free bond is the only traded asset.

A household may not access freely to certain markets. To formalize this, for each agent h, we denote by $N^h \subset \mathbb{R}^J$ the nonempty set of all portfolios of the J assets that agent h can hold. We assume that there is no restriction for the market of risk-free bond. Formally, it is assumed that if $y^h = \left(y_1^h, y_2^h, \ldots, y_J^h\right) \in N^h$ and $\hat{y}_1^h \in \mathbb{R}$, then $\left(\hat{y}_1^h, y_2^h, \ldots, y_J^h\right) \in N^h$. Note in particular that there is no borrowing or lending constraint in terms of the risk-free bond.

The following examples should clarify what N^h is intended to capture: If $N^h = \mathbb{R}^J$, then agent h can trade all assets available in the financial markets; if $N^h = \mathbb{R} \times \{0\}$, where $0 \in \mathbb{R}^{J-1}$, then, among the J traded assets, agent h can trade the risk-free bond but nothing else; and if $N^h = \mathbb{R} \times \mathbb{R}_+^{J-1}$, then agent h can trade any (positive or negative) amount of the risk-free bond, but cannot short sell the other assets. It is of course possible to have the mixture of the last two for various assets other than the risk-free bond. The bound on short sales can be made a strictly negative number. If $J \geq 3$ and $N^h = \mathbb{R} \times \{(y_2^h, \ldots, y_J^h) \in \mathbb{R}^{J-1} \mid y_2^h = \cdots = y_J^h\}$, then agent h is constrained to hold equal quantities of the other assets. We may view the combination of these $J-1$ assets as a mutual fund and agent h as constrained to trade the risk-free bond and the mutual fund, but not the constituent assets.

Note that we do *not* require that the assets are linearly independent, and hence there may be redundant assets. The reader will see that the existence of redundant assets does not matter in our analysis since we will be interested in the bond price only. Note however that even if an asset is a linear combination of other assets, it is not necessarily redundant in our setup because of the trading restriction.

If $J = 1$, then only the risk-free bond is traded and the markets are least complete.

On the other hand, if for every $(\mathbf{z}^h)_{h\in\{1,\dots,H\}}$ with $\mathbf{z}^h \in X$ and $\sum_{h=1}^{H} \mathbf{z}^h = 0$, there exists a $(y^h)_{h\in\{1,\dots,H\}}$ with $y^h \in N^h$ and $\sum_{h=1}^{H} y^h = 0$ such that $\mathbf{z}^h = \sum_{j=1}^{J} y_j^h \mathbf{a}_j$ for every h, then the market structure is fully complete.

To sum up, a market structure of the economy is given by a collection $(\mathbf{A}, N^1, \dots, N^H)$, where, for some positive integer J, $\mathbf{A} \in X^J$ and $N^h \subseteq \mathbb{R}^J$ for each h, satisfying the condition regarding the risk-free bond trade. As the purpose of this paper is to investigate how the bond price is affected by the nature of market structures, we do not assume that $\mathbf{e}^h$ is *hedgeable*, that is, there exists a $y^h \in N^h$ such that $\mathbf{e}^h + \sum_{j=1}^{H} y_j^h \mathbf{a}_j = 0$.

2.4 Utility Maximization and Equilibrium

Given a collection $\mathbf{A} \in X^J$ of J assets, an *asset price vector* is denoted by $q = (q_1, \dots, q_J) \in \mathbb{R}^J$, where q_j is the price for the j-th asset. Since there is no restriction on the payoffs except for the bond, the price may be negative. Every agent h is assumed to be a price taker and his portfolio choice is constrained by the set N^h. Thus his utility maximization problem can be formally written as follows.

$$
\begin{aligned}
\max_{(x_0^h, \mathbf{x}^h, y^h) \in \mathbb{R}\times X \times \mathbb{R}^J} \quad & U^h(x_0^h, \mathbf{x}^h) \\
\text{subject to} \quad & (x_0^h - e_0^h) + \sum_{j=1}^{J} q_j y_j^h \leq 0, \\
& \mathbf{x}^h - \mathbf{e}^h = \sum_{j=1}^{J} y_j^h \mathbf{a}_j, \\
& y^h \in N^h.
\end{aligned}
$$

The first constraint is the budget constraint for the current consumption and asset trades in period 0. The consumption in period 0 is assumed to be the numéraire, whose price equals one. Since it is desirable, the inequality may be replaced with equality. Given that there is no re-trade in period 1, the second constraint simply says that the consumption in that period is derived from the initial endowments and the payoffs from the assets in the portfolio. The third constraint is on the portfolio choice in period 0, as formulated in the previous subsection. Notice that the agent takes the set of feasible portfolios, N^h as given, as in Balasko, Cass, and Siconolfi (1990).[5]

An asset price vector q and an allocation $((x_0^h, \mathbf{x}^h, y^h))_{h\in\{1,\dots,H\}}$ of consumptions and portfolios constitute an *equilibrium* of the economy for the market structure $(\mathbf{A}, N^1, \dots, N^H)$ if, for every h, $(x_0^h, \mathbf{x}^h, y^h)$ is a solution to the above maximization problem and $\sum_{h=1}^{H} y^h = 0$. This equality is the asset market clearing condition, but it implies that the good markets clear as well.

Since we do not impose any restriction on the market structure except for the bond, an equilibrium may not exist. It does exist when there is no individual trading constraint, but this is of course not the only case for existence. Since our results hold whenever there is an equilibrium, we chose not to impose extra restrictions on the market structure a priori.

It is known, and can be shown easily, that any equilibrium allocation is constrained Pareto efficient in the sense that no welfare improving reallocation of assets respecting the individual portfolio choice set N^h.[6] It is fully Pareto efficient if $(\mathbf{A}, N^1, \dots, N^H)$ is a fully complete market structure.

Note that our equilibrium concept is ex ante. If there is a single prior for each agent, ex ante optimal decision implies ex post optimal decision thanks to the recursive structure of the preferences. But for multiple priors, since updating of multiple priors is non trivial, an ex ante optimal decision may involve some ex post suboptimal behavior. Thus the agent may want to re-trade once period one markets open, hence the trade an ex ante equilibrium describes may not be an ex post equilibrium. The issue of dynamic consistency with multiple priors is important but it is beyond the scope of this paper.[7]

Note finally that the equilibrium price of the risk-free discount bond is q_1 and the equilibrium risk-free interest rate is $1/q_1 - 1$. Hence a higher bond price means a lower interest rate, and vice versa.

[5] Calvet, Gonzalez-Eiras, and Sodini (2001) investigated the case where he may choose to expand N^h at some costs.

[6] This can be proved along the lines of the proof of Theorem 12.3 of Magill and Quinzii (1996).

[7] Epstein and Schneider (2003) and Wakai (2003) identified a condition on the set of priors which implies dynamically consistent behavior.

3 Monotonicity of the Equilibrium Bond Prices

A special character of a CARA utility function is that its logarithmic transformation is quasi-linear in the direction of $(1, \mathbf{1})$, even with multiple subjective beliefs. Formally, for each h, define $W^h : \mathbb{R} \times X \to \mathbb{R}$ by $W^h(x_0^h, \mathbf{x}^h) = -\gamma^h \log(-U^h(x_0^h, \mathbf{x}^h))$. This is well defined because $U^h(x_0^h, \mathbf{x}^h) < 0$ and it represents the same preference as U^h because the function $U \mapsto -\gamma^h \log(-U)$ is strictly increasing.

Lemma 1 *The transformed function W^h is quasi-linear in $(1, \mathbf{1})$: that is, $W^h(x_0 + t, \mathbf{x}^h + t\mathbf{1}) = W^h(x_0^h, \mathbf{x}^h) + t$.*

Proof. Let $(x_0^h, \mathbf{x}^h) \in \mathbb{R} \times X$ and $t \in \mathbb{R}$. Since u^h is a CARA utility function,

$$\mathbf{E}^{\pi}\left(u^h(\mathbf{x}^h + t\mathbf{1})\right) = \exp(-t\alpha^h)\mathbf{E}^{\pi}\left(u^h(\mathbf{x}^h)\right) \tag{5}$$

for every $\pi \in \Pi^h$. Hence

$$\mathbf{E}^{\Pi^h}\left(u^h(\mathbf{x}^h + t\mathbf{1})\right) = \exp(-t\alpha^h)\mathbf{E}^{\Pi^h}\left(u^h(\mathbf{x}^h)\right) \tag{6}$$

and thus

$$\mathbf{E}_{u^h}^{\Pi^h}\left(u^h(\mathbf{x}^h + t\mathbf{1})\right) = \mathbf{E}_{u^h}^{\Pi^h}\left(u^h(\mathbf{x}^h)\right) + t. \tag{7}$$

Applying this to (2) we see that $U^h(x_0^h + t, \mathbf{x}^h + t\mathbf{1}) = \exp(-\beta^h t)U^h(x_0^h, \mathbf{x}^h)$, and hence $W^h(x_0 + t, \mathbf{x}^h + t\mathbf{1}) = W^h(x_0^h, \mathbf{x}^h) + t$. ∎

Remark 2 The relation (6) says that the probability measures in Π^h that attain the minimum expected utility in the definition of the minimum expectation operator $\mathbf{E}^{\Pi^h}$ are invariant to additions of the risk-free bond. So although U^h is not a differentiable function on $\mathbb{R} \times X$ because of the minimum operator, it is differentiable in the direction of the risk-free bond. These properties do not generally hold for risky assets.

Remark 3 Let L be a linear subspace of $\mathbb{R} \times X$ that does not contain $(1, \mathbf{1})$ and, along with it, spans the entire $\mathbb{R} \times X$. Let $\widetilde{W} : L \to \mathbb{R}$ be the restriction of W onto L. Then $\widetilde{W}$ is a strictly concave function on L. Let $\left(x_0^h, \mathbf{x}_0^h\right) \in \mathbb{R} \times X$ and $t \in \mathbb{R}$ satisfy $\left(x_0^h - t, \mathbf{x}_0^h - t\mathbf{1}\right) \in L$. Then

$$W\left(x_0^h, \mathbf{x}_0^h\right) = \widetilde{W}\left(x_0^h - t, \mathbf{x}_0^h - t\mathbf{1}\right) + t.$$

Hence W can be decomposed into a strictly concave part and a linear part.

To state the main result of this paper, we introduce the *at-least-as-complete-as* relation between two market structures.

Definition 4 Let $(\mathbf{A}, N^1, \ldots, N^H)$ and $(\hat{\mathbf{A}}, \hat{N}^1, \ldots, \hat{N}^H)$ be two market structures, with $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_J)$ and $\hat{\mathbf{A}} = (\hat{\mathbf{a}}_1, \ldots, \hat{\mathbf{a}}_{\hat{J}})$. Then $(\mathbf{A}, N^1, \ldots, N^H)$ is *at least as complete as* $(\hat{\mathbf{A}}, \hat{N}^1, \ldots, \hat{N}^H)$ if the following condition is satisfied: If $\hat{y}^h \in \hat{N}^h$ for every h and $\sum_{j=1}^{\hat{J}} \hat{y}^h = 0$, then, for every h, there exists a $y^h \in N^h$ such that $\sum_{j=1}^{J} y_j^h \mathbf{a}_j = \sum_{j=1}^{\hat{J}} \hat{y}_j^h \hat{\mathbf{a}}_j$ for every h and $\sum_{j=1}^{J} y^h = 0$.

According to this definition, a market structure is at least as complete as another if every risk allocation attained via the second market structure can also be attained via the first. Note that the attainability consists of the individual spanning condition $\sum_{j=1}^{J} y_j^h \mathbf{a}_j = \sum_{j=1}^{\hat{J}} \hat{y}_j^h \hat{\mathbf{a}}_j$ and the market clearing condition $\sum_{j=1}^{J} y^h = 0$. If all the N^h were equal and linear, the market clearing condition would be superfluous for the subsequent analysis. Otherwise, the individual spanning condition does not automatically imply the market clearing condition, and hence it is necessary to impose this condition in addition to the individual spanning condition.

It is easy to see that if we add a new asset without shrinking the portfolio choice sets for the existing assets, then the resultant market structure is at least as complete as the original one. Hence the case of $J = 1$, in which only the risk-free asset is traded, is the case of the least complete markets. Even if we do not add any new asset, if expand some agent's portfolio choice set without shrinking the others', the resultant market structure is again at least as complete as the original one. However, the resultant market structure may well be at least as complete as the original one even when we replace some existing assets by new ones. In particular, a market structure is at least as complete as any other market structure if and only if it gives rise to the fully complete markets.

The following theorem is the main result of this paper.

Theorem 5 *Let* $(\mathbf{A}, N^1, \ldots, N^H)$ *and* $(\hat{\mathbf{A}}, \hat{N}^1, \ldots, \hat{N}^H)$ *be two market structures, with equilibrium risk-free bond prices* q_1 *and* $\hat{q}_1$*, such that* $(\mathbf{A}, N^1, \ldots, N^H)$ *is at least as complete as* $(\hat{\mathbf{A}}, \hat{N}^1, \ldots, \hat{N}^H)$*. Then* $q_1 \leq \hat{q}_1$*.*

This theorem claims that if a market structure is at least as complete as another, then the risk-free bond price in the first market structure cannot be higher than the risk-free bond price in the second. A less general version of this result was originally proved in Hara (1998). The theorem also generalizes Proposition 1 of Elul (1997), Theorem 3 of Calvet (2001) when restricted to the two-period case, and Theorem 5 of Calvet, Gonzalez-Eiras, and Sodini (2001).

Proof. Let $((x_0^h, \mathbf{x}^h, y^h))_{h \in \{1,\dots,H\}}$ be an allocation corresponding to the equilibrium bond price q_1. By equality (7), for every h, the directional derivative of U^h in the direction of $(0, \mathbf{1})$ at $(x_0^h, \mathbf{x}^h)$ exists and equals $\delta^h D v^h \left(\mathbf{E}_{u^h}^h \left(\mathbf{x}^h\right)\right)$. The directional derivative in the direction of $(1, \mathbf{0})$ exists and equals $D v^h \left(x_0^h\right)$. Also, there is no constraint on the transaction of the risk-free bond (asset 1). Hence utility maximization implies that the following the first-order necessary condition must hold:

$$q_1 = \delta^h \frac{D v^h \left(\mathbf{E}_{u^h}^{\Pi^h} \left(\mathbf{x}^h\right)\right)}{D v^h \left(x_0^h\right)}. \tag{8}$$

Since v^h is an exponential function,

$$\delta^h \frac{D v^h \left(\mathbf{E}_{u^h}^{\Pi^h} \left(\mathbf{x}^h\right)\right)}{D v^h \left(x_0^h\right)} = \delta^h \frac{v^h \left(\mathbf{E}_{u^h}^{\Pi^h} \left(\mathbf{x}^h\right)\right)}{v^h \left(x_0^h\right)}$$

and hence

$$\delta^h v^h \left(\mathbf{E}_{u^h}^{\Pi^h} \left(\mathbf{x}^h\right)\right) = q_1 v^h \left(x_0^h\right).$$

Thus

$$\begin{aligned} W^h(x_0^h, \mathbf{x}^h) &= -\gamma^h \log\left((1+q_1) v^h \left(x_0^h\right)\right) \\ &= -\gamma^h \log\left((1+q_1)\exp(-\beta^h x_0^h)\right) \\ &= -\gamma^h \log(1+q_1) + x_0^h. \end{aligned}$$

Summing these over h, using the market clearing condition, and writing $\gamma = \sum_{h=1}^H \gamma^h$, we obtain

$$\sum_{h=1}^H W^h(x_0^h, \mathbf{x}^h) = -\gamma \log(1+q_1) + \sum_{h=1}^H x_0^h = -\gamma \log(1+q_1) + \sum_{h=1}^H e_0^h. \tag{9}$$

Writing $((\hat{x}_0^h, \hat{\mathbf{x}}^h, \hat{y}^h))_{h \in \{1,\dots,H\}}$ for an allocation corresponding to the equilibrium bond price $\hat{q}$, we can similarly obtain

$$\sum_{h=1}^H W^h(\hat{x}_0^h, \hat{\mathbf{x}}^h) = -\gamma \log(1+\hat{q}_1) + \sum_{h=1}^H e_0^h. \tag{10}$$

By Lemma 1, every W^h is quasi-linear in $(1, \mathbf{1}) \in \mathbb{R} \times N$. Moreover, every equilibrium allocation is constrained Pareto efficient. Thus the allocation $((x_0^h, \mathbf{x}^h, y^h))_{h\in\{1,\ldots,H\}}$ must be a solution to the utilitarian welfare maximization problem:

$$\begin{aligned} \max_{((x_0^h,\mathbf{x}^h,y^h))_{h\in\{1,\cdots,H\}}} \quad & \sum_{h=1}^{H} W^h(x_0^h, \mathbf{x}^h) \\ \text{subject to} \quad & \mathbf{x}^h - \mathbf{e}^h = \sum_{j=1}^{J} y_j^h \mathbf{a}_j \text{ for every } h, \\ & y^h \in N^h \text{ for every } h, \\ & \sum_{h=1}^{H} y^h = 0. \end{aligned} \tag{11}$$

Since $(\mathbf{A}, N^1, \ldots, N^H)$ is at least as complete as $(\hat{\mathbf{A}}, \hat{N}^1, \ldots, \hat{N}^H)$, there exists an allocation $(\widetilde{y}^h)_{h\in\{1,\ldots,H\}}$ of portfolios for $(\mathbf{A}, N^1, \ldots, N^H)$ such that $\widetilde{y}^h \in N^h$ and $\sum_{j=1}^{J} \widetilde{y}_j^h \mathbf{a}_j = \sum_{j=1}^{\hat{J}} \hat{y}_j^h \hat{\mathbf{a}}_j$ for every h, and $\sum_{h=1}^{H} \widetilde{y}^h = 0$. Then $((\hat{x}_0^h, \hat{\mathbf{x}}^h, \widetilde{y}^h))_{h\in\{1,\ldots,H\}}$ satisfies the constraints of the welfare maximization problem. Hence $\displaystyle\sum_{h=1}^{H} W^h(x_0^h, \mathbf{x}^h) \geq \sum_{h=1}^{H} W^h(\hat{x}_0^h, \hat{\mathbf{x}}^h)$. Thus, by (9) and (10),

$$-\gamma \log(1 + q_1) + \sum_{h=1}^{H} e_0^h \geq -\gamma \log(1 + \hat{q}_1) + \sum_{h=1}^{H} e_0^h. \tag{12}$$

Hence $q_1 \leq \hat{q}_1$. ■

Remark 6 By taking $N^h = \mathbb{R}^J$ and $N^h = \mathbb{R}^{\hat{J}}$ for every h, we can see from Theorem 5 that as the market span increases, the equilibrium bond price decreases.

Remark 7 As seen in Remark 3, W^h is quasi linear in $(1, \mathbf{1})$ with a strictly concave non-linear component. Thus, if N^h is convex for every h, then the solution to (11) is unique up to transfers in the direction of in $(1, \mathbf{1})$. This means that the inequality (12) must be strict if at least one agent consume differently in the two equilibria, and so $q_1 < \hat{q}_1$.

Let us give some intuition for the result: With one good but no sequential trades, the equilibrium allocations are constrained Pareto efficient with respect to the given market structure. By the quasi-linearity of the W^h in $(1, \mathbf{1})$ and absence of trade constraints on the risk-free bond, efficiency can be reduced to total welfare maximization. An increase in trading opportunity must increase the total welfare. On the other hand, the total welfare can be directly measured in terms the bond price since the u^h and v^h are exponential functions;

roughly speaking, the price is the same as the marginal utility of risk-free consumption, and the marginal utility works in the same way (but in the opposite direction) as the utility for these functions. So the equilibrium total welfare can be written as a decreasing function of the bond price. Therefore, an increase in the value of the utilitarian social welfare function corresponds to a decrease in bond prices.

Remark 8 Suppose now that the economy were not a pure exchange one but were to involve some firms, whose production technologies are of the multiplicative type as in Diamond (1967). Then, in spite of market incompleteness, the profit maximization is a well defined objective and the equilibrium allocations are constrained Pareto efficient. The proof of Theorem 5, however, would not fully go through: The second equality of (9) no longer holds and equality (12) must be replaced by

$$-\gamma \log(1+q_1) + \sum_{h=1}^{H} x_0^h \geq -\gamma \log(1+\hat{q}_1) + \sum_{h=1}^{H} \hat{x}_0^h.$$

The monotonicity result would therefore remain to hold if $\sum_{h=1}^{H} x_0^h \leq \sum_{h=1}^{H} \hat{x}_0^h$, that is, the aggregate consumption in period 1 is not increased. In other words, if the financial innovation induces the firms to expand their scales of operation, possibly by selling (issuing) the bond, then the bond price must go down and the interest rate must go up. This is a nice, intuitive extension of Theorem 5 to production economies.

Finally, Theorem 5 should not be confused but contrasted with the *invariance* property of *risky* asset prices established by Oh (1996) and his predecessors, the property that with CARA utility functions and normally distributed asset payoffs, the relative prices among risky assets do not depend on the market span.[8]

4 Range of the Equilibrium Bond Prices

Theorem 5 implies that the equilibrium risk-free bond price is lowest when the market structure is fully complete, and it is highest when the market structure is least complete, that is, only the risk-free bond is traded. In order to find the upper and lower bounds on the bond prices independently of market structures, therefore, it is sufficient to identify those prices in the least complete markets and the fully complete markets.

[8] Footnotes 1 and 2 contain more details on this property.

We first study the case where the risk-free bond is the only tradeable asset. We have a clean, closed form expression for the equilibrium price, which then identifies the highest bond price thus the lowest interest rate.

Proposition 9 *Let q_1 be an equilibrium risk-free bond price for the least complete market structure, that is, $J = 1$. Then*

$$q_1 = \delta \frac{\prod_{h=1}^{H} \left(\mathbf{E}^{\Pi^h}(\exp(-\alpha^h \mathbf{e}^h))\right)^{\beta/\alpha^h}}{\exp(-\beta e_0)}, \tag{13}$$

where $\delta = \left(\delta^1\right)^{\beta/\beta^1} \cdots \left(\delta^H\right)^{\beta/\beta^H}$ and $1/\beta = 1/\beta^1 + \cdots + 1/\beta^H$. The expression (13) gives the largest bond price independent of the market structure.

The first part can be shown by directly checking the first order condition, and a proof is given in Appendix. The second part follows from Theorem 5.

Now let us consider the case of the fully complete market structures. This case is harder to identify in our general setup of the recursivity and heterogeneity of subjective sets of probabilities. A short explanation for this is that there is no representative agent. Hence we shall concentrate on an environment which admits a representative agent. Formally, we use the following assumption.

Assumption 10

1. $\dfrac{\alpha^1}{\beta^1} = \cdots = \dfrac{\alpha^H}{\beta^H}$.
2. There exists a probability measure π such that $\Pi^1 = \cdots = \Pi^H = \{\pi\}$.

The first condition is that the ratio between α^h and β^h is common over all agents. Notice that this condition still allows for heterogenous, truly recursive utility functions since α's and β's need not be common. The second condition is the common prior assumption.[9] Note that there is no restriction on subjective discount factors, thus in particular they can be heterogeneous.

Under Assumption 10, a mutual fund theorem type argument applies. Define α, β, δ,

[9] The heterogeneous subjective probabilities investigated in Calvet, Grandmont, and Lemaire (1999) and the multiple subjective probabilities for a single agent investigated in Gilboa and Schmeidler (1989) are, unfortunately, completely eliminated by this condition.

and Π by

$$\begin{aligned}
\frac{1}{\alpha} &= \frac{1}{\alpha^1} + \cdots + \frac{1}{\alpha^H}, \\
\frac{1}{\beta} &= \frac{1}{\beta^1} + \cdots + \frac{1}{\beta^H}, \\
\delta &= \left(\delta^1\right)^{\beta/\beta^1} \cdots \left(\delta^H\right)^{\beta/\beta^H}, \\
\Pi &= \{\pi\}.
\end{aligned}$$

Define then $u : \mathbb{R} \to \mathbb{R}$, $v : \mathbb{R} \to \mathbb{R}$, and $U : \mathbb{R} \times X \to \mathbb{R}$ by $u(w) = -\exp(-\alpha w)$, $v(w) = -\exp(-\beta w)$, and

$$U(x_0, \mathbf{x}) = v(x_0) + \delta v\left(\mathbf{E}_u^{\Pi}(\mathbf{x})\right) = -\exp(-\beta x_0) - \delta\left(\mathbf{E}^{\pi}\left(\exp(-\alpha \mathbf{x})\right)\right)^{\beta/\alpha}. \tag{14}$$

Let $(\lambda^h, \dots, \lambda^H) \in \mathbb{R}^H_{++}$. For each $(x_0, \mathbf{x}) \in \mathbb{R} \times X$, consider the weighted-sum social welfare maximization problem

$$\begin{aligned}
\max_{((x_0^h, \mathbf{x}^h))_{h \in \{1,\dots,H\}}} \quad & \sum_{h=1}^{H} \lambda^h U^h(x_0^h, \mathbf{x}^h) \\
\text{subject to} \quad & \sum_{h=1}^{H} (x_0^h, \mathbf{x}^h) = (x_0, \mathbf{x}).
\end{aligned} \tag{15}$$

Note that when setting up this welfare maximization problem, we are implicitly assuming that the markets are fully complete, as the resource feasibility is the only constraint for this problem.

Theorem 11 *Under Assumption 10, for each $(x_0, \mathbf{x}) \in \mathbb{R} \times X$, define $((x_0^h, \mathbf{x}^h))_{h \in \{1,\dots,H\}}$ by*

$$x_0^h = \frac{\gamma^h}{\gamma} x_0 + \gamma^h \left(\log\left(\frac{\lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H} \frac{\gamma^i}{\gamma} \log\left(\frac{\lambda^i}{\gamma^i}\right) \right), \tag{16}$$

$$\mathbf{x}^h = \frac{\gamma^h}{\gamma} \mathbf{x} + \gamma^h \left(\log\left(\frac{\delta^h \lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H} \frac{\gamma^i}{\gamma} \log\left(\frac{\delta^i \lambda^i}{\gamma^i}\right) \right) \mathbf{1} \tag{17}$$

for each h. Then $((x_0^h, \mathbf{x}^h))_{h \in \{1,\dots,H\}}$ is the solution to the problem (15). Moreover, there exists a constant $\lambda > 0$ such that $\sum_{h=1}^{H} \lambda^h U^h(x_0^h, \mathbf{x}^h) = \lambda U(x_0, \mathbf{x})$ for every $(x_0, \mathbf{x}) \in \mathbb{R} \times X$.

This can be shown by directly computing the first order condition; a proof can be found in Appendix.

The equilibrium bond price can then be derived from Theorem 11.

Proposition 12 *Under Assumption 10, let* q_1 *be an equilibrium risk-free bond price for a fully complete market structure, then*

$$q_1 = \delta \frac{\left(\mathbf{E}^\pi\left(\exp\left(-\alpha \mathbf{e}\right)\right)\right)^{\beta/\alpha}}{\exp(-\beta e_0)}.$$

Therefore, under Assumption 10, we can identify the range of equilibrium bond prices completely. We shall relate it to an index of market incompleteness, as follows.

Proposition 13 *Under Assumption 10, for every market structure* $(\mathbf{A}, N^1, \ldots, N^H)$*, if* q_1 *is an equilibrium risk-free bond price for* $(\mathbf{A}, N^1, \ldots, N^H)$*, then there exists a* θ *in the interval*

$$\left[1, \frac{\prod_{h=1}^{H}\left(\mathbf{E}^\pi(\exp(-\alpha^h \mathbf{e}^h))\right)^{\gamma^h/\gamma}}{\mathbf{E}^\pi\left(\exp\left(-\alpha \mathbf{e}\right)\right)}\right] \tag{18}$$

such that

$$q_1 = \delta \left(\frac{\mathbf{E}^\pi\left(\exp\left(-\alpha \mathbf{e}\right)\right)}{\exp(-\alpha e_0)}\theta\right)^{\beta/\alpha}.$$

This proposition follows from $\alpha/\alpha^h = \gamma^h/\gamma$ for every h under Assumption 10.

Notice that the fraction

$$\frac{\prod_{h=1}^{H}\left(\mathbf{E}^\pi(\exp(-\alpha^h \mathbf{e}^h))\right)^{\gamma^h/\gamma}}{\mathbf{E}^\pi\left(\exp\left(-\alpha \mathbf{e}\right)\right)} \tag{19}$$

is an index measuring the discrepancy of the initial endowment allocation $(\mathbf{e}^h)_{h\in\{1,\ldots,H\}}$ from the Pareto efficient allocations, and hence succinctly summarizing the impact on the price of the risk-free bond when it is the only traded security. Indeed, according to Theorem 11, $(\mathbf{e}^h)_{h\in\{1,\ldots,H\}}$ can constitute a Pareto efficient allocation if and only if there exists a $(\eta^1, \ldots, \eta^H) \in \mathbb{R}^H$ such that $\mathbf{e}^h = (\gamma^h/\gamma)\mathbf{e} + \eta^h \mathbf{1}$ for every h, in which case (19) equals 1; and, according to Proposition 9, the index measures how much higher the bond prices may be if the markets are incomplete.

Note finally that the fraction (19) can be made arbitrarily large. For example, suppose that the $\mathbf{e}^h$ are risky and yet there is no aggregate uncertainty, so that $\sum_{h=1}^{H} \mathbf{e}^h = \eta \mathbf{1}$ for some $\eta \in \mathbb{R}$. If the $\mathbf{e}^h$ are then replaced by $\mathbf{e}^h + \sigma\left(\mathbf{e}^h - \mathbf{E}^\pi(\mathbf{e})\mathbf{1}\right)$ for a common $\sigma > 0$, then $\sum_{h=1}^{H} \mathbf{e}^h$ is not changed but the fraction (19) can be made arbitrarily large by taking σ sufficiently large. In particular, when markets are incomplete, the (real) interest interest rate may be negative even when it would be positive should the market be complete.

5 Concluding Remarks

In an exchange economy with CARA recursive utility functions and multiple subjective beliefs, we showed that the risk-free bond price goes down (and hence the risk-free interest rate goes up) monotonically as the markets become more complete. We then used this result to find the range of bond prices in incomplete markets which depend only on the primitives of the economy and not on market structures.

It was assumed throughout that the state space Ω is a finite set. This implies, among other things, that both the expected utility and the expected marginal utility are finite. If Ω is infinite, then they may be infinite and we may not be able to talk sensibly about the preference ordering or first-order conditions. A list of sufficient conditions for these to be finite in terms of utility function and the underlying probability space can be found in Nielsen (1993, Proposition 1 and 5). The case of special interest for CARA utility functions is where asset payoffs and initial endowments are normally distributed, but it is covered by his results.

An interesting aspect of the proof of our monotonicity result was to exploit the one-to-one correspondence between the welfare improvements of equilibrium allocations and changes in bond prices, owing to CARA utility functions. Since financial innovations often bring about welfare improvements, and since the changes in prices for existing assets are of interest from both positive and normative viewpoints, it would be nice if we could generalize this technique to a wider class of utility functions.

Finally, let us elaborate on the point that the solution (16) and (17) in Theorem 11 can be regarded as a mutual fund theorem for a limited class of recursive utility functions. That is, any increment in the risk-free consumptions on period 0 or the risky consumptions in period 1 is shared by the individual agents according to the proportion $(\gamma^1/\gamma, \ldots, \gamma^H/\gamma)$. This is consistent with the linear sharing rule of Wilson (1968), because $\dfrac{\gamma^h}{\gamma} = \dfrac{1/\beta^h}{1/\beta} = \dfrac{1/\alpha^h}{1/\alpha}$ and thus the proportions of the consumption sharing are those of the risk tolerances $1/\alpha^h$. Our solution, however, differs from Wilson's in that the constant terms for x_0^h and $\mathbf{x}^h$ need not be equal. Indeed,

$$
\begin{aligned}
&\log\left(\frac{\delta^h\lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\left(\frac{\delta^i\lambda^i}{\gamma^i}\right)\\
&= \left(\log\left(\frac{\lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\left(\frac{\lambda^i}{\gamma^i}\right)\right) + \left(\log\delta^h - \sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\delta^i\right).
\end{aligned}
$$

Thus the two constant terms would be equal if and only if $\delta^1 = \cdots = \delta^H$. The difference can therefore be accounted for by the heterogeneity of the subjective time discount factors. In other words, if all agents have the same subjective time discount factor (and Assumption 10 is satisfied), then the mutual fund theorem still holds in its original form, so that $((x_0^h, \mathbf{x}^h))_{h\in\{1,\ldots,H\}}$ is a Pareto-efficient allocation if and only if there exists a $(\eta^1,\ldots,\eta^H) \in \mathbb{R}^H$ such that $\sum_{h=1}^{H}\eta^h = 0$ and

$$
(x_0^h, \mathbf{x}^h) = \frac{\gamma^h}{\gamma}(x_0, \mathbf{x}) + \eta^h(1, \mathbf{1})
$$

for every h.

The representative agent's utility function, on the other hand, does not exactly take the expected utility function. Rather, it takes the recursive utility form, even when the δ^h may differ. This theorem thus provides a sufficient set of conditions under which the recursivity of individual agents' utility functions can be preserved under aggregation.

Appendix

Proof of Proposition 9

Let $((x_0^h, \mathbf{x}^h))_{h\in\{1,\ldots,H\}}$ be a corresponding consumption allocation of the equilibrium risk-free bond price q_1. Then, for every h,

$$
\mathbf{x}^h = \mathbf{e}^h + \frac{e_0^h - x_0^h}{q_1}\mathbf{1}.
$$

Recall Remark 2: the minimizing probability distribution in the definition of $\mathbf{E}_{u^h}^{\Pi^h}$ is not affected by adding any scalar multiple of $\mathbf{1}$, so

$$
\mathbf{E}_{u^h}^{\Pi^h}(\mathbf{x}^h) = \mathbf{E}_{u^h}^{\Pi^h}(\mathbf{e}^h) + \frac{e_0^h - x_0^h}{q_1}.
$$

Hence, by (8),

$$q_1 \exp(-\beta^h x_0^h) = \delta^h \exp\left(-\beta^h \mathbf{E}_{u^h}^{\Pi^h}(\mathbf{e}^h)\right) \exp\left(-\beta^h \frac{e_0^h - x_0^h}{q_1}\right).$$

Take the log of both sides and divide by β^h, then we obtain

$$\frac{1}{\beta^h}\log q_1 - x_0^h = \frac{1}{\beta^h}\log \delta^h - \mathbf{E}_{u^h}^{\Pi^h}(\mathbf{e}^h) - \frac{e_0^h - x_0^h}{q_1}.$$

Take the summation of both sides over h, then we obtain

$$\frac{1}{\beta}\log q_1 - e_0 = \sum_{h=1}^{H} \frac{1}{\beta^h}\log \delta^h - \sum_{h=1}^{H} \mathbf{E}_{u^h}^{\Pi^h}(\mathbf{e}^h).$$

Multiply β to both sides and take the exponential of them, then we obtain

$$q_1 \exp(-\beta e_0) = \prod_{h=1}^{H} \left(\delta^h\right)^{\beta/\beta^h} \prod_{h=1}^{H} \exp\left(-\beta \mathbf{E}_{u^h}^{\Pi^h}(\mathbf{e}^h)\right).$$

Here $\prod_{h=1}^{H} \left(\delta^h\right)^{\beta/\beta^h} = \delta$ and

$$\exp\left(-\beta \mathbf{E}_{u^h}^{\Pi^h}(\mathbf{e}^h)\right) = \mathbf{E}^{\Pi^h}\left(\exp(-\alpha^h \mathbf{e}^h)\right)^{\beta/\alpha^h}.$$

Thus

$$q_1 = \delta \frac{\prod_{h=1}^{H} \left(\mathbf{E}^{\Pi^h}(\exp(-\alpha^h \mathbf{e}^h))\right)^{\beta/\alpha^h}}{\exp(-\beta e_0)}$$

Proof of Theorem 11.

Since the objective function is strictly concave and the constraint functions are linear, it suffices to show that the given $((x_0^h, \mathbf{x}^h))_{h\in\{1,\dots,H\}}$ satisfies the first order condition of the maximization problem. It can be readily checked that the resource feasibility constraints are met. So we need to show that the weighted marginal utility vectors are equalized, i.e., $\lambda^h \frac{\partial U^h(x_0^h, \mathbf{x}^h)}{\partial x_0^h}$ and $\lambda^h \frac{\partial U^h(x_0^h, \mathbf{x}^h)}{\partial \mathbf{x}^h(\omega)}$ are independent of h, for all $\omega \in \Omega$.

¿From (14), and from the definition of x_0^h in (16), recalling $\beta^h\gamma^h = 1$, we have for every h,

$$\begin{aligned}
&\lambda^h \frac{\partial U^h(x_0^h, \mathbf{x}^h)}{\partial x_0^h} \\
=&\lambda^h \beta^h \exp\left(-\beta^h x_0^h\right), \\
=&\exp\left(\log\left(\frac{\lambda^h}{\gamma^h}\right)\right)\exp\left(-\frac{1}{\gamma}x_0\right)\exp\left(-\left(\log\left(\frac{\lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\left(\frac{\lambda^i}{\gamma^i}\right)\right)\right), \\
=&\exp\left(-\frac{1}{\gamma}x_0\right)\exp\left(\sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\left(\frac{\lambda^i}{\gamma^i}\right)\right),
\end{aligned}$$

which is independent of h.

Also, from the definition of $\mathbf{x}^h$ in (17), we have for every h and ω,

$$\begin{aligned}
&\exp\left(-\alpha^h \mathbf{x}^h(\omega)\right) \\
=&\exp\left(-\alpha^h \frac{\gamma^h}{\gamma}\mathbf{x}(\omega)\right)\exp\left(-\alpha^h\gamma^h\left(\log\left(\frac{\delta^h\lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\left(\frac{\delta^i\lambda^i}{\gamma^i}\right)\right)\right), \\
=&\exp\left(-\frac{\alpha^h}{\beta^h\gamma}\mathbf{x}(\omega)\right)\exp\eta^h,
\end{aligned}$$

where $\eta^h := -\alpha^h\gamma^h\left(\log\left(\frac{\delta^h\lambda^h}{\gamma^h}\right) - \sum_{i=1}^{H}\frac{\gamma^i}{\gamma}\log\left(\frac{\delta^i\lambda^i}{\gamma^i}\right)\right)$. Hence

$$\mathbf{E}^\pi\left(\exp\left(-\alpha^h\mathbf{x}^h\right)\right) = \mathbf{E}^\pi\left(\exp\left(-\frac{\alpha^h}{\beta^h\gamma}\mathbf{x}\right)\right)\exp\eta^h.$$

Thus for every h and ω, we have:

$$\begin{aligned}
&\frac{\lambda^h}{\pi(\omega)}\frac{\partial U^h(x_0^h, \mathbf{x}^h)}{\partial \mathbf{x}^h(\omega)} \\
=&-\lambda^h\delta^h\frac{\beta^h}{\alpha^h}\left(\mathbf{E}^\pi\left(\exp\left(-\alpha^h\mathbf{x}^h\right)\right)\right)^{\frac{\beta^h}{\alpha^h}-1}\left(-\alpha^h\right)\exp\left(-\alpha^h\mathbf{x}^h(\omega)\right), \\
=&\left(\lambda^h\delta^h\beta^h\left(\exp\eta^h\right)^{\frac{\beta^h}{\alpha^h}}\right)\left(\mathbf{E}^\pi\left(\exp\left(-\frac{\alpha^h}{\beta^h\gamma}\mathbf{x}(\omega)\right)\right)\right)^{\frac{\beta^h}{\alpha^h}-1}\exp\left(-\frac{\alpha^h}{\beta^h\gamma}\mathbf{x}(\omega)\right).
\end{aligned}$$

This can be seen to be independent of h as follows: The second and the third terms above are clearly so by Assumption 10. The first term is

$$\lambda^h\delta^h\beta^h\frac{\prod_{i=1}^{H}\left(\frac{\lambda^i\delta^i}{\gamma^i}\right)^{\gamma^i/\gamma}}{\frac{\lambda^h\delta^h}{\gamma^h}} = \prod_{i=1}^{H}\left(\frac{\lambda^i\delta^i}{\gamma^i}\right)^{\gamma^i/\gamma},$$

which is also independent of h. In conclusion, the weighted marginal utility is equalized at any ω, as we wanted to show.

As for the representative agent's utility function,

$$v^h(x_0^h) = -\exp\left(-\beta x_0\right) \frac{\prod_{i=1}^{H} \left(\frac{\lambda^i}{\gamma^i}\right)^{\gamma^i/\gamma}}{\frac{\lambda^h}{\gamma^h}}$$

and hence

$$\sum_{h=1}^{H} \lambda^h v^h(x_0^h) = -\exp\left(-\beta x_0\right) \gamma \prod_{i=1}^{H} \left(\frac{\lambda^i}{\gamma^i}\right)^{\gamma^i/\gamma}.$$

On the other hand,

$$v^h\left(\mathbf{E}_{u^h}^{\Pi^h}(\mathbf{x}^h)\right) = -\left(\mathbf{E}^{\pi}(\exp(-\alpha \mathbf{x}))\right)^{\beta^h/\alpha^h} \frac{\prod_{i=1}^{H} \left(\frac{\lambda^i \delta^i}{\gamma^i}\right)^{\gamma^i/\gamma}}{\frac{\lambda^h \delta^h}{\gamma^h}}$$

and hence

$$\sum_{h=1}^{H} \lambda^h \delta^h v^h\left(\mathbf{E}_{u^h}^{\Pi^h}(\mathbf{x}^h)\right) = -\delta\left(\mathbf{E}^{\pi}(\exp(-\alpha \mathbf{x}))\right)^{\beta/\alpha} \gamma \prod_{i=1}^{H} \left(\frac{\lambda^i}{\gamma^i}\right)^{\gamma^i/\gamma}.$$

Thus

$$\sum_{h=1}^{H} \lambda^h U^h(x_0^h, \mathbf{x}^h) = \left(\gamma \prod_{i=1}^{H} \left(\frac{\lambda^i}{\gamma^i}\right)^{\gamma^i/\gamma}\right) \left(-\exp(-\beta x_0) - \delta\left(\mathbf{E}^{\pi}(\exp(-\alpha \mathbf{x}))\right)^{\beta/\alpha}\right).$$

Hence, by letting

$$\lambda = \gamma \prod_{i=1}^{H} \left(\frac{\lambda^i}{\gamma^i}\right)^{\gamma^i/\gamma},$$

the proof is completed.

Proof of Proposition 12

Apply the equality (8) for the risk-free bond price to the representative agent's utility function identified by Theorem 11, then

$$\begin{aligned} q_1 &= \delta \frac{Dv\left(\mathbf{E}_u^{\pi}(\mathbf{e})\right)}{Dv(e_0)}, \\ &= \delta \frac{\exp\left(-\beta\left(-\frac{1}{\alpha}\log\left(\mathbf{E}^{\pi}\left(\exp\left(-\alpha \mathbf{e}\right)\right)\right)\right)\right)}{\exp(-\beta e_0)}, \\ &= \delta \frac{\left(\mathbf{E}^{\pi}\left(\exp\left(-\alpha \mathbf{e}\right)\right)\right)^{\beta/\alpha}}{\exp(-\beta e_0)}. \end{aligned}$$

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