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Abstract

In a model of a two-period exchange economy under uncertainty, we find both upper and lower bounds for the risk free interest rate when the agents' utility functions exhibit constant absolute risk aversion. These bounds are independent of the degree of market incompleteness, and so in particular these results show to what extent market incompleteness can explain the risk-free rate puzzle in this class of general equilibrium models with heterogeneous agents. A general method of finding these bounds without the assumption of constant absolute risk aversion is also presented.

JEL Classification Code: D52, D91, E21, E44, G12.

Keywords: The risk-free rate puzzle, constant absolute risk aversion, incomplete markets, general equilibrium.

1 Introduction

In this paper, we consider a model of an exchange economy under uncertainty with two consumption periods and one physical good, where consumption smoothing over time and uncertainty is done by asset transactions in financial markets. The preference relation of each agent is represented by a time-independent, additively separable utility function and the discount factor is common across them. Markets may be incomplete, and initial endowments may not be marketable.

In this setting, the equilibrium prices depend delicately on the structure of the incomplete markets in general, and exact prices cannot be obtained without knowing specific structure of markets. But even in the situation where it is regarded plausible to assume that the markets are incomplete, the exact structure of markets is difficult to observe in the context of financial markets; it is one thing to find that some markets are missing and so some types of risks cannot be insured, but it is another to identify exactly which type of risk is uninsurable. Thus, in order to learn equilibrium prices, it is desirable to know theoretical ranges of possible equilibrium prices, i.e., upper and lower bounds of equilibrium prices, which *do not* depend on the fine details of the market structure. We are interested in finding such bounds.

In this paper we concentrate on bounds for the risk free rate of return. This is of special interest since there has been extensive research under the name of the risk free rate puzzle, given by Weil (1992). Kocherlakota (1996) provides an excellent survey on this topic. So, in this context, just as Weil's (1992) original contribution, our aim is to provide a benchmark for the question of to what extent the market incompleteness can possibly explain the observed risk-free interest rate in general equilibrium models with heterogeneous agents with time separable utility functions.

The contribution of this paper is two fold. The first contribution of this paper is to identify the upper and lower bounds for the bond price, which only depend on the primitives of the economy, when every agent's utility function exhibits constant absolute risk aversion (CARA for short). The bounds are succinctly related to the degree of risk aversion and the risk properties of initial endowments. We emphasize that they are independent of market incompleteness; that is, we do not assume anything as to what kind of risky assets are available for trade.

The second contribution is to provide a general method of finding upper and lower bounds when there is no condition imposed on the utility functions other than the convexity of derivatives (the prudence of an agent's utility function in Kimball's (1992) sense), and there is no assumption on the incompleteness of markets. As a simple application of this general method, we show that the equilibrium price of the risk-free bond is no lower than the discount factor, provided the derivative of every agent's utility function is a convex function and the expected aggregate endowment in the second period is no larger than the first-period aggregate endowment.

As far as we know, no existing contribution has clarified bounds on the bond price (or lower bound on the interest rate) in a succinct way. So we believe that not only these results complement Weil's original contribution by further clarifying the theoretical explanatory power of market incompleteness for the risk free rate puzzle, but also these will serve as a valuable tool for finding a rough estimate of equilibrium interest rates, since computation of an equilibrium price system is not necessarily a straightforward task when markets are incomplete. In a broader perspective, we believe that the analysis of this paper suggests a new and important line of research in the so called general equilibrium with incomplete markets (GEI) literature. The existence and inefficiency results have been established in general setups, but when it comes to the detailed pricing implications, mostly computational approaches with specific market structures are prevalent. This paper is one of the first attempts to fill the gap for a deeper understanding of GEI.

Let us briefly mention related works. Levine and Zame (1998, 2002) considered an infinite-horizon economy under uncertainty with heterogeneous agents to investigate how the possibility of intertemporal income transfers weakens equilibrium implications of incomplete markets. A key step of their analysis is to find an upper bound on the interest rate. Our technique is inspired by theirs, though we do not need to make any a priori distinction between the cases with and without the aggregate risk as they did.[1] Willen (1998) uses a similar technique, again for the case without aggregate risk, in the context of international trade between two countries with differing market incompleteness. The result for the bounds with CARA utility functions generalizes an earlier result shown in Elul (1997) on the risk-free rate puzzle, who identified several conditions under which introducing a new security raises the equilibrium risk-free interest

[1] To be exact, our technique was inspired by a working paper version of Levine and Zame (2002), and they subsequently adopted our argument with acknowledgement in the published version.

rate.[2]

The next section presents the model of this paper. Section 3 deals with the case of CARA utility functions and find upper and lower bounds for the risk-free bond price. Section 4 discusses a general method of finding bounds on the bond price and shows that the bond price cannot be lower than the common discount factor. Section 5 concludes with an extension to infinite dimensional cases, and also suggests a couple of directions of future research.

2 The Model

There are two trading periods, 0 and 1, and there is a single perishable good in each period. There is no uncertainty in the first period, when the consumption good and assets are exchanged. At the beginning of the second period, the assets pay off, and then consumption takes place. The uncertainty in the second period is described by a finite state space Ω, and each state $\omega \in \Omega$ occurs with probability $\mu(\omega) > 0$. We often refer to each function from Ω to $\mathbb{R}$ as a random variable. Denote by $\mathbf{1}$ the function from Ω to $\mathbb{R}$ that takes constant value one. The constant variable $\mathbf{1}$ will be interpreted as the risk-free discount bond. Let X be a linear subspace in the set of all random variables such that $\mathbf{1} \in X$. We take the commodity space to be $\mathbb{R} \times X$ and denote by $\mathbf{E}$ the expectation operator with respect to μ. A generic element of $\mathbb{R} \times X$ will be denoted by $(x_0, \mathbf{x})$, where x_0 corresponds to consumption in the first period, and $\mathbf{x}$ is a random variable that corresponds to consumption in the second period.

There are H agents in the economy. Each *agent*, indexed $h \in \{1, \cdots, H\}$, is characterized by:

- Time invariant von Neumann Morgenstern utility function u^h. It is increasing, strictly concave and continuously differentiable. Its derivative Du^h is assumed to be a convex function; that is, it exhibits prudence.

- Initial endowment vector is in the consumption set; that is, $(e_0^h, \mathbf{e}^h) \in \mathbb{R} \times X$.

We assume that the agents have a common discount factor $\delta > 0$. Thus the preference relation of agent h is represented by the expected utility function $U^h : \mathbb{R} \times X \to \mathbb{R}$ defined by

[2] The key step for this result was reported in an unpublished paper of Hara (1998).

$$U^h(x_0^h, \mathbf{x}^h) \equiv u^h\left(x_0^h\right) + \delta \sum_{\omega\in\Omega} u^h\left(\mathbf{x}^h(\omega)\right)\mu(\omega) = u^h\left(x_0^h\right) + \delta\mathbf{E}\left(u^h\left(\mathbf{x}^h\right)\right). \tag{1}$$

The agents trade assets in period 0. The *market span* is a linear subspace M of the commodity space X. An element of M corresponds to a vector of returns of some portfolio of the assets. The arbitrage free prices of the portfolios are described by a linear function $p: M \to \mathbb{R}$, which is referred to as a *state price function.* Since the prices of underlying assets generating the market span M can be recovered from a state price function, we do not model individual specifications of these assets explicitly. This also facilitates a simpler exposition for our results.

The agents are assumed to be price takers. Agent h's utility maximization problem is, given market span M and a state price function p:

$$\begin{array}{ll} \displaystyle\max_{(x_0^h, \mathbf{x}^h)\in\mathbb{R}\times X} & U^h(x_0^h, \mathbf{x}^h) \\ \text{subject to:} & \mathbf{x}^h - \mathbf{e}^h \in M, \\ & (x_0^h - e_0^h) + p(\mathbf{x}^h - \mathbf{e}^h) = 0. \end{array}$$

The first constraint implies that the net trade vector $\mathbf{x}^h - \mathbf{e}^h$ can be achieved through asset trades, and the second constraint is the budget constraint. Notice that the first-period consumption is assumed to be the numéraire, whose price equals one. Under our assumptions, since $p(\mathbf{z})$ units of period 0 consumption must be given up for a budget feasible net trade $\mathbf{z}$, the first order necessary and sufficient conditions for the maximization are the feasiblity and $p(\mathbf{v}) = \delta\frac{\mathbf{E}\left[\mathbf{z}Du^h\left(\mathbf{x}^h\right)\right]}{Du^h\left(x_0^h\right)}$ for any $\mathbf{z} \in M$.

The case of complete markets corresponds to the case where the market span M coincides with the commodity space X. Since our purpose is to characterize the equilibrium price of bond without reference to the structure of markets, the market span M should not be related to the other primitives of the economy a priori. So in particular we do *not* require $\mathbf{e}^h \in M$. Besides, the results in Section 3 would be much simpler, but uninteresting, if $\mathbf{e}^h \in M$ were required.

We write $\overline{e}^h = \mathbf{E}\left(\mathbf{e}^h\right)$, $\mathbf{e} = \sum_h \mathbf{e}^h$, $\overline{e} = \mathbf{E}(\mathbf{e})$, and $e_0 = \sum_h e_0^h$. So $\overline{e}^h$ is agent h's expected endowment in the second period, $\mathbf{e}$ is the aggregate endowment in the second period, which is a random variable, $\overline{e}$ is the expected aggregate endowment in the second period, and e_0 the aggregate endowment in the first period.

We say that a state price function p and a consumption allocation $((x_0^h, \mathbf{x}^h))_{h\in\{1,\dots,H\}}$ constitute an *equilibrium* for the economy with market span M if, for every h, $(x_0^h, \mathbf{x}^h)$ is a solution to the above maximization problem and $\sum_{h=1}^H (x_0^h, \mathbf{x}^h) = \sum_{h=1}^H (e_0^h, \mathbf{e}^h)$. It can then be shown that the asset markets clear automatically. It is known that an equilibrium exists, and any equilibrium is constrained efficient in the sense that no welfare improving reallocation of goods respecting the market span, since the market span is fixed.

So the equilibrium price of the risk-free discount bond is $p(\mathbf{1})$ and the equilibrium risk-free interest rate is $p(\mathbf{1})^{-1} - 1$. Hence a lower interest rate means a higher bond price, and vice versa. Note that the first order condition for utility maximization implies that the equilibrium bond price is equal to the discounted expected marginal utility; that is,

$$\delta \frac{\mathbf{E}\left(Du^h\left(\mathbf{x}^h\right)\right)}{Du^h\left(x_0^h\right)} = p\left(\mathbf{1}\right) \tag{2}$$

holds in equilibrium.

To conclude section, a few remarks on the model are due. First, note that the time-separability and time-invariance of the utility functions U^h are maintained. Dropping these properties and using recursive or habit-formation utility functions would result in a different equilibrium level of the risk-free interest rate, which is a method of solving the risk-free rate puzzle that we shall not pursue here. Secondly, the linearity of M and p means that there are no transaction costs; in particular there is no short sales constraint. This is an important assumption of this model, because introducing transaction costs is known to partially solve the equity premium and risk-free rate puzzles. Finally, the agents uses a common probability μ as well as a common discount factor δ. Calvet, Grandmont, and Lemaire (1999) analyzed the consequences of dropping the assumption of a common probability. As Gollier (2001) mentioned, the subjective time discount factors could be formally identified with subjective probabilities, and so the difficulties arising from incorporating heterogeneous time discount factors are the same as those arising from heterogeneous beliefs.

3 Bounds with CARA Utility Functions

In this section, we consider the case of CARA utility functions. We start with such a restrictive class because sharper results obtain in this case, and these also provide intuitions for general cases considered in the next section.

We assume that von Neumann Morgenstern utility function u^h has a constant coefficient $\alpha^h > 0$ of absolute risk aversion, so that $u^h(w) = -\exp\left(-\alpha^h w\right)$ for every h. The reciprocal $1/\alpha^h$ of the absolute risk aversion is called the absolute risk tolerance and denoted by γ^h. The utility function can now be written as

$$U^h(x_0^h, \mathbf{x}^h) = -\exp(-\alpha^h x_0^h) - \delta \mathbf{E}(\exp(-\alpha^h \mathbf{x}^h)).$$

Notice that $Du^h(w) = \alpha^h \exp\left(-\alpha^h w\right)$ is a convex function, and so the prudence assumption of the previous section is met. The condition (2) in this case is reduced to:

$$p(\mathbf{1}) = \frac{\delta^h \mathbf{E}\left[\exp(-\alpha^h \mathbf{x}^h)\right]}{\exp(-\alpha^h x_0^h)}. \tag{3}$$

3.1 Monotonicity of the risk-free interest rates

A special character of a CARA utility function is that its logarithmic transformation is quasi-linear in the direction of $(1, \mathbf{1})$. Formally, for each h, define $W^h : \mathbb{R} \times X \to \mathbb{R}$ by $W^h(x_0^h, \mathbf{x}^h) = (-1/\alpha^h)\log(-U^h(x_0^h, \mathbf{x}^h))$. This is well defined because $U^h(x_0^h, \mathbf{x}^h) < 0$ and it represents the same preference as U^h because the function $u \mapsto (-1/\alpha^h)\log(-u)$ is strictly increasing.

Lemma 1 *The transformed function W^h is quasi-linear in $(1, \mathbf{1})$.*

Proof. For any $(x_0^h, \mathbf{x}^h) \in \mathbb{R} \times X$, taking into account the orthogonal decomposition $(x_0^h, \mathbf{x}^h) = \left\{(x_0^h, \mathbf{x}^h) - t(1, \mathbf{1})\right\} + t(1, \mathbf{1})$ where $t = \frac{1}{2}\left(x_0^h + \mathbf{E}\left[\mathbf{x}^h\right]\right)$, we have

$$W^h(x_0^h, \mathbf{x}^h) \tag{4}$$
$$= -\frac{1}{\alpha^h}\log\{\exp(-\alpha^h x_0^h) + \delta \mathbf{E}\left[\exp(-\alpha^h \mathbf{x}^h)\right]\} \tag{5}$$
$$= -\frac{1}{\alpha^h}\log\left\{\exp(-\alpha^h x_0^h - t)\exp\left(-\alpha^h t\right) + \delta \mathbf{E}\left[\exp(-\alpha^h\left(\mathbf{x}^h - t\mathbf{1}\right)\exp\left(-\alpha^h t\mathbf{1}\right)\right)\right]\right\} \tag{6}$$
$$= -\frac{1}{\alpha^h}\log\left\{\exp(-\alpha^h x_0^h - t) + \delta \mathbf{E}\left[\exp(-\alpha^h\left(\mathbf{x}^h - t\mathbf{1}\right)\right)\right]\right\} + t. \tag{7}$$

So W^h is linear in the direction of $(1, \mathbf{1})$, as we wanted. ■

This fact can also be confirmed by showing that the derivative of W^h in the direction of $(1, \mathbf{1})$ is one. Notice also that the non-linear part of the expression above is strictly concave.

To obtain a lower and an upper bounds for the risk free rate, we first establish the following result:[3]

[3] This result was originally proved in Hara (1998). This is a generization of Proposition 1 of Elul (1997).

Proposition 2 *Let M and N be two market spans such that $\mathbf{1} \in M \subset N$. Let $p : M \to \mathbb{R}$ be an equilibrium state price function for M and $q : N \to \mathbb{R}$ be an equilibrium state price function for N. Then $p(\mathbf{1}) \geq q(\mathbf{1})$.*

Proof. Let $((x_0^h, \mathbf{x}^h))_{h\in\{1,\cdots,H\}}$ be a consumption allocation corresponding an equilibrium state price p. From (3), we can also write W^h for every h as follows:

$$W^h(x_0^h, \mathbf{x}^h) = -\gamma^h \log((1 + p(\mathbf{1})) \exp(-\alpha^h x_0^h)) = -\gamma^h \log(1 + p(\mathbf{1})) + x_0^h.$$

Summing these over h, using the market clearing condition, and writing $\gamma = \sum_{h=1}^H \gamma^h$, we obtain:

$$\sum_{h=1}^H W^h(x_0^h, \mathbf{x}^h) = -\gamma \log(1 + p(\mathbf{1})) + \sum_{h=1}^H x_0^h = -\gamma \log(1 + p(\mathbf{1})) + \sum_{h=1}^H e_0^h. \tag{8}$$

Writing $((\hat{x}_0^h, \hat{\mathbf{x}}^h))_{h\in\{1,\cdots,H\}}$ for a consumption allocation for q, we can similarly obtain:

$$\sum_{h=1}^H W^h(\hat{x}_0^h, \hat{\mathbf{x}}^h) = -\gamma \log(1 + q(\mathbf{1})) + \sum_{h=1}^H e_0^h. \tag{9}$$

By Lemma 1, every W^h is quasi-linear in $(1, \mathbf{1}) \in \mathbb{R} \times N$ and the constrained efficiency of every equilibrium allocation, the allocation $((\hat{x}_0^h, \hat{\mathbf{x}}^h))_{h\in\{1,\cdots,H\}}$ must be a solution to the utilitarian welfare maximization problem:

$$\begin{array}{ll} \text{Max}_{((z_0^h,\mathbf{z}^h))_{h\in\{1,\cdots,H\}}} & \sum_{h=1}^H W^h(z_0^h, \mathbf{z}^h) \\ \text{subject to} & \mathbf{z}^h - \mathbf{e}^h \in N \text{ for every } h, \\ & \sum_{h=1}^H (z_0^h, \mathbf{z}^h) = \sum_{h=1}^H (e_0^h, e^h). \end{array} \tag{10}$$

Since $M \subset N$, the equilibrium allocation $((x_0^h, \mathbf{x}^h))_{h\in\{1,\cdots,H\}}$ for M satisfies the constraints in (10). Hence $\sum_{h=1}^H W^h(x_0^h, \mathbf{x}^h) \leq \sum_{h=1}^H W^h(\hat{x}_0^h, \hat{\mathbf{x}}^h)$. Thus, by (8) and (9),

$$-\gamma \log(1 + p(\mathbf{1})) + \sum_{h=1}^H e_0^h \leq -\gamma \log(1 + q(\mathbf{1})) + \sum_{h=1}^H e_0^h. \tag{11}$$

Hence $p(\mathbf{1}) \geq q(\mathbf{1})$. ■

Remark 3 Each W^h is quasi linear in $(1, \mathbf{1})$ with strictly concave non-linear component, so is the sum. Therefore, the solution to (10) is uniquely determined up to transfers in the direction of in $(1, \mathbf{1})$. This means that the inequality (11) must be strict if at least one agent consume differently in the two equilibrium, and so $p(\mathbf{1}) > q(\mathbf{1})$.

The result says less complete the markets M are, the lower the risk-free interest rate is. This monotonicity is a remarkable property of the CARA economies and it does not hold in the general setting. This should not be confused but contrasted with the *invariance* property of *risky* asset prices established by Oh (1996) and his predecessors, the property that with CARA utility functions and normally distributed asset payoffs, the relative prices among risky assets[4] do not depend on the market span.

Proposition 2 implies that the equilibrium risk-free interest rate is highest when $M = X$, i.e., the markets are complete, and it is lowest when M coincides with the line spanned by $\mathbf{1}$, i.e., the risk-free bond is the only asset traded in markets. In order to find the upper and lower bounds on the interest rates that are independent of market spans, therefore, it is sufficient to identify those rates with the complete markets and with the least complete markets, which we shall do in the following subsections.

3.2 Bounds on the risk-free rate

Even for the CARA economies, the equilibrium bond price delicately depends on the structure of the market span M when markets are incomplete. But with the complete markets, the mutual fund theorem enables us to obtain an explicit formula for the bond price, which depends on the initial endowments only through the aggregate values e_0 and $\mathbf{e}$; and the aggregate risk tolerance $\sum_h \gamma^h$ how they are distributed among the agents is irrelevant for the complete market bond price. So we can use the explicit formula to obtain a desired bound.

Proposition 4 *Denote $\gamma = \sum_h \gamma^h$. Let $p : X \to \mathbb{R}$ be an equilibrium state price function for the complete market span X, then*

$$p(\mathbf{1}) = \delta \mathbf{E}\left(\exp\left(-\frac{1}{\gamma}\left(\mathbf{e} - e_0\mathbf{1}\right)\right)\right).$$

Proof. We shall explicitly construct an equilibrium where the price of bond is given as in the statement. This is sufficient since it can be shown that an equilibrium is unique from the fact that the agents' preferences are smooth and quasi-linear with respect to $(1, \mathbf{1})$.

[4] To be exact, we need to assume that the payoffs of the risky assets have zero mean, because otherwise the change in the risk-free interest rate would affect relative prices of risky assets according to how large their means are.

First for each h, find $(x_0^h, \mathbf{x}^h) \in \mathbb{R} \times X$ such that

$$\mathbf{x}^h - x_0^h \mathbf{1} = \frac{\gamma^h}{\gamma} (\mathbf{e} - e_0 \mathbf{1}), \tag{12}$$

and

$$x_0^h + \delta \mathbf{E} \left(\mathbf{x}^h \exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \right) = e_0^h + \delta \mathbf{E} \left(\mathbf{e}^h \exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \right), \tag{13}$$

hold simultaneously. Such an $(x_0^h, \mathbf{x}^h)$ exists uniquely, since if we substitute $\mathbf{x}^h$ in (13) with $\mathbf{x}^h = x_0^h \mathbf{1} + (\gamma^h/\gamma)(\mathbf{e} - e_0 \mathbf{1})$, the left hand side is increasing in x_0^h, and it goes to $+\infty$ or $-\infty$ as x_0^h approaches to $+\infty$ or $-\infty$. We shall show that the profile of consumption bundles, $((x_0^h, \mathbf{x}^h))_{h \in \{1,\dots,H\}}$, and the state price function given by

$$\mathbf{z} \mapsto \delta \mathbf{E} \left(\mathbf{z} \exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \right). \tag{14}$$

constitute an equilibrium.

Take the summation of both sides of equalities (12), and we have

$$\sum_h \mathbf{x}^h - \mathbf{e} = \left(\sum_h x_0^h - e_0 \right) \mathbf{1}. \tag{15}$$

From (13), we have

$$\delta \mathbf{E} \left(\left(\mathbf{x}^h - \mathbf{e}^h \right) \exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \right) = -\left(x_0^h - e_0^h \right),$$

for each h, and so summing these up, and using (15), we have

$$\begin{aligned} &\delta \mathbf{E} \left(\left(\sum_h \mathbf{x}^h - \mathbf{e} \right) \exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \right) \\ &= \delta \left(\sum_h x_0^h - e_0 \right) \mathbf{E} \left(\exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \right) \\ &= -\left(\sum_h x_0^h - e_0 \right). \end{aligned}$$

Since $\mathbf{E}(\exp(-(1/\gamma)(\mathbf{e} - e_0 \mathbf{1}))) > 0$, this equality implies $\sum_h x_0^h = e_0$, and so that $\sum_h \mathbf{x}^h = \mathbf{e}$ from (15). Hence the allocation is feasible.

The equality (12) also implies that, for all h,

$$\begin{aligned} \delta \frac{Du^h(\mathbf{x}^h)}{Du^h(x_0^h)} &= \delta \exp \left(-\alpha^h \left(\mathbf{x}^h - x_0^h \mathbf{1} \right) \right) \\ &= \delta \exp \left(-\alpha^h \frac{\gamma^h}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right) \\ &= \delta \exp \left(-\frac{1}{\gamma} (\mathbf{e} - e_0 \mathbf{1}) \right), \end{aligned}$$

which does not depend on h. Thus the allocation $\left((x_0^h, \mathbf{x}^h)\right)_{h\in\{1,\cdots,H\}}$ is Pareto efficient and its supporting state price function coincides with the state price function defined by (14). Finally, the equality (13) implies that every agent's budget constraint is satisfied. Hence the state price function is an equilibrium price function. The proof is thus completed. ■

Let us now consider the lowest risk free rate, for the case where M is equal to the line spanned by $\mathbf{1}$. i.e., the risk free asset is the only marketable asset.

For future reference, for each h, define

$$c^h \equiv \frac{\mathbf{E}\left(Du^h\left(\mathbf{e}^h\right)\right)}{Du^h\left(\overline{e}^h\right)} = \frac{\mathbf{E}\left(\exp\left(-\alpha^h \mathbf{e}^h\right)\right)}{\exp\left(-\alpha^h \overline{e}^h\right)} = \mathbf{E}\left(-\alpha^h\left(\mathbf{e}^h - \overline{e}^h \mathbf{1}\right)\right).$$

So c^h is proportional to the first order condition (2) evaluated at the (average) endowment vector. We shall see in the next section that c^h measures agent h's prudence evaluated at his initial endowment, and so we shall refer to c^h as the *prudence at the initial endowments*.[5] Another way to look at this number is to apply the second-order Taylor approximation $\exp(w) \approx 1 + w + 2^{-1}w^2$. Then $c^h \approx 1 + 2^{-1}\left(\alpha^h\right)^2 \mathrm{Var}(\mathbf{e}_h)$, or $\alpha^h \mathbf{S}(\mathbf{e}_h) \approx \left(2(c^h - 1)\right)^{1/2}$, where $\mathbf{S}$ denotes the standard deviation. The numbers c^h thus measure the variability of his second-period initial endowments weighted by the constant coefficients of absolute risk aversion. As shown by Duffie and Jackson (1990), Demange and Laroque (1995), Ohashi (1995), Rahi (1995), and others, they help provide a necessary and sufficient condition for the optimal asset structure when only a limited number of assets can be traded in markets.

Now we are ready to state our result for the lowest risk free rate: let $\alpha_{\max} = \max\{\alpha^1, \dots, \alpha^H\}$ and $\alpha_{\min} = \min\{\alpha^1, \dots, \alpha^H\}$, and set

$$b^A = \begin{cases} \exp\left(-\alpha_{\max}\dfrac{\overline{e} - e_0}{H}\right) & \text{if } \overline{e} - e_0 \leq 0, \\ \exp\left(-\alpha_{\min}\dfrac{\overline{e} - e_0}{H}\right) & \text{if } \overline{e} - e_0 > 0. \end{cases} \tag{16}$$

Proposition 5 *Let M be the line spanned by $\mathbf{1}$ and $p : M \to \mathbb{R}$ be an equilibrium state price function for M. Then:*

1. $p(\mathbf{1}) \leq \delta b^A \max\{c^1, \dots, c^H\}$.

[5] This is related to but different from the absolute prudence of Kimball (1990), $-D^3u^h(x^h)/D^2u^h(x^h)$ in that it is measured for a random variable and therefore depends on his absolute prudence at different consumption levels.

2. If, moreover, $e^0 = \overline{e}$ *and* $c^1 = \cdots = c^H$*, then* $p(\mathbf{1}) = \delta c^h$ *for every* h*.*

Proof. Let $\left((x_0^h, \mathbf{x}^h)\right)_{h=1}^H$ be the associated equilibrium allocation.

1. From the resource constraint there exists an h such that $\overline{x}^h - x_0^h \geq (\overline{e} - e_0)/H$. Then for that h,

$$\frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} = \frac{\exp\left[-\alpha^h \overline{x}^h\right]}{\exp\left[-\alpha^h x_0^h\right]} = \exp\left[-\alpha^h(\overline{x}^h - x_0^h)\right] \leq \exp\left[-\frac{\alpha^h(\overline{e} - e_0)}{H}\right]. \tag{17}$$

By the definition of b, we have

$$\exp\left[-\frac{\alpha^h(\overline{e} - e_0)}{H}\right] \leq b.$$

Hence

$$\frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} \leq b \tag{18}$$

for some h.

On the other hand, since M is spanned by $\mathbf{1}$, we can write $\mathbf{x}^h = \mathbf{e}^h + y^h\mathbf{1}$ for some $y^h \in \mathbb{R}$. By construction $\bar{x}^h = \bar{e}^h + y^h$, and so we have $\mathbf{x}^h - \overline{x}^h\mathbf{1} = \mathbf{e}^h - \overline{e}^h\mathbf{1}$. Hence

$$\begin{aligned} \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)} &= \frac{\mathbf{E}\left[\exp\left(-\alpha^h\mathbf{x}^h\right)\right]}{\exp\left(-\alpha^h\overline{x}^h\right)} & (19)\\ &= \frac{\mathbf{E}\left[\exp\left(-\alpha^h\mathbf{e}^h\right)\exp\left(-\alpha^h y^h\mathbf{1}\right)\right]}{\exp\left(-\alpha^h\overline{e}^h\right)\exp\left(-\alpha^h y^h\right)} \\ &= \frac{\mathbf{E}\left(\exp\left(-\alpha^h\mathbf{e}^h\right)\right)}{\exp\left(-\alpha^h\overline{e}^h\right)} \\ &= c^h & (20) \end{aligned}$$

for every h.

By the first-order condition for the bond demand, we have

$$p(\mathbf{1}) = \delta\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(x_0^h)} = \delta\frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)}\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)}.$$

By (18) and (20), therefore,

$$p(\mathbf{1}) \leq \delta b^A c.$$

2. The symmetric argument is applicable. Since $e^0 = \overline{e}$, we have $b^A = 1$ and

$$\frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} \geq 1$$

for some h. By equality (20), we have $p(\mathbf{1}) \geq \delta c^h$. Thus

$$p(\mathbf{1}) \geq \delta\min\{c^1, \ldots, c^H\}.$$

Since $\min\{c^1, \ldots, c^H\} = \max\{c^1, \ldots, c^H\}$, this and the first part establish the second part. ■

By gathering the preceding results, we can now give the upper and lower bounds on the equilibrium bond price.

Proposition 6 *Let M be a market span such that $\mathbf{1} \in M$ and $p : M \to \mathbb{R}$ be an equilibrium state price function for M, then*

$$\delta \mathbf{E}\left(\exp\left(-\frac{1}{\gamma}(\mathbf{e} - e_0\mathbf{1})\right)\right) \leq p(\mathbf{1}) \leq \delta b^A \max\{c^1, \ldots, c^H\}.$$

Proof. This can be obtained by combining Propositions 2, 4, and 5. ■

3.3 Negative risk-free interest rates

The bound we obtained delicately depends on the prudence at the initial endowments c^h. Since this parameter requires information on agents' individual risks, one may wonder if this can be replaced with some aggregate property of the economy. In this subsection, we argue that this task will be difficult by considering a subclass of CARA economies. To give the idea first, notice that the second part of Proposition 5 states that the upper bound of the bond price in the first part is indeed attained if the expected aggregate endowment is constant over time, and all agents have the same prudence at the initial endowments c^h. So if c^h can get arbitrary large whereas the aggregate distribution of initial endowments remains constant, it shows that the equilibrium bond price in incomplete markets can be arbitrarily large (so the rate of interest is even negative – recall that the good is not storable) while the bond price in the complete markets stays at the discount factor δ.

Now we formally state the class of economies. For each h, fix initial endowments $(e_0^h, \mathbf{e}^h) \in \mathbb{R} \times X$ such that (1) each $\mathbf{e}^h$ have the same distribution with a positive variance and (2) $\mathbf{e} = e_0\mathbf{1}$. Set $\alpha^1 = \cdots = \alpha^H \equiv \alpha$. An example of the initial endowments that satisfies the first two conditions is that $\Omega = \{1, \ldots, H\}$, $\mu(\omega) = 1/H$, $\mathbf{e}^h(\omega) = 1$ if $\omega = h$ and $\mathbf{e}^h(\omega) = 0$ otherwise, and $e_0^h = 1$ for every h. For each $s > 0$, define the *s-stretched economy* as the economy consisting of the H agents, with initial endowments $\left(e_0^h, \mathbf{e}^h + s\left(\mathbf{e}^h - \overline{e}^h\mathbf{1}\right)\right) \in \mathbb{R} \times X$ and the same constant coefficients $\alpha^h = \alpha$ of absolute risk aversion as before. Thus we have a series of CARA economies indexed by s. Note that the aggregate endowment of the s-stretched economy is independent of s, and it is $(e_0, \mathbf{e})$ for every $s > 0$. So one may hope to find a lower bound for the risk free rate which depends on $(e_0, \mathbf{e})$, as well δ and α.

Proposition 7 *Let M be the line spanned by $\mathbf{1}$ and $p_s : M \to \mathbb{R}$ be an equilibrium state price function for M and let $q_s : X \to \mathbb{R}$ be an equilibrium state price function with the complete markets of the s-stretched economy. Then:*

1. $p_s(\mathbf{1}) \to \infty$ *as* $s \to \infty$.

2. $q_s(\mathbf{1}) = \delta$ *for every* $s > 0$.

Part 1 of this proposition claims that the equilibrium bond price with the least complete market becomes unboundedly large, while the bond price in the complete markets is equal to δ regardless of the values of s. In particular, even when the bond price equals to the common discount factor δ at the complete-market equilibrium, the bond price may be greater than one and the risk-free interest rate may well be negative.

Proof. For each $s > 0$, define

$$\begin{aligned} c(s) &= \frac{\mathbf{E}\left(\exp\left(-\alpha^h\left(\mathbf{e}^h + s\left(\mathbf{e}^h - \bar{e}^h\mathbf{1}\right)\right)\right)\right)}{\exp\left(-\alpha^h\mathbf{E}\left(\mathbf{e}^h + s\left(\mathbf{e}^h - \bar{e}^h\mathbf{1}\right)\right)\right)} \\ &= \frac{\mathbf{E}\left(\exp\left(-\alpha^h\left(\mathbf{e}^h + s\left(\mathbf{e}^h - \bar{e}^h\mathbf{1}\right)\right)\right)\right)}{\exp\left(-\alpha^h\bar{e}^h\right)}. \end{aligned}$$

This is well defined since all the $\mathbf{e}^h + s\left(\mathbf{e}^h - \bar{e}^h\mathbf{1}\right)$ have the same distribution and all the α^h are equal, the last expression does not depend on h. Part 2 of Proposition 5 implies that $p_s(\mathbf{1}) = \delta c(s)$ for every s. Moreover, writing $B^h = \{\omega \in \Omega \mid \mathbf{e}^h(\omega) - \bar{e}^h \leq 0\}$, we also have

$$\begin{aligned} c(s) &= \mathbf{E}\left(\exp\left(-\alpha^h(s+1)\left(\mathbf{e}^h - \bar{e}^h\mathbf{1}\right)\right)\right) \\ &\geq \mu(B^h)\mathbf{E}\left(\exp\left(\alpha^h(s+1)\left(\bar{e}^h\mathbf{1} - \mathbf{e}^h\right)\right) \mid B^h\right) \\ &\geq \mu(B^h)\mathbf{E}\left(\alpha^h(s+1)\left(\bar{e}^h\mathbf{1} - \mathbf{e}^h\right) \mid B^h\right) \\ &\geq \mu(B^h)\alpha^h(s+1)\,\mathbf{E}\left(\bar{e}^h\mathbf{1} - \mathbf{e}^h \mid B^h\right) \end{aligned}$$

Since $\mathbf{e}^h$ has a positive variance, $\mu(B^h) > 0$ and $\mathbf{E}\left(\bar{e}^h\mathbf{1} - \mathbf{e}^h \mid B^h\right) > 0$. Hence

$$\mu(B^h)\alpha^h(s+1)\,\mathbf{E}\left(\bar{e}^h\mathbf{1} - \mathbf{e}^h \mid B^h\right)$$

as $s \to \infty$. Thus, as $s \to \infty$, $c(s) \to \infty$ so $p_s(\mathbf{1}) = c(s) \to \infty$.

2. Since the aggregate endowment of the s-stretched economy equals $(e_0, \mathbf{e})$ for every $s > 0$, Proposition 4 implies that

$$p_s(\mathbf{1}) = \delta\mathbf{E}\left(\exp\left(-\frac{1}{\gamma}\left(\mathbf{e} - e_0\mathbf{1}\right)\right)\right).$$

Since $\mathbf{e} = e_0 \mathbf{1}$ by the second condition, $p_s(\mathbf{1}) = \delta$. ■

4 A General Method of Finding Bounds

In this section, inspired by the proof of Proposition 5, we discuss a general method of finding upper and lower bounds for the risk-free interest rate, without assuming constant absolute risk aversion. As an easy consequence of this, we show that if the expected aggregate endowment is non-increasing over time, then the equilibrium risk-free bond price is not lower than the common discount factor δ.

Recall that the first order condition (2) holds under our maintained assumption, which can be rewritten as, for *every* agent h,

$$p(\mathbf{1}) = \delta \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(x_0^h)} = \delta \frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)}, \tag{21}$$

where $\bar{x}^h = \mathbf{E}\left(\mathbf{x}^h\right)$. Since the equality holds for all h, if we can find bounds for $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(x_0^h)$ for *some* h, these bounds serve as bounds for the equilibrium risk-free interest rate, or, equivalently, for the risk-free bond price. The right-hand side of (21) shows that it is the product of two factors, and we shall discuss the properties of these in turn.

The first factor $Du^h(\overline{x}^h)/Du^h(x_0^h)$ is the intertemporal marginal rate of substitution. If $e_0 \geq \overline{e}$, then $x_0^h \geq \overline{x}^h$ for *some* h and hence this factor is no smaller than 1 for this h. Similarly, if $e_0 \leq \overline{e}$, then this factor is no larger than 1 for *some* h. By closely examining the risk attitude of this agent, we can find a bound better than 1.

Note first that if a von Neumann Morgenstern utility function u is more risk averse than another v, then, for every w_0 and w_1 with $w_0 \geq w_1$, we have

$$\frac{Du(w_1)}{Du(w_0)} \geq \frac{Dv(w_1)}{Dv(w_0)}.$$

Let $e_0 \geq \overline{e}$, then the agents must consume more in the first period in equilibrium, and so there is an agent who does so more than the average; that is, there is an $\overline{h}$ such that $x_0^{\overline{h}} - \overline{x}^{\overline{h}} \geq H^{-1}(e_0 - \overline{e}) \geq 0$. If agent $\overline{h}$ is more risk averse than an agent with constant absolute risk aversion with coefficient α on the interval $[\overline{x}^{\overline{h}}, x_0^{\overline{h}}]$ then using the inequality above for $v(w) = -\exp(-\alpha w)$, we have

$$\frac{Du^{\overline{h}}(\overline{x}^{\overline{h}})}{Du^{\overline{h}}(x_0^{\overline{h}})} \geq \frac{\exp\left[-\alpha \overline{x}^{\overline{h}}\right]}{\exp\left[-\alpha x_0^{\overline{h}}\right]} \geq \exp\left(\alpha \frac{e_0 - \overline{e}}{H}\right) \geq 1.$$

So the general recipe for finding a tighter lower bound for the term $Du^{\overline{h}}(\overline{x}^{\overline{h}})/Du^{\overline{h}}(x_0^{\overline{h}})$ when $e_0 \geq \overline{e}$ is to identify agent $\overline{h}$ who consumes more in the first period relative to the other agents and measure the minimum absolute risk aversion α on the interval $\left[\overline{x}^{\bar{h}}, x_0^{\bar{h}}\right]$. Then the lower bound for the bond price is improved form one to $\exp\left(\alpha H^{-1}\left(e_0 - \overline{e}\right)\right)$.

For some utility functions, the *relative* risk aversion may vary in a much narrower range over relevant wealth levels than the absolute risk aversion. We can then improve the above bound by using the intertemporal ratio $e_0/\overline{e}$ of aggregate endowments. Indeed, if the (generally non-constant)*relative* risk aversions of $u^{\overline{h}}$ are larger than α, then we have

$$\frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} \geq \left(\frac{e_0}{\overline{e}}\right)^{\alpha} \geq 1.$$

Symmetric bounds can be obtained for the case of $e_0 \leq \overline{e}$, though we then need to use the *maximum* absolute risk aversions. We shall be precise on this point in the proof of Proposition 8.

The second factor $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$ shows how much, in ratio, the marginal utility from the bond is increased by the risk present in the second-period consumption. By Jensen's Inequality, this is no smaller than 1. It measures the degree of prudence of Kimball (1990). Indeed, if a von Neumann Morgenstern utility function u is more risk averse and more prudent than another v, then, for every $\mathbf{x} \in X$ with $\mathbf{E}(\mathbf{x}) = \overline{x}$, it can be shown that

$$\frac{\mathbf{E}\left(Du(\mathbf{x})\right)}{Du(\overline{x})} \geq \frac{\mathbf{E}\left(Dv(\mathbf{x})\right)}{Dv(\overline{x})}.$$

Once, for example, we know that the (generally non-constant) absolute risk aversions and prudence of *every* u^h lie in an interval $[\alpha_{\min}, \alpha_{\max}]$, we can conclude that

$$\frac{\mathbf{E}\left(\exp(-\alpha_{\min}\mathbf{x}^h)\right)}{\exp(-\alpha_{\min}\overline{x}^h)} \leq \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)} \leq \frac{\mathbf{E}\left(\exp(-\alpha_{\max}\mathbf{x}^h)\right)}{\exp(-\alpha_{\max}\overline{x}^h)}$$

for *every* h, which provides both the upper and lower bounds of $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$.

Again, for some utility functions, the relative risk aversion and *relative prudence* may vary in a much narrower range than the absolute risk aversion and absolute prudence. The relative prudence is defined by

$$-\frac{D^3u(x)\,x}{D^2u(x)}$$

and for the utility function $u(x) = (1-\alpha)^{-1}\left(x^{1-\alpha} - 1\right)$ exhibiting constant relative risk aversion α, it is constantly equal to $\alpha + 1$. Moreover, then,

$$\frac{\mathbf{E}\left(Du(\mathbf{x})\right)}{Du(\overline{x})} = \overline{x}^{\alpha}\mathbf{E}\left(\mathbf{x}^{-\alpha}\right).$$

Note the relative and absolute risk aversions, as well as the relative and absolute prudence, give rise to the same ordering between any two utility functions. This implies that we can apply the same argument as for the absolute risk aversion and absolute prudence: If the relative risk aversion of every h lie in an interval $[\alpha_{\min}, \alpha_{\max}]$ and the relative prudence of every h lie in an interval $[\alpha_{\min} + 1, \alpha_{\max} + 1]$, then

$$\left(\overline{x}^h\right)^{\alpha_{\min}}\mathbf{E}\left(\left(\mathbf{x}^h\right)^{-\alpha_{\min}}\right) \leq \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)} \leq \left(\overline{x}^h\right)^{\alpha_{\max}}\mathbf{E}\left(\left(\mathbf{x}^h\right)^{-\alpha_{\max}}\right)$$

for every h.

We have thus obtained bounds for the first factor $Du^h(\overline{x}^h)/Du^h(x_0^h)$ for *some* h and bounds for the second factor $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$ for *every* h. By multiplication, we can obtain bounds for the bond price $p(\mathbf{1})$. The following proposition is a straightforward consequence of the above method.

Proposition 8 *Let M be a market span such that $\mathbf{1} \in M$. Also let a state price function $p: M \to \mathbb{R}$ and a consumption allocation $\left(\left(x_0^h, \mathbf{x}^h\right)\right)_{h\in\{1,\cdots,H\}}$ constitute an equilibrium for M.*

1. *Suppose that, for every h, the coefficients of absolute risk aversions over the interval $\left[\overline{x}^{\bar{h}}, x_0^{\bar{h}}\right]$ lie in the interval $[\alpha_{\min}, \alpha_{\max}]$ and define b^A as in equality (16). Then $p(\mathbf{1}) \geq \delta b^A$.*
2. *Suppose that, for every h, the relative risk aversion over the interval $\left[\overline{x}^{\bar{h}}, x_0^{\bar{h}}\right]$ lie in the interval $[\alpha_{\min}, \alpha_{\max}]$ and the relative prudence over the same interval lie in $[\alpha_{\min} + 1, \alpha_{\max} + 1]$, and define b^R by*

$$b^R = \begin{cases} \left(\dfrac{e_0}{\overline{e}}\right)^{\alpha_{\max}} & \text{if } \overline{e} \leq e_0, \\ \left(\dfrac{e_0}{\overline{e}}\right)^{\alpha_{\min}} & \text{if } \overline{e} > e_0 \end{cases}$$

   Then $p(\mathbf{1}) \geq \delta b^R$.

If $e_0 \geq \overline{e}$, then $b^A \geq 1$ and $b^R \geq 1$, and hence $p(\mathbf{1}) \geq \delta$, that is, the equilibrium bond price must not be lower but may be higher than the common discount factor δ. It is easy to show that if $\sum_h \mathbf{e}^h = e_0\mathbf{1}$ (i.e., the total endowments are time independent and there is no aggregate uncertainty) and the markets are complete, then $p(\mathbf{1}) = \delta$. This proposition thus implies that if the aggregate endowment is deterministic and stationary, then the incompleteness of the markets must not raise the risk-free interest rate.

Proof. Since Du^h is convex for every h, we have

$$\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h\left(\bar{x}^h\right)} \geq 1. \tag{22}$$

Let $h = \bar{h}$ be such that $\overline{x}^{\bar{h}} - x_0^{\bar{h}} \leq H^{-1}\left(\overline{e} - e_0\right)$, and thus

$$\frac{Du^{\bar{h}}\left(\bar{x}^{\bar{h}}\right)}{Du^{\bar{h}}\left(x_0^{\bar{h}}\right)} \geq \frac{Du^{\bar{h}}\left(x_0^{\bar{h}} + H^{-1}\left(\overline{e} - e_0\right)\right)}{Du^{\bar{h}}\left(x_0^{\bar{h}}\right)}. \tag{23}$$

If $\overline{e} - e_0 \geq 0$, then

$$\frac{Du^{\bar{h}}\left(x_0^{\bar{h}} + H^{-1}\left(\overline{e} - e_0\right)\right)}{Du^{\bar{h}}\left(x_0^{\bar{h}}\right)} \geq \exp\left(-\alpha_{\max}\frac{\overline{e} - e_0}{H}\right).$$

If $e_0 - \overline{e} < 0$, then

$$\frac{Du^{\bar{h}}\left(x_0^{\bar{h}} + H^{-1}\left(\overline{e} - e_0\right)\right)}{Du^{\bar{h}}\left(x_0^{\bar{h}}\right)} \geq \exp\left(-\alpha_{\min}\frac{\overline{e} - e_0}{H}\right)$$

Hence, by plugging (22) and (23) into (21), we complete the proof.

2. This part can be proved analogously. ■

Notice that the equality (21) shows that if $e_0 \leq \overline{e}$ and the Du^h are *concave* in the relevant interval of wealth levels, then $p(\mathbf{1}) \leq \delta$ would follow. The more interesting case, however, is where the Du^h are *convex* as assumed earlier, because most frequently applied utility functions, such as those exhibiting constant absolute or relative risk aversions, have this property.

Note that, in the proof of Proposition 8, we did not use any bound on the prudence measure $\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h\left(\bar{x}^h\right)}$ except that it is no less than one; in particular, we did not use any information of the second-period equilibrium allocation $\left(\mathbf{x}^1, \ldots, \mathbf{x}^H\right)$. The general approach we described above, however, shows that such a piece of information would improve the lower bound on the equilibrium bond price. For this reason, the bounds obtained in Proposition 8 should not be expected to be tight. The task of finding the equilibrium allocation or the (bounds for) prudence measure $\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h\left(\bar{x}^h\right)}$ is not straightforward in general. Sharper results obtain for the case of CARA utility functions because the monotonicity of the risk-free rate (Proposition 2) implies that the lowest rate is attained when the risk-free bond is the only tradeable asset, at which the prudence measure $\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)}$ is the same as the prudence at the initial endowments, $c^h = \frac{\mathbf{E}\left(Du^h(\mathbf{e}^h)\right)}{Du^h(\overline{e}^h)}$.

5 Conclusion

In this paper, we have found the upper and lower bounds on the risk-free interest rates in a two-period model with incomplete asset markets. The upper bound was given for general utility functions, while the lower bound was only for CARA utility functions. We also discussed a general method of finding upper and lower bounds. These results will be useful in illustrating the risk-free rate puzzle in tractable general equilibrium models with incomplete asset markets.

It is assumed throughout this paper that the state space Ω is a finite set. This implies, among other things, that the expected utility $\mathbf{E}\left(u^h\left(\mathbf{x}^h\right)\right)$, which appeared in the definition of U^h (equality (1)), and the expected marginal utility $\mathbf{E}\left(Du^h\left(\mathbf{x}^h\right)\right)$, which appeared in the first-order condition for the utility maximization problem on the risk-free bond (equality (2)), are finite. If Ω is infinite, then the expected utility or expected marginal utility (or both) may be infinite and we may not be able to talk sensibly about the preference ordering or first-order conditions. A list of sufficient conditions for these to be finite in terms of utility function and the underlying probability space can be found in Nielsen (1993, Proposition 1 and 5). A lesson to be learnt from his results is that the finiteness of the expected utility does not automatically imply that of the marginal utility. However, in the case of CARA utility functions, as in Section 3, if the expected utility $\mathbf{E}\left(u^h\left(\mathbf{x}^h\right)\right)$ is finite for every second-period consumption $\mathbf{x} \in X$, then so is the expected marginal utility $\mathbf{E}\left(Du^h\left(\mathbf{x}^h\right)\right)$, since the derivative of the exponential function is the exponential function itself. Moreover, then, every $\mathbf{x}^h \in X$ and $Du^h\left(\mathbf{x}^h\right)$ are square-integrable and the Gateaux derivative of U^h at $\mathbf{x}^h$ in the direction of $\mathbf{z} \in X$ is given by $\mathbf{E}\left(\mathbf{z}Du^h\left(\mathbf{x}^h\right)\right)$. Hence the first-order condition is still given by $p(\mathbf{z}) = \delta\frac{\mathbf{E}\left(\mathbf{z}Du^h\left(\mathbf{x}^h\right)\right)}{Du^h(x_0^h)}$ for every $\mathbf{z} \in M$.

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