Consistency, Converse Consistency, and Aspirations in Coalitional Games*

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Abstract

In the problems of choosing "aspirations" for coalitional games, we study two axioms, "MW-consistency" and "converse MW-consistency", introduced by Moldovanu and Winter (1994). We mainly consider two domains: the domain of all NTU games and the domain of all TU games. In particular, we study which subsolutions of the aspiration correspondence satisfy *MW-consistency* and/or *converse MW-consistency*. We also provide axiomatic characterizations of the aspiration kernel and the aspiration nucleolus on the domain of all TU games.

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1 Introduction

A "coalitional game" associates with each coalition of agents a set of feasible payoff vectors. The interpretation is that if the members of a coalition cooperate, they can achieve any payoff vector in their feasible set. The analysis of coalitional games aims at answering two questions:

(i) Which coalitions form?

(ii) What is the payoff of each member of a coalition that forms?

Most of the studies of coalitional games have been carried out under the assumption that the grand coalition (*i.e.* the set of all agents) eventually forms. Then, the analysis reduces to determining payoffs for the agents subject to feasibility for the grand coalition.

In this paper, we consider the problems of choosing "aspirations" for coalitional games (Bennett [2], [3], [4]; Bennett and Zame [5]; Cross [7]; Moldovanu and Winter [14]; Sharkey [26]). Unlike the standard approach, the "aspiration approach" does not impose the assumption that the grand coalition eventually forms. Given a set of agents N and a coalitional game for N, an "aspiration" is a payoff vector $x = (x_i)_{i \in N}$ that satisfies the following two conditions. The first condition, individual feasibility, is that for each agent i, there is at least one coalition S for which agent i's payoff x_i is jointly compatible with those of other members, namely, $i \in S$ and $(x_j)_{j \in S}$ is feasible for S. Note that individual feasibility is weaker than the feasibility condition of the standard approach.[1] The second condition, coalitional rationality, is that no coalition can improve upon its component of the payoff vector, namely, for each coalition S, there is no payoff vector y that is feasible for S and for all $i \in S$, $y_i > x_i$. Coalitional rationality is part of the definition of the core (Gillies [9]).[2]

An aspiration summarizes predictions about which coalitions are likely to form and what the resulting payoffs of their members will be. An aspiration x can be interpreted as follows: for each agent i, x_i is the payoff that she asks in return for her cooperation (*i.e.* the price of her cooperation). A coalition S forms only if the prices of its members' cooperation are jointly

[1] In the standard approach, a payoff vector is feasible if it is feasible for the grand coalition.

[2] Given a game, a payoff vector is in the core of the game if it is feasible for the grand coalition and satisfies coalitional rationality.

compatible, namely, $(x_i)_{i \in S}$ is feasible for S. The studies of the aspirations have revealed that the set of aspirations is closely related to the outcomes obtained from two alternative approaches: "multi-coalitional bargaining approach" and "noncooperative approach".[3]

A "bargaining problem" consists of a set of agents, a set of feasible payoff vectors for the grand coalition, and an element of it called the "disagreement point." The interpretation is that the agents can obtain any payoff vector in the feasible set if they unanimously agree upon it. In the case of disagreement, however, they receive their disagreement payoffs. Note that a bargaining problem can be considered as a coalitional game. A "bargaining solution" assigns to each bargaining problem a feasible payoff vector. In the literature, many interesting bargaining solutions have been proposed, the Nash solution, the Kalai-Smorodinsky solution, and the egalitarian solution being three well-known examples.[4] The "multi-coalitional bargaining approach" to study coalitional games involves constructing a bargaining problem and specifying a bargaining solution for each coalition. Consider a game for the set of agents N. For each coalition, while its feasible set is specified by the game, the disagreement point is determined endogenously. Roughly speaking, given a payoff vector, the disagreement point for each coalition is determined as a list of "outside options" for the members of the coalition. Given a bargaining solution assigned to each coalition, a payoff vector x is a "multi-coalitional bargaining outcome" if for each coalition S, $(x_i)_{i \in S}$ is chosen by the assigned bargaining solution for the bargaining problem with the disagreement associated with x.[5] It so happens that every multi-coalitional bargaining outcome is an aspiration. Conversely, each aspiration can be obtained as a multi-coalitional bargaining outcome for some initial specification of bargaining solutions.

The "non-cooperative approach" analyzes the following coalition formation game: a randomly chosen agent proposes a coalition to be formed and a feasible payoff distribution for its members. The proposal is accepted if every member of the coalition agrees upon it. Otherwise, in the next period the first agent who rejected the proposal makes a new proposal. The game

[3]See Bennett [3].

[4]See Thomson [29] for a compact exposition of the axiomatic studies of bargaining problems.

[5]Such a payoff vector turns out to be a fixed point of a dynamic process in which conjectures about agreement and disagreement outcomes in each coalition are determined endogenously.

ends when a proposal is accepted. It turns out that the set of aspirations of the original coalitional game is equal to the set of stationary subgame perfect equilibrium proposals of this noncooperative game. More precisely, a payoff vector x is an aspiration if and only if it is a stationary subgame perfect strategy for each agent i to propose a coalition S and the payoff vector $(x_i)_{i \in S}$ such that $(x_i)_{i \in S}$ is feasible for S, and to accept any proposal that offers her at least x_i.[6]

The observations in the above two paragraphs strongly suggest that the set of aspirations is an appropriate object to focus on if one wants to analyze coalitional games without imposing the assumption that the grand coalition eventually forms. As the set of aspirations might be very large in general, we are also interested in possible refinements of it. Our approach to study these refinements is axiomatic. Given a class of coalitional games, a "solution" is a correspondence that associates with every game in the class a non-empty set of payoff vectors. The main objective of the axiomatic study of coalitional games is to explore the implications of desirable properties of solutions. We focus on the implications of two properties: consistency and converse consistency.[7]

Consistency deduces from the desirability of a payoff vector in a game the desirability of its restrictions to all subgroups of agents in the associated "reduced games." Suppose that a set of agents N is facing a game and a payoff vector x is agreed upon. Suppose then that some agents leave. Then let us reevaluate the situation from the viewpoint of the remaining agents N'. Namely, for each coalition $S \subseteq N'$, let us identify what S can obtain without any help from other agents in N'. In this context, since any agent i in $N \setminus N'$ has agreed upon x, it is natural to assume that she is willing to cooperate with S if offered x_i. Additionally, suppose that S can choose such "partners" from $N \setminus N'$. The revised feasible set for S would be the set of payoff vectors that S can obtain in this manner. This operation defines a game in which the set of agents is N'. We refer to this game as a "MW-reduced game" since it is introduced by Moldovanu and Winter [14].[8] *MW-consistency* states that in this reduced game, the original agreement should be confirmed, namely,

[6]The particular strategic game described here is due to Selten [23]. The literature following the paper includes Chatterjee, Dutta, Ray, and Sengupta [6], Perry and Reny [21], and Moldovanu and Winter [15, 16].

[7]See Thomson [30] for an extensive survey of studies on these properties applied to various models of game theory and economics.

[8]For TU games, it is originally introduced by Bennett [2, 3] and Winter [31].

$(x_i)_{i \in N'}$ should be agreed upon.

Converse consistency deduces the desirability of a payoff vector in a game from the desirability of its restrictions to all pairs of agents in the associated two-agent reduced games. Consider a coalitional game for N and a payoff vector x under evaluation. Suppose that for each pair of agents $\{i, j\}$ in N, (x_i, x_j) is chosen for the MW-reduced game associated with x and $\{i, j\}$. Then, *converse MW-consistency* states that x should be chosen for the original game.

In the standard approach, the way of reevaluating the situation of the remaining agents we described above was first introduced by Davis and Maschler [8] for TU games[9] and by Greenberg [10] and Peleg [19] for NTU games. For each strict subset of the remaining agents, the revised feasible sets in the standard approach and the aspiration approach coincide. However, for the coalition of all remaining agents (*i.e.* the revised grand coalition), the revised feasible set in the standard approach is the "slice" of the feasible set for the original grand coalition at the payoff vector chosen for the original game.[10]

On the domain of all NTU games, the aspiration correspondence satisfies both *MW-consistency* and *converse MW-consistency* (Moldovanu and Winter [14]). This result holds even on the domain of all TU games. The studies of the aspiration correspondence have revealed that it admits several interesting "subsolutions." On the domain of all NTU games, the partnered aspiration solution, the balanced aspiration solution, and the equal gains aspiration solution are such examples. On the domain of all TU games, in addition to the above three solutions, the aspiration kernel and the aspiration nucleolus are studied. On the domain of all NTU games, the partnered aspiration solution satisfies both *MW-consistency* and *converse MW-consistency* (Moldovanu and Winter [14]). In this paper, we analyze which of other subsolutions of the aspiration correspondence satisfy *MW-consistency* and/or *converse MW-consistency*. As byproducts of this exercise, we obtain interesting axiomatic characterizations of the aspiration kernel and the aspiration nucleolus on the domain of all TU games.

The paper is organized as follows. In Section 2, we introduce NTU games and examples of solutions. In Section 3, we introduce TU games and ex-

[9]In a transferable utility (TU) coalitional game, the set of feasible payoff vectors for each coalition is represented by a real number representing what the coalition can achieve on its own, its "worth."

[10]See Section 4 for the formal definition.

amples of solutions. In Section 4, we study *MW-consistency* and *converse MW-consistency* in NTU games and TU games. In Section 5, we provide axiomatic characterizations of the aspiration kernel and the aspiration nucleolus. In Section 6, we briefly discuss whether the aspiration kernel can be extended to NTU games so that the resulting extension is *MW-consistent*.

2 NTU games and solutions

There is an infinite set of "potential" agents, indexed by the natural numbers $\mathbb{N}$. Let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathbb{N}$. Given a countable set A, let $\mathbb{R}^A$ denote the Cartesian product of $|A|$ copies of the set of real numbers $\mathbb{R}$, indexed by the members of A. We use $\subset$ for strict set inclusion and $\subseteq$ for weak set inclusion. To simplify the notation, given $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $S \subset N$, we often write $x_S \equiv (x_i)_{i \in S}$ and $x(S) \equiv \sum_S x_i$.

Given $N \in \mathcal{N}$, a **non-transferable utility (NTU) coalitional game for $\boldsymbol{N}$** is a list $V = (V(S))_{S \subseteq N}$ such that for all $S \subseteq N$, $V(S)$ is a non-empty subset of $\mathbb{R}^S$ that is comprehensive and bounded from above, namely, it satisfies (i) for all $x \in V(S)$, if $y \in \mathbb{R}^S$ is such that for all $i \in S$, $x_i \geq y_i$, then $y \in V(S)$, and (ii) there exist $p \in \mathbb{R}^S_{++}$ and $\overline{x} \in \mathbb{R}^S$ such that for all $x \in V(S)$, $p \cdot x \leq p \cdot \overline{x}$. For all $N \in \mathcal{N}$, let $\mathcal{W}^N_{all}$ denote the class of all NTU games for N, and $\mathcal{W}_{all} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{W}^N_{all}$.

Let $N \in \mathcal{N}$ and $V \in \mathcal{W}^N_{all}$. A payoff vector $x \in \mathbb{R}^N$ is **individually feasible in $\boldsymbol{V}$** if for all $i \in N$, there exists $S \subseteq N$ such that $i \in S$ and $x_S \in V(S)$. It is **coalitionally rational in $\boldsymbol{V}$** if for all $S \subseteq N$, there is no $y \in V(S)$ such that for all $i \in S$, $y_i > x_i$. An **aspiration for $\boldsymbol{V}$** is a payoff vector in $\mathbb{R}^N$ satisfying individual feasibility and coalitional rationality.

Let $\mathcal{W}$ be an arbitrary class of NTU games. A **solution on $\boldsymbol{\mathcal{W}}$** is a correspondence from $\bigcup_{N \in \mathcal{N}} \mathcal{W}^N$ to $\bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ that associates with each $N \in \mathcal{N}$ and each $V \in \mathcal{W}^N$ a non-empty set of payoff vectors satisfying individual feasibility. We use φ to denote a generic solution.

The aspiration correspondence (Albers [1]; Bennett [3]; Cross [7]) is a solution that selects for each game its set of aspirations.

Aspiration correspondence, *Asp*: For all $N \in \mathcal{N}$ and all $V \in \mathcal{W}^N$, $Asp(V)$ is the set of aspirations for V.

On the domain of all NTU games, the aspiration correspondence is well-defined, namely, it is nonempty-valued (Bennett and Zame [5]).

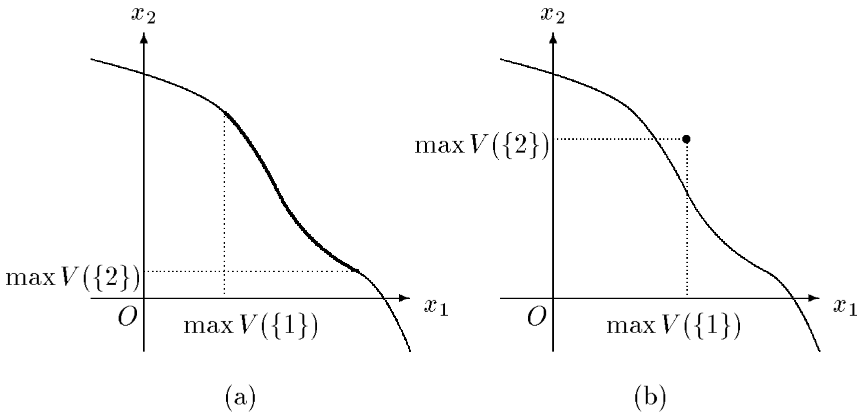

Figure 1: Aspirations for two-agent NTU games: In panel (a), the set of aspirations is indicated by the thick curve. In panel (b), it consists of a single point $(\max V(\{1\}), \max V(\{2\}))$.

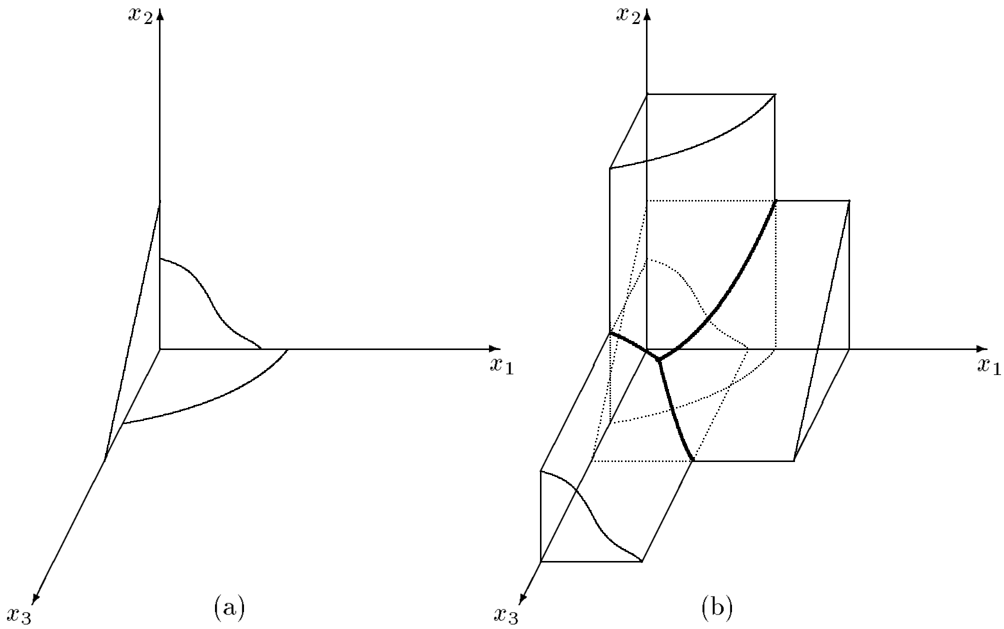

Figure 2: Aspirations for three-agent NTU games: In the picture, $N = \{1, 2, 3\}$, $V(\{1\}) = V(\{2\}) = V(\{3\}) = -\mathbb{R}_+$, and $V(N) = -\mathbb{R}_+$. The set of aspiration for V is the union of three thick curves in panel (b).

We are mainly interested in "subsolutions" of the aspiration correspondence. On the domain of all NTU games, well-known examples of such solutions are the partnered aspiration solution (Albers [1]), the equal gains aspiration solution (Bennett [3]), and the balanced aspiration solution (Cross [7]). Next, we define these solutions.

Given $N \in \mathcal{N}$, $V \in \mathbb{W}^N_{all}$, and $x \in \mathbb{R}^N$, the **set of generating coalitions for $\boldsymbol{V}$ and $\boldsymbol{x}$** is defined by

$$\mathcal{GC}(V, x) \equiv \{S \subseteq N \mid x_S \in V(S)\}.$$

Given a game and a payoff vector for it, generating coalitions are those coalitions whose members' payoffs are jointly compatible. Thus, if each agent demands her component of the payoff vector, the generating coalitions are the only coalitions that are likely to form. For an aspiration for the game, each agent is a member of at least one generating coalition and each generating coalition distributes payoffs efficiently among its members.

Consider a game V, an aspiration x, a generating coalition S, and two agents i and j in S. Suppose that agent i's payoff x_i is so high that while agent j has an alternative coalition where she can attain her payoff x_j, agent i needs agent j in order to attain x_i. In such a situation, agent j might insist that she should receive more and agent i should receive less. In this sense, the aspiration x is "unstable." In order for x to be "stable," if agent j has an alternative coalition in $\mathcal{GC}(V, x)$ without agent i, then agent i should also have an alternative coalition in $\mathcal{GC}(V, x)$ without agent j. This idea is formalized in the definition of the following solution:

Partnered aspiration solution, $\boldsymbol{ParAsp}$: For all $N \in \mathcal{N}$ and all $V \in \mathbb{W}^N$,

$$ParAsp(V) \equiv \left\{ x \in Asp(V) \;\middle|\; \begin{array}{l} \text{for all } S \in \mathcal{GC}(V,x) \text{ and all } i, j \in S, \text{ if there} \\ \text{is } T \in \mathcal{GC}(V,x) \text{ s.t. } i \in T \text{ and } j \notin T, \text{ then} \\ \text{there is } T' \in \mathcal{GC}(V,x) \text{ s.t. } j \in T' \text{ and } i \notin T' \end{array} \right\}.$$

On the domain of all NTU games, the partnered aspiration solution is well-defined (Bennett and Zame [5]; Sharkey [26]).[11]

The next refinement is based on the premise that agents when bargaining tend to share the gains equally. In our context, by forming a coalition, the agents forego the payoffs that they could have attained by forming alternative

[11] Bennett and Zame [5] prove the non-emptiness of the set $ParAsp(V)$ on the domain of strictly comprehensive NTU games.

coalitions. Therefore, each agent's largest payoff from alternative coalitions serves as an "outside option." Formally, given $N \in \mathcal{N}$, $V \in \mathbb{V}^N_{all}$, $x \in \mathbb{R}^N$, $S \in \mathcal{GC}(V,x)$, and $i \in S$, the **outside option for *i* relative to *V*, *x*, and *S*** is defined by

$$d_i^S(V,x) \equiv \max_{\substack{T \ni i \\ T \neq S}} \left\{ y_i \in \mathbb{R} \;\middle|\; (y_i, x_{T \setminus \{i\}}) \in V(T) \right\}.$$

On the domain of all NTU games, the solution that selects those aspirations such that in each generating coalition the agents equally share the gains from their outside options is well-defined (Bennett and Zame [5]).

Equal gains aspiration solution, *EqAsp*: For all $N \in \mathcal{N}$ and all $V \in \mathbb{V}^N$,

$$EqAsp(V) \equiv \left\{ x \in Asp(V) \;\middle|\; \begin{array}{l} \text{for all } S \in \mathcal{GC}(V,x) \text{ and all } i,j \in S, \\ x_i - d_i^S(V,x) = x_j - d_j^S(V,x) \end{array} \right\}.$$

The next refinement we introduce is based on the idea that competition among the coalitions for "scarce" agents drives up the payoff demands of these agents, and drives down the payoff demands of other agents. Cross [7] and Bennett [3] argue that this competition leads to a "balanced" structure of generating coalitions.

Given $N \in \mathcal{N}$, a collection of coalitions $\mathcal{B} \subseteq 2^N$ is **strictly balanced on *N*** if there is $\delta = (\delta_S)_{S \in \mathcal{B}} \in \mathbb{R}^{\mathcal{B}}_{++}$ such that for all $i \in N$,

$$\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \delta_S = 1.$$

It is **weakly balanced on *N*** if there is $\delta = (\delta_S)_{S \in \mathcal{B}} \in \mathbb{R}^{\mathcal{B}}_{+}$ such that for all $i \in N$,

$$\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \delta_S = 1.$$

On the domain of all NTU games, the solution that selects those aspirations such that the associated collections of generating coalitions are weakly balanced is well-defined (Cross [7]; Sharkey [26]).[12]

Balanced aspiration solution, *BalAsp*: For all $N \in \mathcal{N}$ and all $V \in \mathbb{V}^N$,

$$BalAsp(V) \equiv \left\{ x \in Asp(V) \;\middle|\; \mathcal{GC}(V,x) \text{ is weakly balanced} \right\}.$$

[12]Cross [7] considers TU games only.

For each game, the core (Gillies [9]) selects the set of payoff vectors that are feasible for the grand coalition and coalitionally rational. It is well-known that the core may assign an empty set so that it is not a well-defined solution on $\mathbb{V}_{all}$.

Core, C: For all $N \in \mathcal{N}$ and all $V \in \mathbb{V}^N$,

$$C(V) \equiv \{x \in V(N) \mid x \text{ is coalitionally rational in } V\}.$$

3 TU games and solutions

Given $N \in \mathcal{N}$, a **transferable utility (TU) coalitional game for N** is a vector $v \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$. For each $S \subseteq N$, the number $v(S)$ represents what coalition S can obtain on its own, its "worth". Let $\mathcal{V}_{all}^N$ denote the class of all TU games for N, and $\mathcal{V}_{all} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}_{all}^N$.

As before, given $N \in \mathcal{N}$ and $v \in \mathcal{V}_{all}^N$, an **aspiration for v** is a payoff vector $x \in \mathbb{R}^N$ satisfying individual feasibility and coalitional rationality. For TU games, the definitions of these two conditions are simplified as follows:

(i) (individual feasibility) for all $i \in N$, there exists $S \subseteq N$ such that $i \in S$ and $x(S) \leq v(S)$;

(ii) (coalitional rationality) for all $S \subseteq N$, $x(S) \geq v(S)$.

Let $\mathcal{V}$ be an arbitrary class of TU games. A **solution on $\mathcal{V}$** is a correspondence that associates with each $N \in \mathcal{N}$ and each $v \in \mathcal{V}^N$ a non-empty set of payoff vectors $x \in \mathbb{R}^N$ satisfying individual feasibility.

On the domain of all TU games, the partnered aspiration solution, the balanced aspiration solution, and the equal gains aspiration solution are defined in the same way as on the domain of all NTU games. Here we just point out that the definitions of the outside options are simplified as follows: given $N \in \mathcal{N}$, $v \in \mathcal{V}_{all}^N$, $x \in \mathbb{R}^N$, $S \in \mathcal{GC}(v,x)$, and $i \in S$,

$$d_i^S(v,x) \equiv \max_{\substack{T \ni i \\ T \neq S}} \big[v(T) - x(T \setminus \{i\})\big].$$

On the domain of all TU games, the aspiration correspondence, the partnered aspiration solution, and the equal gains aspiration are well-defined (Bennett [3]). On this domain, in addition to these solutions, the aspiration nucleolus and the aspiration kernel (Bennett [2]) have been studied. Now, we define these solutions.

Given $v \in \mathcal{V}^N$ and $x \in \mathbb{R}^N$, let $e(v,x)$ be the vector in $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ defined by setting for all $S \in 2^N \setminus \{\emptyset\}$, $e_S(v,x) \equiv v(S) - x(S)$. The number $e_S(v,x)$ represents the "dissatisfaction of S in v at x."

Given $N \in \mathcal{N}$ and $z \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, let $\theta(z) \in \mathbb{R}^{2^{|N|}-1}$ be obtained by rearranging the coordinates of z in non-increasing order. For all $z, z' \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, $\boldsymbol{z}$ **is lexicographically smaller than** $\boldsymbol{z'}$ if either (i) $\theta_1(z) < \theta_1(z')$ or (ii) there is $k > 1$ such that $\theta_k(z) < \theta_k(z')$ and for all $k' < k$, $\theta_{k'}(z) = \theta_{k'}(z')$.

Given $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{all}$, and $x \in \mathbb{R}^N$, x is an **preimputation for** $\boldsymbol{v}$ if $x(N) = v(N)$. Let $PreI(v)$ denote the set of preimputations for v. We refer to the mapping $PreI$ as the "preimputation correspondence".

For each TU game, the prenucleolus (Schmeidler [22]) selects a payoff vector that lexicographically minimizes the dissatisfactions of the coalitions over the set of preimputations.

Prenucleolus, $\boldsymbol{PreNuc}$**:** For all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$,

$$PreNuc(v) \equiv \left\{ x \in PreI(v) \;\middle|\; \begin{array}{l} \text{for all } y \in PreI(v) \setminus \{x\}, \\ e(v,x) \text{ is lexicographically smaller than } e(v,y) \end{array} \right\}.$$

On the domain of all TU games, the prenucleolus is *single-valued* (Schmeidler [22]).

Similarly, for each TU game, the aspiration nucleolus (Bennett [2]) selects a payoff vector that "lexicographically minimizes" the dissatisfactions of the coalitions over the set of aspirations.

Aspiration nucleolus, $\boldsymbol{AspNuc}$**:** For all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$,

$$AspNuc(v) \equiv \left\{ x \in Asp(v) \;\middle|\; \begin{array}{l} \text{for all } y \in Asp(v) \setminus \{x\}, \\ e(v,x) \text{ is lexicographically smaller than } e(v,y) \end{array} \right\}.$$

On the domain of all TU games, the aspiration nucleolus is *single-valued* (Bennett [2]; Sharkey [26]). So, we write $x = AspNuc(v)$ instead of $\{x\} = AspNuc(v)$.

Given $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{all}$, $x \in \mathbb{R}^N$, and $i, j \in N$ with $i \neq j$, let

$$s_{ij}(v,x) \equiv \max_{\substack{S \ni i \\ S \not\ni j}} \big[v(S) - v(S) \big].$$

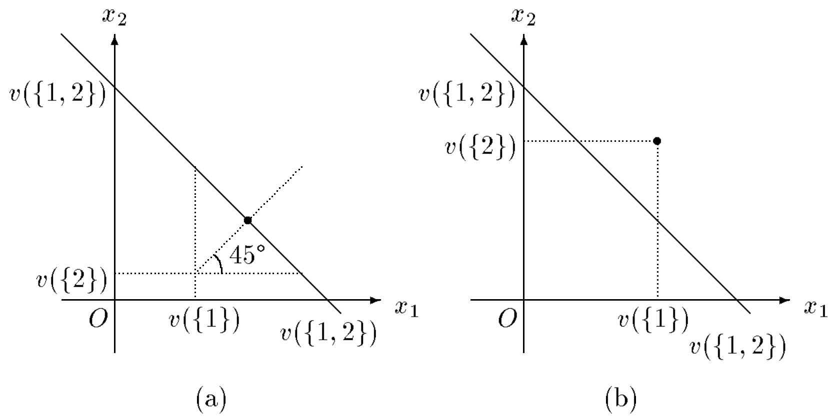

Figure 3: The aspiration nucleolus and the aspiration kernel in two-agent TU games. (a) If $v(\{1,2\}) \geq v(\{1\}) + v(\{2\})$, then the aspiration nucleolus selects the "standard solution" payoff vector: each agent is given first her individual worth, and then what remains is divided equally. (b) Otherwise, it selects $(v(\{1\}), v(\{2\}))$. It is easy to see that the aspiration nucleolus and the aspiration kernel coincide for the two-agent case.

The number $s_{ij}(v,x)$ represents maximum payoff that agent i can obtain without cooperation of agent j, supposing that other agents agree upon x. The prekernel (Davis and Maschler [8]) is defined as follows:

Prekernel, *PreKer*: For all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$,

$$PreKer(v) \equiv \big\{x \in PreI(v) \;\big|\; \text{for all } i,j \in N,\ s_{ij}(v,x) = s_{ji}(v,x)\big\}.$$

It is well-known that the prenucleolus is a subsolution of the prekernel. Since the prenucleolus is a well-defined solution on the domain of all TU games, so is the prekernel. (See Figures 3 and 4.)

Similarly, the aspiration kernel (Bennett [2]) is defined as follows:

Aspiration kernel, *AspKer*: For all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$,

$$AspKer\,(v) \equiv \left\{ x \in Asp\,(v) \;\middle|\; \begin{array}{l} \text{for all } S \in \mathcal{GC}\,(v,x) \text{ and all } i,j \in S \\ s_{ij}(v,x) = s_{ji}(v,x) \end{array} \right\}.$$

The following lemma reveals a relation among the aspiration nucleolus, the aspiration kernel, and the balanced aspiration solution.

Lemma 3.1 (Sharkey [26]) *For all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N_{all}$,*

$$AspNuc(v) \in AspKer(v) \cap BalAsp(v).$$

Since the aspiration nucleolus is well-defined on the domain of all TU games, so is the aspiration kernel.

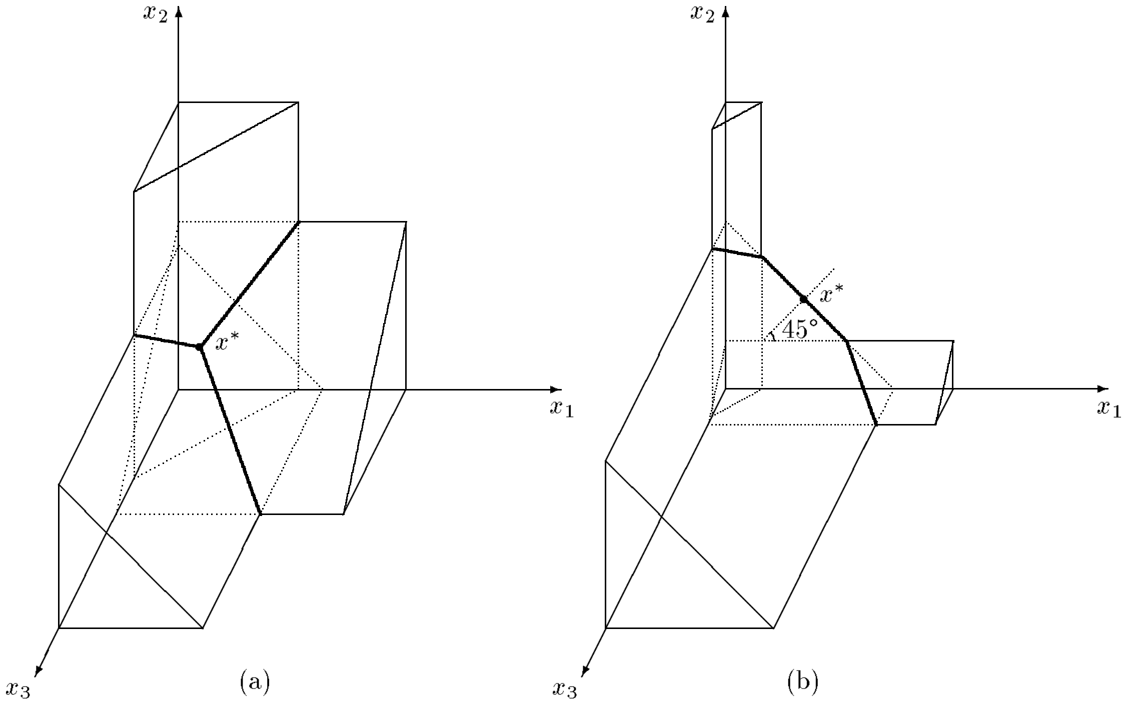

Figure 4: The aspiration nucleolus and the aspiration kernel in three-agent TU games. In both panels, $V(\{1,2,3\}) = -\mathbb{R}_+^{\{1,2,3\}}$ and the payoff vector chosen by the aspiration nucleolus is indicated as x^*. It can be shown that for the three-agent case, the aspiration kernel is single-valued so that it coincides with the aspiration nucleolus.

4 Consistency and converse consistency

Given $N \in \mathcal{N}$, $V \in \mathbb{V}^N$, $x \in \mathbb{R}^N$, and $N' \subset N$, the **MW-reduced game of V relative to x and N'**, denoted $r^x_{N'}(V)$, is defined by setting for all $S \subseteq N'$,

$$r^x_{N'}(V)(S) \equiv \bigcup_{T \subseteq N \setminus N'} \left\{ y \in \mathbb{R}^S \middle| (y, x_T) \in V(S \cup T) \right\}.$$

MW-consistency (Moldovanu and Winter [14]) says that if a payoff vector is chosen for a game, then the restriction of it to any subgroup should be chosen for the associated MW-reduced game.

MW-consistency: For all $N \in \mathcal{N}$, all $V \in \mathbb{V}^N$, all $x \in \varphi(V)$, and all $N' \subset N$, we have $r^x_{N'}(V) \in \mathbb{V}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(V))$.

Converse MW-consistency (Moldovanu and Winter [14]) says that if a payoff vector for a game is such that its restriction to any pair of agents is chosen for the associated two-agent MW-reduced game, then it should be chosen for the original game.

Converse MW-consistency: For all $N \in \mathcal{N}$, all $V \in \mathbb{W}^N$, and all $x \in \mathbb{R}^N$, if for all $N' \subset N$ with $|N'| = 2$, we have $r^x_{N'}(V) \in \mathbb{W}^{N'}$ and $x_{N'} \in \varphi\left(r^x_{N'}(V)\right)$, then $x \in \varphi(V)$.

For TU games, the definitions of MW-reduced games are simplified as follows: given $N \in \mathcal{N}$, $v \in \mathcal{V}^N$, $x \in \mathbb{R}^N$, and $N' \subset N$, the **MW-reduced game of $\boldsymbol{v}$ relative to $\boldsymbol{x}$ and $\boldsymbol{N'}$**, denoted $r^x_{N'}(v)$, is defined by setting for all $S \subseteq N'$,

$$r^x_{N'}(v)(S) \equiv \max_{T \subseteq N \setminus N'} \left[v(S \cup T) - x(T)\right].$$

As mentioned in the introduction, most of studies on coalitional games are carried out under the assumption that in each game, the grand coalition eventually forms. In this "standard approach," a definition of reduced games was first introduced by Davis and Maschler [8] for TU games and by Greenberg [10] and Peleg [18] for NTU games. Let us provide the formal definition of it and review some interesting results concerning it.

Given $N \in \mathcal{N}$, $V \in \mathbb{W}^N$, $x \in V(x)$, and $N' \subset N$, the **DM-reduced game of $\boldsymbol{V}$ relative to $\boldsymbol{x}$ and $\boldsymbol{N'}$**, denoted $\hat{r}^x_{N'}(V)$, is defined by setting for all $S \subseteq N'$,

$$\hat{r}^x_{N'}(V)(S) \equiv \begin{cases} \left\{y \in \mathbb{R}^{N'} \;\middle|\; \left(y, x_{N\setminus N'}\right) \in V(N)\right\} & \text{if } S = N', \\ \displaystyle\bigcup_{T \subseteq N \setminus N'} \left\{y \in \mathbb{R}^S \;\middle|\; (y, x_T) \in V(S \cup T)\right\} & \text{otherwise.} \end{cases}$$

Given $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{all}$, $x \in \mathbb{R}^N$, and $N' \subset N$, the **DM-reduced game of $\boldsymbol{v}$ relative to $\boldsymbol{x}$ and $\boldsymbol{N'}$** is simplified as follows: for all $S \subset N'$,

$$\hat{r}^x_{N'}(v)(S) \equiv \begin{cases} v(N) - x(N \setminus N') & \text{if } S = N', \\ \displaystyle\max_{T \subseteq N \setminus N'} \left[v(S \cup T) - x(T)\right] & \text{otherwise.} \end{cases}$$

DM-consistency: For all $N \in \mathcal{N}$, all $v \in \mathbb{W}^N$, all $x \in \varphi(V)$, and all $N' \subset N$, we have $\hat{r}^x_{N'}(V) \in \mathbb{W}^{N'}$ and $x_{N'} \in \varphi\left(\hat{r}^x_{N'}(V)\right)$.

Given $N \in \mathcal{N}$ and $V \in \mathbb{W}^N$, let

$$PreI(V) \equiv \left\{x \in V(N) \;\middle|\; \text{there is no } y \in V(N) \text{ such that for all } i \in N,\ y_i > x_i \right\}.$$

Converse DM-consistency: For all $N \in \mathcal{N}$, all $V \in \mathbb{W}^N$, and all $x \in PreI(V)$, if for all $N' \subset N$ with $|N'| = 2$, we have $\hat{r}^x_{N'}(V) \in \mathbb{W}^{N'}$ and $x_{N'} \in \varphi\left(\hat{r}^x_{N'}(V)\right)$, then $x \in \varphi(V)$.

On the domain of strictly comprehensive NTU games with a non-empty core, the core is *DM-consistent* and *conversely DM-consistent* (Peleg [19]). On the domain of all TU games, the prekernel is *DM-consistent* and *conversely DM-consistent* (Davis and Maschler [8]; Peleg [19]). On the same domain, the prenucleolus is *DM-consistent* (Sobolev [28]).

4.1 NTU games

On the domain of all NTU games, the aspiration correspondence and the partnered aspiration solution are *MW-consistent* and *conversely MW-consistent.* It follows from this result that each of these two solutions is the unique *MW-consistent* and *conversely MW-consistent* extension of its two-agent version to the n-agent case (Theorems 4.5 and 4.6 in Moldovanu and Winter [14]).

In this section, we analyze whether other subsolutions of the aspiration correspondence satisfy these properties.

As the following example shows, even on the domain of all TU games, the equal gains aspiration solution violates *MW-consistency* and *converse MW-consistency.*

Example 4.1 Let $N \equiv \{1,2,3\}$. Consider the following TU game: for all $S \subseteq N$,

$$v(S) \equiv \begin{cases} 6 & \text{if } S \in \{\{1,2,3\},\{1,2\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \equiv (4,2,0)$ and $y \equiv (3,3,0)$. Then $x \in Asp(v)$ and

$$\mathcal{GC}(v,x) = \mathcal{GC}(v,y) = \{\{3\}, \{1,2\}, \{1,2,3\}\}.$$

It can be also shown that $x \in EqAsp(v)$. (See the appendix for the proof.)

Note that

$$\begin{aligned} r^x_{\{1,2\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,3\}) - x_3\} = \max\{0, 0-0\} = 0, \\ r^x_{\{1,2\}}(v)(\{2\}) &= \max\{v(\{2\}), v(\{2,3\}) - x_3\} = \max\{0, 0-0\} = 0, \\ r^x_{\{1,2\}}(v)(\{1,2\}) &= \max\{v(\{1,2\}), v(\{1,2,3\}) - x_3\} = \max\{6, 6-0\} = 6. \end{aligned}$$

Note also that $EqAsp\big(r^x_{\{1,2\}}(v)\big) = \{(3,3)\} \not\ni (x_1, x_2)$. Thus, the equal gains aspiration solution violates *MW-consistency.*

As shown in the appendix, y satisfies the hypothesis of *converse MW-consistency* with respect to the equal gains aspiration solution; namely, for all

$i, j \in N$ with $i \neq j$, $(y_i, y_j) \in EqAsp(r^y_{\{i,j\}}(v))$. However, we have $\{1,2,3\} \in \mathcal{GC}(v, y)$,

$$\begin{aligned}
\max_{\substack{S \ni 1 \\ S \neq \{1,2,3\}}} [v(S) - y(S)] &= \max\{0-0,\ 0-3,\ 6-3\} = 3, \\
\max_{\substack{S \ni 3 \\ S \neq \{1,2,3\}}} [v(S) - y(S)] &= \max\{0-3,\ 0-0,\ 0-3\} = 0,
\end{aligned}$$

so that $y \notin EqAsp(v)$. Thus, the equal gains aspiration solution violates *converse MW-consistency.* □

Next, we analyze properties of the balanced aspiration solution.

Proposition 4.1 *On the domain of all NTU games, the balanced aspiration solution is MW-consistent.*

Proof. Let $N \in \mathcal{N}$, $V \in \mathbb{W}^N_{all}$, $x \in BalAsp(V)$, and $\mathcal{B} \equiv \mathcal{GC}(V, x)$. Then there exists a list of non-negative weights $(\delta_S)_{S \in \mathcal{B}}$ such that for all $i \in N$,

$$\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \delta_S = 1.$$

Let $N' \subset N$ and $\mathcal{B}' \equiv \mathcal{GC}(r^x_{N'}(V), x_{N'})$. For all $S \in \mathcal{B}'$, let

$$\lambda_S \equiv \sum_{\substack{T \subseteq N \setminus N' \\ \text{s.t. } S \cup T \in \mathcal{B}}} \delta_{S \cup T}.$$

Note that

$$\mathcal{B}' = \big\{S \in 2^N \setminus \{\emptyset\} \,\big|\, \text{for some } T \subseteq N \setminus N',\ S \cup T \in \mathcal{B}\big\}.$$

Thus, for all $i \in N'$,

$$\sum_{\substack{S \in \mathcal{B}' \\ S \ni i}} \lambda_S = \sum_{\substack{S \in \mathcal{B}' \\ S \ni i}} \sum_{\substack{T \subseteq N \setminus N' \\ \text{s.t. } S \cup T \in \mathcal{B}}} \delta_{S \cup T} = \sum_{\substack{R \in \mathcal{B} \\ R \ni i}} \delta_R = 1.$$

This means that $\mathcal{B}'$ is weakly balanced on N'. Thus, $x_{N'} \in BalAsp(r^x_{N'}(V))$. ■

The following example shows that even on the domain of all TU games, the balanced aspiration solution violates *converse MW-consistency.*

Domain: $\mathcal{W}_{all}$	*Asp*	*ParAsp*	*EqAsp*	*BalAsp*
MW-CONSISTENCY	Yes	Yes	No (ex. 4.1)	Yes (prop. 4.1)
CONVERSE MW-CONSISTENCY	Yes	Yes	No (ex. 4.1)	No (ex. 4.2)

Table 1: MW-consistency and converse MW-consistency on the domain of all NTU games.

Example 4.2 Let $N \equiv \{1,2,3\}$. Consider the following TU game: for all $S \subseteq N$,

$$v(S) \equiv \begin{cases} 1 & \text{if } |S| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \equiv (1,1,0)$. Then $x \in Asp(v)$ and $\mathcal{GC}(v,x) = \big\{\{1,3\},\{2,3\},\{3\}\big\}$. Note that for two-agent case, the aspiration correspondence and the balanced aspiration solution coincide. Therefore, as the aspiration correspondence is *MW-consistent*, for all $N' \subset N$ with $|N'| = 2$, $x_{N'} \in Asp(r^x_{N'}(v)) = BalAsp(r^x_{N'}(v))$. However, since $\mathcal{GC}(v,x)$ is not weakly balanced, $x \notin BalAsp(v)$. Thus, the balanced aspiration solution violates *converse MW-consistency.* □

Table 1 summarizes our results for NTU games.

4.2 TU games

Next, we focus on TU games.

Given $N \in \mathcal{N}$, $v \in \mathcal{V}^N$, $x \in \mathbb{R}^N$, and $\alpha \in \mathbb{R}$, let

$$\mathcal{S}_\alpha(v,x) \equiv \big\{S \subset N \;\big|\; e_S(v,x) \geq \alpha\big\}.$$

The following lemma plays an essential role in the proof of the characterization of the prenucleolus in Sobolev [28].

Lemma 4.1 (Kohlberg [12]) *For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $x \in PreI(v)$, we have $x = PreNuc(v)$ if and only if for all $\alpha \in \mathbb{R}$ with $\mathcal{S}_\alpha(v,x) \neq \emptyset$, $\mathcal{S}_\alpha(v,x)$ is strictly balanced on N.*

A similar lemma for the aspiration nucleolus is available.

Lemma 4.2 (Sharkey [26]) *For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $x \in Asp(v)$, we have $x = AspNuc(v)$ if and only if for all $\alpha \in \mathbb{R}$ with $\mathcal{S}_\alpha(v,x) \neq \emptyset$, $\mathcal{S}_\alpha(v,x)$ is strictly balanced on N.*

Now, we use Lemma 4.2 to prove the following result:

Proposition 4.2 *On the domain of all TU games, the aspiration nucleolus is MW-consistent.*

Proof. Let $N \in \mathcal{N}$, $v \in \mathcal{V}_{all}$, $x = AspNuc(v)$, and $N' \subset N$. Let $\alpha \in \mathbb{R}$ be such that $\mathcal{S}_\alpha(r^x_{N'}(v), x_{N'}) \neq \emptyset$. By the definition of $r^x_{N'}(v)$, for all $S \in \mathcal{S}_\alpha(r^x_{N'}(v), x_{N'})$, there exists $T \subseteq N \setminus N'$, which may be empty, such that $r^x_{N'}(v)(S) = v(S \cup T) - x(T)$. Since $v(S \cup T) - x(T) \geq \alpha$, $\mathcal{S}_\alpha(v, x) \neq \emptyset$. To simplify the notation, let $\mathcal{B} \equiv \mathcal{S}_\alpha(v, x)$ and $\mathcal{B}' \equiv \mathcal{S}_\alpha(r^x_{N'}(v), x_{N'})$. By Lemma 4.2, $\mathcal{B}$ is strictly balanced on N. Thus, there exists a list of positive weights $(\delta_S)_{S \in \mathcal{B}}$ such that for all $i \in N$,

$$\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \delta_S = 1.$$

For all $S \in \mathcal{B}'$, let

$$\lambda_S \equiv \sum_{\substack{T \subseteq N \setminus N' \\ \text{s.t. } S \cup T \in \mathcal{B}}} \delta_{S \cup T}.$$

Note that

$$\mathcal{B}' = \big\{ S \in 2^N \setminus \{\emptyset\} \;\big|\; \text{for some } T \subseteq N \setminus N',\, S \cup T \in \mathcal{B} \big\}.$$

Thus, for all $i \in N'$,

$$\sum_{\substack{S \in \mathcal{B}' \\ S \ni i}} \lambda_S = \sum_{\substack{S \in \mathcal{B}' \\ S \ni i}} \sum_{\substack{T \subseteq N \setminus N' \\ \text{s.t. } S \cup T \in \mathcal{B}}} \delta_{S \cup T} = \sum_{\substack{R \in \mathcal{B} \\ R \ni i}} \delta_R = 1.$$

This means that $\mathcal{B}'$ is strictly balanced on N'.

Thus, for all $\alpha \in \mathbb{R}$ with $\mathcal{S}_\alpha(r^x_{N'}(v), x_{N'}) \neq \emptyset$, $\mathcal{S}_\alpha(r^x_{N'}(v), x_{N'})$ is strictly balanced on N'. By Lemma 4.2, $x_{N'} = AspNuc(r^x_{N'}(v))$. ■

As the following example shows, the aspiration nucleolus violates *converse MW-consistency.*

Example 4.3 Let $N \equiv \{1, 2, 3, 4\}$. Consider the following TU game for N: for all $S \subseteq N$,

$$v(S) \equiv \begin{cases} 6 & \text{if } S \in \big\{\{1,2,3\}, \{1,2,4\}\big\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \equiv (3,3,0,0)$ and $y \equiv (2,2,2,2)$. It can be shown that $x \in AspKer(v)$, $y \in Asp(v)$, and $e(v,y)$ is lexicographically smaller than $e(v,x)$. Thus, $x \neq AspNuc(v)$. Since the aspiration kernel is *MW-consistent* and it coincides with the aspiration nucleolus for the two-agent case, for all $i,j \in N$ with $i \neq j$,

$$(x_i, x_j) \in AspKer\big(r^x_{\{i,j\}}(v)\big) = \big\{AspNuc\big(r^x_{\{i,j\}}(v)\big)\big\}\,.$$

Thus, x satisfies the hypothesis of *converse MW-consistency* for the aspiration nucleolus. Since $x \neq AspNuc(v)$, we concludes that the aspiration nucleolus violates *converse MW-consistency.* □

The following lemma essentially implies that on the domain of all TU games, the aspiration kernel is *MW-consistent* and *conversely MW-consistent.*

Lemma 4.3 (Peleg [19]) *For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N_{all}$, all $x \in \mathbb{R}^N_+$, all $N' \subset N$, and all $i,j \in N'$ with $i \neq j$, we have $s_{ij}\big(r^x_{N'}(v), x_{N'}\big) = s_{ij}(v,x)$.*

Proposition 4.3 *On the domain of all TU games, the aspiration kernel is MW-consistent and conversely MW-consistent.*

Proof. (*MW-consistency*) Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{all}$, $x \in AspKer(v)$, and $N' \subset N$. Since the aspiration correspondence is *MW-consistent*, $x_{N'} \in Asp(r^x_{N'}(v))$. Let $S \in \mathcal{GC}(r^x_{N'}(v), x_{N'})$ and $i,j \in S$ with $i \neq j$. Then, by the definition of $r^x_{N'}(v)(S)$, there exists $T \subseteq N \setminus N'$, which may be empty, such that $x(S) \leq r^x_{N'}(v)(S) = v(S \cup T) - x(T)$. Since $x \in AspKer(v)$, $S \cup T \in \mathcal{GC}(v,x)$, and $i,j \in S \cup T$, $s_{ij}(v,x) = s_{ji}(v,x)$. By Lemma 4.3, $s_{ij}(r^x_{N'}(v), x_{N'}) = s_{ji}(r^x_{N'}(v), x_{N'})$.

(*Converse MW-consistency*) Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{all}$, and $x \in \mathbb{R}^N$ be such that for all $N' \subset N$ with $|N'| = 2$, $x_{N'} \in AspKer(r^x_{N'}(v))$. Since the aspiration correspondence is *conversely MW-consistent* and the aspiration kernel is its subsolution, $x \in Asp(v)$. Let $S \in \mathcal{GC}(v,x)$ and $i,j \in S$ with $i \neq j$. Then $x_i + x_j \leq v(S) - x(S \setminus \{i,j\}) \leq r^x_{\{i,j\}}(v)(\{i,j\})$ so that $\{i,j\} \in \mathcal{GC}\big(r^x_{\{i,j\}}(v), x_i, x_j\big)$. Since $(x_i,x_j) \in AspKer\big(r^x_{\{i,j\}}(v)\big)$, $s_{ij}\big(r^x_{\{i,j\}}(v), x_i, x_j\big) = s_{ji}\big(r^x_{\{i,j\}}(v), x_i, x_j\big)$. By Lemma 4.3, $s_{ij}(v,x) = s_{ji}(v,x)$. ■

5 Two axiomatic characterizations

In this section, we focus on the domain of all TU games, and analyze the implications of *MW-consistency*, *converse MW-consistency*, and the following three basic axioms:

Domain: $\mathcal{V}_{all}$	$PreI$	$PreKer$	$PreNuc$
equal treatment of equals	No	Yes	Yes
anonymity	Yes	Yes	Yes
zero-independence	Yes	Yes	Yes
single-valuedness	No	No	Yes
DM-CONSISTENCY	Yes	Yes	Yes
CONVERSE DM-CONSISTENCY	Yes	Yes	No

Table 2: Properties of subsolutions of the preimputation correspondence on the domain of all TU games.

Domain: $\mathcal{V}_{all}$	Asp	$ParAsp$	$EqAsp$	$BalAsp$	$AspKer$	$AspNuc$
equal treatment of equals	No	No	Yes	No	Yes	Yes
anonymity	Yes	Yes	Yes	Yes	Yes	Yes
zero-independence	Yes	Yes	Yes	Yes	Yes	Yes
single-valuedness	No	No	No	No	No	Yes
MW-CONSISTENCY	Yes	Yes	No (ex. 4.1)	Yes (prop. 4.1)	Yes (prop. 4.3)	Yes (prop. 4.2)
CONVERSE MW-CONSISTENCY	Yes	Yes	No (ex. 4.1)	No (ex. 4.2)	Yes (prop. 4.3)	No (ex. 4.3)

Table 3: Properties of subsolutions of the aspiration correspondence on the domain of all TU games.

Equal treatment of equals: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}$, and all $i, j \in N$, if for all $S \subseteq N \setminus \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then for all $x \in \varphi(v)$, $x_i = x_j$.

Anonymity: For all $N, N' \in \mathcal{N}$ with $|N| = |N'|$, all $v, v' \in \mathcal{V}^N$, and all bijections $b\colon N \to N'$, if for all $S \subseteq N'$, $v'(S) = v(\{b(i) \,|\, i \in S\})$, then

$$\varphi(v') = \left\{x \in \mathbb{R}^{N'} \;\middle|\; \text{there exists } y \in \varphi(v) \text{ such that for all } i \in N,\ x_i = y_{b(i)}\right\}.$$

Zero-independence: For all $N \in \mathcal{N}$, $v, w \in \mathcal{V}$, and all $x \in \mathbb{R}^N$, if for all $S \subseteq N$, $w(S) = v(S) + \sum_S x_i$, then for all $y \in \varphi(v)$, $x + y \in \varphi(w)$.

On the domain of all TU games, the prekernel is the only subsolution of the preimputation correspondence satisfying *equal treatment of equals, zero-independence, DM-consistency*, and *converse DM-consistency* (Peleg [19]). (See Table 2.) It so happens that a similar result holds for the aspiration kernel.

Theorem 5.1 *On the domain of all TU games, the aspiration kernel is the only subsolution of the aspiration correspondence satisfying* equal treatment of equals, zero-independence, MW-consistency, *and* converse MW-consistency.

Proof. Clearly, the aspiration kernel is a subsolution of the aspiration correspondence satisfying *equal treatment of equals* and *zero-independence*. By Proposition 4.3, it also satisfies *MW-consistency* and *converse MW-consistency*.

Conversely, let φ be a subsolution of the aspiration correspondence satisfying these four axioms. Clearly, φ coincides with the aspiration kernel for the two-agent case. Let $N \in \mathcal{N}$ with $|N| \geq 3$, and $v \in \mathcal{V}_{all}^N$. First, we show that $\varphi(v) \subseteq AspKer(v)$. Let $x \in \varphi(v)$. By *MW-consistency* of φ, for all $N' \subset N$ with $|N'| = 2$, we have $x_{N'} \in \varphi(r_{N'}^x(v)) = AspKer(r_{N'}^x(v))$. By *converse MW-consistency* of the aspiration kernel, $x \in AspKer(v)$.

Next, we show that $AspKer(v) \subseteq \varphi(v)$. Let $y \in AspKer(v)$. Since the aspiration kernel is *MW-consistent*, for all $N' \subset N$ with $|N'| = 2$,

$$y_{N'} \in \varphi(r_{N'}^y(v)) = AspKer(r_{N'}^y(v)).$$

By *converse MW-consistency* of φ, $y \in \varphi(v)$. Thus, $\varphi(v) = AspKer(v)$. ∎

On the domain of all TU games, the prenucleolus is the only subsolution of the preimputation correspondence satisfying *single-valuedness, anonymity, zero-independence*, and *DM-consistency* (Sobolev [28]). (See Table 2.)

It turns out that, by using Lemma 4.2 and by following the argument in Sobolev [28], one can obtain a similar axiomatic characterization of the aspiration nucleolus. (Since the proof is very long, we provide it in the appendix.)

Theorem 5.2 *On the domain of all TU games, the aspiration nucleolus is the only subsolution of the aspiration correspondence satisfying single-valuedness, anonymity, zero-independence, and MW-consistency.*

6 Bilateral bargaining and NTU aspiration kernels

In this section, we consider two correspondences based on the idea of bilateral bargaining. Consider an NTU game and an aspiration for it. Let S be a generating coalition and i, j two members of it. We determine agent i's outside option as the largest payoff she can obtain without cooperating with agent j. Doing the same for agent j, we require their payoffs to be an outcome of a bilateral bargaining problem in which the disagreement point is the vector of the outside options. Based on this idea, we propose two generalizations of the aspiration kernel to NTU games. In the first generalization, the "egalitarian bargaining solution" (Kalai [11]) is used to solve each bilateral bargaining problem. In the second generalization, the "Nash bargaining solution" (Nash [17]) is used to solve each bilateral bargaining problem.[13] Formally, the disagreement outcome for each pair of agents is defined as follows: for all $N \in \mathcal{N}$, all $V \in \mathbb{W}_{all}$, all $x \in \mathbb{R}^N$, and all $i, j \in N$ with $i \neq j$,

$$\widehat{d}_{ij}(V, x) \equiv \max_{\substack{T \subset N \\ T \ni i \\ T \not\ni j}} \left\{ y_i \in \mathbb{R} \;\middle|\; (y_i, x_{T \setminus \{i\}}) \in V(T) \right\}.$$

Egalitarian aspiration kernel, *Egal-AspKer*: For all $N \in \mathcal{N}$ and all $V \in \mathbb{W}^N$,

$$\textit{Egal-AspKer}(v) \equiv \left\{ x \in \textit{Asp}(V) \;\middle|\; \begin{array}{l} \text{for all } S \in \mathcal{GC}(V, x) \text{ and all } i, j \in S, \\ x_i - \widehat{d}_{ij}(V, x) = x_j - \widehat{d}_{ji}(V, x) \end{array} \right\}.$$

Nash aspiration kernel, *Nash-AspKer*: For all $N \in \mathcal{N}$ and all $V \in \mathbb{W}^N$,

$$\textit{Nash-AspKer}(v) \equiv \left\{ x \in \textit{Asp}(V) \;\middle|\; \begin{array}{l} \text{for all } S \in \mathcal{GC}(V, x) \text{ and all } i, j \in S, \ (x_i, x_j) \\ \text{maximizes } \left(y_i - \widehat{d}_{ij}(V, x)\right)\left(y_j - \widehat{d}_{ji}(V, x)\right) \\ \text{subject to } y_i \geq \widehat{d}_{ij}(V, x),\ y_j \geq \widehat{d}_{ji}(V, x), \text{ and} \\ (y_i, y_j, x_{S \setminus \{i,j\}}) \in V(S) \end{array} \right\}.$$

The following lemma generalizes Lemma 4.3. It says that the disagreement point is invariant under the MW-reduction operation.

Lemma 6.1 *For all $N \in \mathcal{N}$, all $V \in \mathbb{W}^N_{all}$, all $x \in \mathbb{R}^N_+$, all $N' \subset N$, and all $i, j \in N'$ with $i \neq j$, we have $\widehat{d}_{ij}(V, x) = \widehat{d}_{ij}(r^x_{N'}(V), x_{N'})$.*

Proof: Let $N \in \mathcal{N}$, $V \in \mathbb{W}^N_{all}$, $x \in \mathbb{R}^N$, $N' \subset N$, $S \in \mathcal{GC}\left(r^x_{N'}(v), x_{N'}\right)$, and

[13] Similar generalizations of the prekernel for NTU games are studied by Moldovanu [13], Serrano [24], and Serrano and Shimomura [25].

$i, j \in S$ with $i \neq j$. Then

$$\begin{aligned} \widehat{d}_{ij}(r^x_{N'}(V), x_{N'}) &= \max_{\substack{T \subset N' \\ T \ni i \\ T \not\ni j}} \left\{ y_i \;\middle|\; (y_i, x_{T \setminus \{i\}}) \in r^x_{N'}(V)(T) \right\} \\ &= \max_{\substack{T \subset N' \\ T \ni i \\ T \not\ni j \\ R \subseteq N \setminus N'}} \left\{ y_i \;\middle|\; (y_i, x_{T \setminus \{i\}}, x_R) \in V(T \cup R) \right\} \\ &= \max_{\substack{Q \subset N \\ Q \ni i \\ Q \not\ni j}} \left\{ y_i \;\middle|\; (y_i, x_{Q \setminus \{i\}}) \in V(Q) \right\} \\ &= \widehat{d}_{ij}(V, x). \end{aligned}$$

■

On the domain of strictly comprehensive NTU games, there is no subsolution of the preimputation correspondence that satisfies *DM-consistency* and coincides with the prekernel for the two-agent TU games (Moldovanu [13]; Serrano [24]).

By using Lemma 6.1, it can be shown that on the domain of strictly comprehensive NTU games, the egalitarian aspiration kernel and the Nash aspiration kernel are *MW-consistent* and *conversely MW-consistent*, ***if they are well-defined.*** Unfortunately, it so happens that they are not.

Example 1 Let $N \equiv \{1, 2, 3\}$ and $V \in \mathcal{W}^N_{all}$ be defined as follows: for all $i \in N$, $V(\{i\}) \equiv \{x \in \mathbb{R} | \; x_i \leq 0\}$; $V(\{1, 2\}) \equiv \left\{x \in \mathbb{R}^{\{1,2\}} \;\middle|\; x_2 \leq -2x_1 + 4\right\}$; $V(\{1, 3\}) \equiv \left\{x \in \mathbb{R}^{\{1,3\}} \;\middle|\; x_1 \leq -2x_3 + 4\right\}$; $V(\{2, 3\}) \equiv \left\{x \in \mathbb{R}^{\{2,3\}} \;\middle|\; x_3 \leq -2x_2 + 4\right\}$; $V(N) \equiv \left\{x \in \mathbb{R}^{\{1,2,3\}} \;\middle|\; x_1 + x_2 + x_3 \leq 6\right\}$.

It can be shown that for V, both the egalitarian aspiration kernel and the Nash aspiration kernel assign an empty set.[14] □

The above example leads us to the following conjecture:

Conjecture 6.1 *On the domain of strictly comprehensive NTU games, there is no solution that satisfies MW-consistency and coincides with the aspiration kernel for the two-agent TU games.*

Proving or disproving Conjecture 6.1 seems to be an important next step towards the full understanding of *MW-consistency* for NTU games.

[14] The proof of this claim is available from authors on request.

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Appendix

Proofs of the claims used in example 4.1: First, we show that $x \in EqAsp(v)$. Note that

$$\begin{aligned}
\max_{\substack{S \ni 1 \\ S \neq \{1,2\}}} [v(S) - x(S)] &= \max\{0-4,\, 0-4,\, 6-6\} = 0, \\
\max_{\substack{S \ni 2 \\ S \neq \{1,2\}}} [v(S) - x(S)] &= \max\{0-2,\, 0-2,\, 6-6\} = 0, \\
\max_{\substack{S \ni 1 \\ S \neq \{1,2,3\}}} [v(S) - x(S)] &= \max\{0-4,\, 6-6,\, 0-4\} = 0, \\
\max_{\substack{S \ni 2 \\ S \neq \{1,2,3\}}} [v(S) - x(S)] &= \max\{0-2,\, 6-6,\, 0-2\} = 0, \\
\max_{\substack{S \ni 3 \\ S \neq \{1,2,3\}}} [v(S) - x(S)] &= \max\{0-0,\, 0-4,\, 0-3\} = 0.
\end{aligned}$$

Thus, $x \in EqAsp(v)$.

Next, we show that for all $i, j \in N$ with $i \neq j$, $(y_i, y_j) \in EqAsp(r^y_{\{i,j\}}(v))$. For coalition $\{1,2\}$,

$$\begin{aligned}
r^y_{\{1,2\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,3\}) - y_3\} = \max\{0, 0-0\} = 0, \\
r^y_{\{1,2\}}(v)(\{2\}) &= \max\{v(\{2\}), v(\{2,3\}) - y_3\} = \max\{0, 0-0\} = 0, \\
r^y_{\{1,2\}}(v)(\{1,2\}) &= \max\{v(\{1,2\}), v(\{1,2,3\}) - y_3\} = \max\{6, 6-0\} = 6,
\end{aligned}$$

and $(y_1, y_2) = (3,3) \in EqAsp\big(r^y_{\{1,2\}}(v)\big)$.

For coalition $\{1,3\}$,

$$\begin{aligned}
r^y_{\{1,3\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,2\}) - y_2\} = \max\{0, 6-3\} = 3, \\
r^y_{\{1,3\}}(v)(\{3\}) &= \max\{v(\{3\}), v(\{2,3\}) - y_2\} = \max\{0, 0-3\} = 0, \\
r^y_{\{1,3\}}(v)(\{1,3\}) &= \max\{v(\{1,3\}), v(\{1,2,3\}) - y_2\} = \max\{0, 6-3\} = 3,
\end{aligned}$$

and $(y_1, y_3) = (3,0) \in EqAsp\big(r^y_{\{1,3\}}(v)\big)$.

For coalition $\{2,3\}$,

$$\begin{aligned}
r^y_{\{2,3\}}(v)(\{2\}) &= \max\{v(\{2\}), v(\{1,2\}) - y_1\} = \max\{0, 6-3\} = 3, \\
r^y_{\{2,3\}}(v)(\{3\}) &= \max\{v(\{3\}), v(\{1,3\}) - y_1\} = \max\{0, 0-3\} = 0, \\
r^y_{\{2,3\}}(v)(\{2,3\}) &= \max\{v(\{2,3\}), v(\{1,2,3\}) - y_1\} = \max\{0, 6-3\} = 3,
\end{aligned}$$

and $(y_2, y_3) = (3, 0) \in EqAsp\big(r^y_{\{2,3\}}(v)\big)$. ■

Next, we provide the proof of Theorem 5.2. As mentioned before, the proof is similar to the proof of a theorem in Sobolev [28], which is written in Russian. The proof of Sobolev's theorem (in English) can be found in Peleg [20]. Essential parts of Peleg's proof is reproduced in Snijders [27]. Here, we provide a complete proof.

Proof of Theorem 5.2: Clearly, the aspiration nucleolus is a subsolution of the aspiration correspondence satisfying *anonymity* and *zero-independence*. It is also *single-valued* (Bennett [2]; Sharkey [26]) and, by Proposition 4.2, *MW-consistent*.

Conversely, let φ be a subsolution of the aspiration correspondence satisfying the four axioms. Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{all}$, and $x = AspNuc(v)$. We show, in seven steps, that $x = \varphi(v)$.

Let $A \equiv \{\alpha \in \mathbb{R} \mid \text{for some } S \subset N,\ \alpha = e_S(v,x)\}$ and $(\alpha_1, \alpha_2, \cdots, \alpha_{|A|}) \in \mathbb{R}^{|A|}$ be an enumeration of A with $\alpha_1 > \alpha_2 > \cdots > \alpha_{|A|}$. To simplify the notation, for each $k \in \{1, 2, \ldots, |A|\}$, we write $\mathcal{S}_k \equiv \mathcal{S}_{\alpha_k}(v, x)$.

Given $k \in \{1, 2, \ldots, |A|\}$ and $i \in N$, let $\mathcal{S}^i_k \equiv \{S \in \mathcal{S}_k \mid i \in S\}$. By Lemma 4.2, $\mathcal{S}_k$ is strictly balanced on N. Moreover, the associated weights can be chosen to be rational. Thus, there exist a natural number μ_k and a list of natural numbers $(\mu_S)_{S \in \mathcal{S}_k}$ such that for all $i \in N$, $\sum_{S \in \mathcal{S}^i_k} \mu_S = \mu_k$. Let $\mathcal{B}_k$ be the partition of N such that for all $B \in \mathcal{B}_k$ and all $i, j \in N$, we have $i, j \in B$ if and only if $\mathcal{S}^i_k = \mathcal{S}^j_k$. Let $\beta_k \equiv \max_{B \in \mathcal{B}_k} |B|$, $\gamma_k \equiv \sum_{S \in \mathcal{S}_k} \mu_S$, and $\lambda_k \equiv \binom{\gamma_k}{\mu_k}$.

Step 1: Given $k \in \{1, 2, \ldots, K\}$, we construct $M_k \in \mathcal{N}$ and $\mathcal{T}_k \subseteq 2^{M_k} \setminus \{\emptyset\}$ that satisfy the following conditions:

(i) $N \subseteq M_k$;

(ii) $|M_k| = \beta_k \cdot \lambda_k$;

(iii) for all $S \in \mathcal{S}_k$, there exists $T \in \mathcal{T}_k$ with $T \cap N = S$;

(iv) for all $S \in 2^N \setminus \{N, \emptyset\}$ and all $T \in \mathcal{T}_k$, if $T \cap N = S$, then $S \in \mathcal{T}_k$;

(v) for all $i \in M_k$, we have $|\mathcal{T}^i_k| = \mu_k$ and $\big|\{j \in M_k \mid \mathcal{T}^j_k = \mathcal{T}^i_k\}\big| = \beta_k$, where $\mathcal{T}^i_k \equiv \{S \in \mathcal{T}_k \mid i \in S\}$ and $\mathcal{T}^j_k \equiv \{S \in \mathcal{T}_k \mid j \in S\}$.

Let $(B_1, B_2, \ldots, B_{|\mathcal{B}_k|})$ be an enumeration of $\mathcal{B}_k$. For each $h \in \{1, 2, \ldots, \lambda_k\}$, we construct a set D_h of agents as follows: For each $h \in \{1, 2, \ldots, |\mathcal{B}_k|\}$, if $|B_h| = \beta_k$, then let $D_h \equiv B_h$; if $|B_h| < \beta_k$, then choose $\beta_k - |B_h|$ agents from $\mathbb{N} \setminus N$, and add them to B_h to define D_h. For each $h \in \{|\mathcal{B}_k| + 1, |\mathcal{B}_k| + 2, \ldots, \lambda_k\}$, choose β_k agents from $\mathbb{N} \setminus N$ to define D_h. Since the set of potential agents is countably infinite, it is clear that, in the above constructions of D_h's, we can make them mutually exclusive. Then, let

$$M_k \equiv D_1 \cup D_2 \cup \cdots \cup D_{\lambda_k}.$$

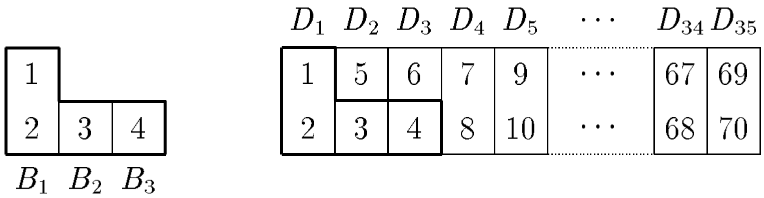
Note that $\{D_1, D_2, \ldots, D_{\lambda_k}\}$ is a partition of M_k. By construction, M_k satisfies conditions (i) and (ii).

Next, imagine that there are γ_k empty "rooms." We will fill these rooms with (appropriately replicated) groups in $\{D_1, D_2, \ldots, D_{\lambda_k}\}$, and each room will correspond to an element of $\mathcal{T}_k$. For each $S \in \mathcal{S}_k$, create μ_S copies of the set $\bigcup_{\substack{h \in \{1,\ldots,|\mathcal{B}_k|\} \\ \text{s.t. } D_h \cap S \neq \emptyset}} D_h$. The total number of these copies being $\gamma_k = \sum_{S \in \mathcal{S}_k} \mu_S$, we can put them into different rooms. Recall that for each $i \in N$, $\sum_{S \in \mathcal{S}_k^i} \mu_S = \mu_k$. This implies that for each $h \in \{1, \ldots, |\mathcal{B}_k|\}$, group D_h belongs to exactly μ_k rooms. Next, for each $h \in \{|\mathcal{B}_k| + 1, \ldots, \lambda_k\}$, create μ_k copies of D_h. Since $\lambda_k = \binom{\gamma_k}{\mu_k}$, we can allocate them so that all γ_k rooms are distinct and contain the same number of groups. It is easy to see that $\mathcal{T}_k$ thus constructed satisfies conditions (iii), (iv), and (v). (The above construction of M_k and $\mathcal{T}_k$ is illustrated in Figure 5 for a "simple" case.)

Step 2: Given $k \in \{1, 2, \ldots, |A|\}$, let $\mathcal{D}_k \equiv \{D_1, D_2, \ldots, D_{\lambda_k}\}$, M_k, and $\mathcal{T}_k$ be as constructed in Step 1. We show that for all $i, j \in M_k$ with $i \neq j$, there exists a permutation $\pi_{M_k} \colon M_k \to M_k$ such that $\pi_{M_k}(i) = j$ and for all $T \in \mathcal{T}_k$, $\pi_{M_k}(T) \in \mathcal{T}_k$.

Let $i, j \in M_k$ with $i \neq j$. By condition (v) of $\mathcal{T}_k$ in Step 1, $|\mathcal{T}_k^i| = |\mathcal{T}_k^j| = \mu_k$. Thus, there exists a permutation $\pi_0 \colon \mathcal{T}_k \to \mathcal{T}_k$ such that $\pi_0(\mathcal{T}_k^i) = \mathcal{T}_k^j$. Note that $\mathcal{T}_k$ has a property that if μ_k distinct coalitions in $\mathcal{T}_k$ are chosen, then there exists exactly one group in $\mathcal{D}_k$ that is a subset of all of these μ_k coalitions. Thus, π_0 induces a permutation on $\mathcal{D}_k$, denoted $\pi_{\mathcal{D}_k}$. Formally, $\pi_{\mathcal{D}_k} : \mathcal{D}_k \to \mathcal{D}_k$ is defined by setting for all $D \in \mathcal{D}_k$,

$$\pi_{\mathcal{D}_k}(D) \equiv \bigcap_{\substack{T \in \mathcal{T}_k \\ T \supseteq D}} \pi_0(T).$$



$\mathcal{S}_k$	{1,2,3}	{1,2}	{3,4}	{3}	{4}
	$\mu_{\{1,2,3\}} = 1$	$\mu_{\{1,2\}} = 2$	$\mu_{\{3,4\}} = 1$	$\mu_{\{3\}} = 1$	$\mu_{\{4\}} = 2$
$\mathcal{T}_k$	1 5 7 9 11 2 3 8 10 12 13 15 17 19 21 14 16 18 20 22 23 25 27 29 31 24 26 28 30 32	1 7 9 11 13 2 8 10 12 14 33 35 37 39 41 34 36 38 40 42 43 45 47 49 51 44 46 48 50 52 1 15 17 19 21 2 16 18 20 22 23 35 37 39 53 34 36 38 40 54 55 57 59 61 63 56 58 60 62 64	5 6 7 15 23 3 4 8 16 24 25 33 41 43 45 26 34 42 44 46 53 55 57 65 67 54 56 58 66 68	5 9 17 27 29 3 10 18 28 30 35 41 47 49 53 36 42 48 50 54 59 61 65 67 69 60 62 66 68 70	6 11 19 23 27 4 12 20 24 28 31 37 43 47 51 32 38 44 48 52 55 59 63 65 69 56 60 64 66 70 6 13 21 25 29 4 14 22 26 30 31 39 45 49 51 32 40 46 50 52 57 61 63 67 69 58 62 64 68 70

Figure 5: Step 1 of the proof of Theorem 5.2: In the above example, $N = \{1,2,3,4\}$, $\mathcal{S}_k = \{\{1,2,3\},\{1,2\},\{3,4\},\{3\},\{4\}\}$, $\mu_k = 3$, $\mu_{\{1,2,3\}} = \mu_{\{3,4\}} = \mu_{\{3\}} = 1$, $\mu_{\{1,2\}} = \mu_{\{4\}} = 2$. Thus, $\beta_k = \max\{2,1\} = 2$, $\gamma_k = 1+1+1+2+2 = 7$, and $\lambda_k = \binom{\gamma_k}{\mu_k} = \binom{7}{3} = 35$.

Note that each coalition in $\mathcal{T}_k$ can be viewed as a coalition of groups. For each $T \in \mathcal{T}_k$, its image under $\pi_{\mathcal{D}_k}$ can be defined by

$$\pi_{\mathcal{D}_k}(T) \equiv \bigcup_{\substack{D \in \mathcal{D}_k \\ D \subseteq T}} \pi_{\mathcal{D}_k}(D).$$

Now, we show that for all $T \in \mathcal{T}_k$, $\pi_{\mathcal{D}_k}(T) \in \mathcal{T}_k$. Let $T \in \mathcal{T}_k$. Then, by the definition of $\pi_{\mathcal{D}_k}$, for all $D \in \mathcal{D}_k$ with $D \subseteq T$, we have $\pi_{\mathcal{D}_k}(D) \subseteq \pi_0(T)$. Thus,

$$\pi_{\mathcal{D}_k}(T) = \bigcup_{\substack{D \in \mathcal{D}_k \\ D \subseteq T}} \pi_{\mathcal{D}_k}(D) \subseteq \pi_0(T).$$

Since both $\pi_{\mathcal{D}_k}(T)$ and $\pi_0(T)$ consist of the same number of groups, we have $\pi_{\mathcal{D}_k}(T) = \pi_0(T)$. Thus, $\pi_{\mathcal{D}_k}(T) \in \mathcal{T}_k$.

Note that if $D \in \mathcal{D}_k$ and $i \in D$, then $j \in \pi_{\mathcal{D}_k}(D)$. Note also that for all $T \in \mathcal{T}_k$, $\pi_{\mathcal{D}_k}(T) \in \mathcal{T}_k$. By construction, each group in $\mathcal{D}_k$ contains exactly β_k agents. For each $D \in \mathcal{D}_k$ with $i \in D$, choose a bijection $b_D \colon D \to \pi_{\mathcal{D}_k}(D)$ with $b_D(i) = j$. For each $D \in \mathcal{D}_k$ with $i \notin D$, choose an arbitrary bijection $b_D \colon D \to \pi_{\mathcal{D}_k}(D)$. Given the list $(b_D)_{D \in \mathcal{D}_k}$ of such bijections, define the permutation $\pi_{M_k} \colon M_k \to M_k$ by setting for all $D \in \mathcal{D}_k$ and all $h \in D$, $\pi_{M_k}(h) \equiv b_D(h)$. Since $\mathcal{D}_k$ is a partition of M_k, π_{M_k} is well-defined. Clearly, $\pi_{M_k}(i) = j$. Let $T \in \mathcal{T}_k$. Then

$$\pi_{M_k}(T) = \bigcup_{h \in T} \pi_{M_k}(h) = \bigcup_{\substack{D \in \mathcal{D}_k \\ D \subseteq T}} \bigcup_{h \in D} b_D(h) = \bigcup_{\substack{D \in \mathcal{D}_k \\ D \subseteq T}} \pi_{\mathcal{D}_k}(D) = \pi_{\mathcal{D}_k}(T) \in \mathcal{T}_k.$$

Thus, π_{M_k} is a desired permutation on M_k.

Step 3: For all $k \in \{1, 2, \ldots, |A|\}$, let M_k and $\mathcal{T}_k$ be as constructed in Step 1. Here, we construct a set $\boldsymbol{M}$ and a partition of $2^{\boldsymbol{M}} \setminus \{\boldsymbol{M}, \emptyset\}$.

Let

$$\boldsymbol{M} \equiv M_1 \times M_2 \times \cdots \times M_{|A|}.$$

For all $k \in \{1, 2, \ldots, |A|\}$, let

$$\boldsymbol{\mathcal{S}}_k \equiv \left\{ \boldsymbol{S} \subset \boldsymbol{M} \;\middle|\; \begin{array}{l} \text{for some } T \in \mathcal{T}_k, \\ \boldsymbol{S} = M_1 \times \cdots \times M_{k-1} \times T \times M_{k+1} \times \cdots \times M_{|A|} \end{array} \right\}.$$

Finally, let

$$\boldsymbol{\mathcal{S}}_{|A|+1} \equiv 2^{\boldsymbol{M}} \setminus \Big(\{\boldsymbol{M}, \emptyset\} \cup \big(\textstyle\bigcup_{k=1}^{|A|} \boldsymbol{\mathcal{S}}_k \big) \Big).$$

In order to stress that fact that $\boldsymbol{M}$ is a Cartesian product of the sets of agents, we write its subsets and its elements in bold faces. Note that, since the set of potential agents is countably infinite, in the presence of *anonymity*, $\boldsymbol{M}$ can be viewed as an element in $\mathcal{N}$.

Stet 4: We show that for all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{M}$, there exists a permutation $\pi_{\boldsymbol{M}}\colon \boldsymbol{M} \to \boldsymbol{M}$ such that (i) $\pi_{\boldsymbol{M}}(\boldsymbol{i}) = \boldsymbol{j}$ and (ii) for all $k \in \{1, 2, \ldots, |A| + 1\}$ and all $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_k$, we have $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \boldsymbol{\mathcal{S}}_k$.

Let $\boldsymbol{i} = (i_1, i_2, \ldots, i_{|A|}) \in \boldsymbol{M}$ and $\boldsymbol{j} = (j_1, j_2, \ldots, j_{|A|}) \in \boldsymbol{M}$. By Step 2, for all $k \in \{1, 2, \ldots, |A|\}$, there exists a permutation $\pi_{M_k}\colon M_k \to M_k$ such that $\pi_{M_k}(i_k) = j_k$ and for all $T \in \mathcal{T}_k$, $\pi_{M_k}(T) \in \mathcal{T}_k$. Let $\pi_{\boldsymbol{M}}\colon \boldsymbol{M} \to \boldsymbol{M}$ be the permutation defined by setting for all $\boldsymbol{h} = (h_1, h_2, \ldots, h_{|A|}) \in \boldsymbol{M}$,

$$\pi_{\boldsymbol{M}}(\boldsymbol{h}) \equiv \left(\pi_{M_1}(h_1), \pi_{M_2}(h_2), \ldots, \pi_{M_{|A|}}(h_{|A|})\right).$$

Clearly, $\pi_{\boldsymbol{M}}(\boldsymbol{i}) = \boldsymbol{j}$. Moreover, for all $k \in \{1, 2, \ldots, |A|\}$ and all $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_k$, we have $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \boldsymbol{\mathcal{S}}_k$. Thus, $\pi_{\boldsymbol{M}}$ induces a permutation on $\bigcup_{k=1}^{|A|} \boldsymbol{\mathcal{S}}_k$. Therefore, for all $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_{|A|+1}$, $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \boldsymbol{\mathcal{S}}_{|A|+1}$.

Step 5: Let $w \in \mathcal{V}_{all}^{\boldsymbol{M}}$ be defined as follows: (i) $w(\boldsymbol{M}) \equiv v(N) - x(N)$; (ii) for all $k \in \{1, 2, \ldots, |A|\}$ and all $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_k$, $w(\boldsymbol{S}) \equiv \alpha_k$; and (iii) for all $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_{|A|+1}$, $w(\boldsymbol{S}) \equiv \min\{\alpha_{|A|}, v(N) - x(N)\}$. We show that for all $\boldsymbol{i} \in \boldsymbol{M}$, $\varphi_{\boldsymbol{i}}(w) = 0$.

Let $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{M}$. By Step 4, there exists a permutation $\pi_{\boldsymbol{M}}\colon \boldsymbol{M} \to \boldsymbol{M}$ such that $\pi_{\boldsymbol{M}}(\boldsymbol{i}) = \boldsymbol{j}$, and for all $k \in \{1, 2, \ldots, |A| + 1\}$ and all $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_k$, we have $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \boldsymbol{\mathcal{S}}_k$. Let $w' \in \mathcal{V}_{all}^{\boldsymbol{M}}$ be defined by setting for all $\boldsymbol{S} \subseteq \boldsymbol{M}$, $w'(\boldsymbol{S}) \equiv w((\pi_{\boldsymbol{M}})^{-1}(\boldsymbol{S}))$. By *anonymity*, $\varphi_{\boldsymbol{i}}(w) = \varphi_{\boldsymbol{j}}(w')$. Let $\boldsymbol{S} \subset \boldsymbol{M}$. Since $\bigcup_{k=1}^{|A|+1} \boldsymbol{\mathcal{S}}_k = 2^{\boldsymbol{M}} \setminus \{\boldsymbol{M}, \emptyset\}$, there exists $k \in \{1, 2, \ldots, |A| + 1\}$ such that $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_k$. Since $(\pi_{\boldsymbol{M}})^{-1}(\boldsymbol{S}) \in \boldsymbol{\mathcal{S}}_k$, by the definitions of w and w', $w'(\boldsymbol{S}) = w(\boldsymbol{S})$. Clearly, $w'(\boldsymbol{M}) = w(\boldsymbol{M})$. Thus, $w' = w$. Therefore, $\varphi_{\boldsymbol{i}}(w) = \varphi_{\boldsymbol{j}}(w') = \varphi_{\boldsymbol{j}}(w)$. This holds for all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{M}$.

Since $x \in Asp(v)$, x is individually feasible and coalitionally rational. By coalitional rationality, for all $k \in \{1, 2, \ldots, |A|\}$, $\alpha_k \leq 0$. Together with individual feasibility, we have either $\alpha_1 = 0$ or $v(N) = x(N)$. Thus, we have (i) for all $\boldsymbol{S} \subseteq \boldsymbol{M}$, $w(\boldsymbol{S}) \leq 0$, and (ii) there exists $\boldsymbol{T} \subseteq \boldsymbol{M}$ such that $\boldsymbol{T} \neq \emptyset$ and $w(\boldsymbol{T}) = 0$. Let $\boldsymbol{i} \in \boldsymbol{T}$. Then, since $\varphi(w) \in Asp(w)$,

$$0 = w(\boldsymbol{T}) = \sum_{\boldsymbol{j} \in \boldsymbol{T}} \varphi_{\boldsymbol{j}}(w) = |\boldsymbol{T}| \cdot \varphi_{\boldsymbol{i}}(w).$$

Since $|\boldsymbol{T}| > 0$, $\varphi_{\boldsymbol{i}}(w) = 0$. Thus, for all $\boldsymbol{i} \in \boldsymbol{M}$, $\varphi_{\boldsymbol{i}}(w) = 0$.

Step 6: Let

$$\boldsymbol{M'} \equiv \big\{(i, i, \ldots, i) \in M \;\big|\; i \in N\big\}.$$

Clearly, $|\boldsymbol{M'}| = |N|$. Let $\boldsymbol{b}\colon N \to \boldsymbol{M'}$ be the bijection defined by setting for all $i \in N$, $\boldsymbol{b}(i) \equiv (i, i, \ldots, i)$. We show that for all $S \subseteq N$, $r^{\varphi(w)}_{\boldsymbol{M'}}(w)(\boldsymbol{b}(S)) = v(S) - x(S)$.

Let $S \subset N$. Then for some $k \in \{1, 2, \ldots, |A|\}$, $v(S) - x(S) = \alpha_k$. By property (iii) of $\mathcal{T}_k$ in Step 1, there exists $T \in \mathcal{T}_k$ such that $T \cap N = S$. Let

$$\boldsymbol{R} \equiv M_1 \times \cdots \times M_{k-1} \times T \times M_{k+1} \times \cdots \times M_{|A|}.$$

Then $\boldsymbol{b}(S) \subseteq \boldsymbol{R}$. Moreover, since $T \cap N = S$, $\boldsymbol{R} \setminus \boldsymbol{b}(S) \subseteq \boldsymbol{M} \setminus \boldsymbol{M'}$. Thus,

$$r^{\varphi(w)}_{\boldsymbol{M'}}(w)(\boldsymbol{b}(S)) = \max_{\boldsymbol{Q} \subseteq \boldsymbol{M} \setminus \boldsymbol{M'}} w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) \geq w\big(\boldsymbol{b}(S) \cup (\boldsymbol{R} \setminus \boldsymbol{b}(S))\big) = w(\boldsymbol{R}).$$

Since $\boldsymbol{R} \in \boldsymbol{\mathcal{S}}_k$, $w(\boldsymbol{R}) = \alpha_k$. Thus,

$$r^{\varphi(w)}_{\boldsymbol{M'}}(w)(\boldsymbol{b}(S)) \geq w(\boldsymbol{R}) = \alpha_k = v(S) - x(S).$$

Now, we claim that the opposite (weak) inequality also holds. Let $\boldsymbol{Q} \subseteq \boldsymbol{M} \setminus \boldsymbol{M'}$. If $\boldsymbol{b}(S) \cup \boldsymbol{Q} \in \boldsymbol{\mathcal{S}}_{|A|+1}$, then $w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) = \alpha_{|A|} \leq \alpha_k = v(S) - x(S)$. If there exists $k' \leq |A|$ such that $\boldsymbol{b}(S) \cup \boldsymbol{Q} \in \boldsymbol{\mathcal{S}}_{k'}$, then there exists $T' \in \mathcal{T}_{k'}$ such that

$$\boldsymbol{b}(S) \cup \boldsymbol{Q} = M_1 \times \cdots \times M_{k'-1} \times T' \times M_{k'+1} \times \cdots \times M_{|A|}.$$

Since $\boldsymbol{Q} \subseteq \boldsymbol{M} \setminus \boldsymbol{M'}$ and $S \subseteq T'$, we have $T' \cap N = S$. Thus, by property (iv) of $\mathcal{T}_{k'}$ in Step 1, we have $S \in \mathcal{S}_{k'}$, so that $w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) = \alpha_{k'} \leq v(S) - x(S)$. So, in both of above cases, for all $\boldsymbol{Q} \subseteq \boldsymbol{M} \setminus \boldsymbol{M'}$, $w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) \leq v(S) - x(S)$. This implies that

$$\hat{r}^{\varphi(w)}_{\boldsymbol{M'}}(w)(\boldsymbol{b}(S)) = \max_{\boldsymbol{Q} \subseteq \boldsymbol{M} \setminus \boldsymbol{M'}} w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) \leq v(S) - x(S).$$

By definition, $w(\boldsymbol{M}) = v(N) - x(N)$. Note that for all $k \in \{1, 2, \ldots, |A|\}$, $N \notin \mathcal{T}_k$. Thus, for all $\boldsymbol{S} \supseteq \boldsymbol{M'}$, we have $\boldsymbol{S} \in \boldsymbol{\mathcal{S}}_{|A|+1}$ and, therefore,

$$w(\boldsymbol{S}) = \min\{\alpha_{|A|},\, v(N) - x(N)\} \leq v(N) - x(N).$$

Thus,

$$r_{\boldsymbol{M'}}^{\varphi(w)}(w)(\boldsymbol{b}(N)) = \max_{\boldsymbol{Q}\subseteq\boldsymbol{M}\setminus\boldsymbol{M'}} w(\boldsymbol{b}(N)\cup\boldsymbol{Q}) = v(N) - x(N).$$

Altogether, for all $S \subseteq N$,

$$r_{\boldsymbol{M'}}^{\varphi(w)}(w)(\boldsymbol{b}(S)) = v(S) - x(S).$$

Step 7: By *max consistency*, for all $\boldsymbol{i} \in \boldsymbol{M'}$, $\varphi_{\boldsymbol{i}}\big(\hat{r}_{\boldsymbol{M'}}^{\varphi(w)}(w)\big) = \varphi_{\boldsymbol{i}}(w) = 0$.

Finally, by *anonymity* and *zero-independence* of φ, we deduce that for all $i \in N$,

$$\varphi_i(v) = \varphi_{\boldsymbol{b}(i)}\big(\hat{r}_{\boldsymbol{M'}}^{\varphi(w)}(w)\big) + x_i = 0 + x_i = x_i.$$

Thus, $\varphi(v) = AspNuc(v)$. ■