Matching with interdependent choices*

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Abstract

We consider a many-to-one matching problem, for example, a problem involving workers and firms, in which firms' choices are interdependent. We propose two conditions on an aggregate choice function called *independence of rejected additional applications* and *interdependent substitutability*, and then show that it is sufficient for the existence of stable matchings for any workers' preference profile. Interestingly, properties of stable matchings shown in the previous literature sometimes follow and sometimes not.

1 Introduction

Since Gale and Shapley (1962), theory of stable matchings has expanded in multiple directions, including matching with contracts, matching with constraints, and what covers a rich class of matching markets. A standard matching model considers a situation that each worker has preferences over the set of firms, each firm has preferences over the subsets of the set of workers, and based on those preferences each worker matches with at most one firm and each firm matches with a set of workers. Contrast to this, in particular, a matching model with contracts increases one more dimension indicating how to match to a standard matching model, and a matching model with constraints exogenously constrains a matching outcome space. Though these extensions are indeed helpful, almost all models share the same assumption that choices (or preferences) are independent. That is, no matter which workers the other firms choose, a firm's choice never changes. This assumption is not only theoretically limited but also, as we discuss later, practically restrictive. This paper differs from the previous literature in that choices are interdependent. The main contributions of the present paper are a new modeling of a matching problem through interdependent choices, and the introduction of a sufficient condition for the existence of stable matchings.

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We consider a many-to-one matching problem such as the problem faced firms and workers in which each worker matches at most one firm and each firm is able to match with multiple workers. Although we assume that workers' choices or preferences are independent, we allow firms' choices to be interdependent. In the previous literature, each firm has a choice function from a set of workers to itself, thus the aggregate choice function is the Cartesian product of single firms' choice functions. Instead we model that firms' choices directly by means of a single aggregate choice function from the Cartesian product of sets of workers to itself. Hence our formulation of firms' choices subsumes their choices in the former modelings. Moreover, we can describe complex firms' choices. For example, *complementarity* in firms' choices or *externality* between firms, or *state-dependent choices* are well defined.

Our solution concept is stability. In theory of matchings, the notion of stability has played a central role. A matching is "stable" if it prevents from profitable deviation by workers and (or) firms. We say that a matching is *set-wise stable* if there is no additional applications by workers such that a set of firms and the workers finds that all of them are at least as well off and at least one of them better off. This reduces to the usual definition of stability in the standard matching problem.

As is well known, even with firms' choices independent, there is in general no stable matching for some workers' preference profile. The property of individual firm's choice function called *substitutability* guarantees the existence of stable matchings for any workers' preference profiles (Kelso and Crawford (1982); Roth (1985)). In our more general domains, the two properties of the firms' aggregate choice function are sufficient for the existence of stable matchings. The first property is called *independence of rejected additional applications*. This says that if the aggregate choice from some application set larger than another is included in the choice from a subset of the set, then the two choices actually coincide. The second property is called *interdependent substitutability* which resembles substitutability in independent choice models. This requires that given an application set, for any application sets between the given application set and any matchings potentially generated from the given application set, rejected applications are monotonic. As a property on an aggregate choice function, interdependent substitutability is weaker than substitutability of all independent choice functions.

Under the two properties above, we show the existence of stable matchings for all workers' preference profiles. The Gale and Shapley's worker-proposing (firm-proposing) deferred acceptance algorithm (Gale and Shapley (1962)) is known to find a stable matching for a model with independent choice functions. We generalize the firm-proposing deferred acceptance algorithm for our model, which is also computationally solvable. Importantly, we show that it is possible to generalize the worker proposing deferred acceptance algorithm for our model, but the aggregate choice function being interdependent requires that combinations of applications be checked, which raises computational difficulties.

There is another way, via a fixed point method, of finding stable matchings. It is based on

Tarski's fixed point theorem (Tarski (1955)). We could construct a complete lattice similar to Adachi (2000) or Hatfield and Milgrom (2005), but a critical difficulty arises, namely, that do not ensure monotonic function on the lattice. This is because our interdependent substitutability partially requires monotonic relations on the lattice, by contrast to that substitutability on all firms' independent choice functions does fully.[1]

We identify which results or properties of the theory of stable matchings established in the previous literature still follow. The consequences seem to be slightly negative. In welfare analyses on stable matchings, Since the set of stable matchings coincide with the core in the standard problem, they are efficient. Though our stability notion does not coincide with the core, we show that any stable matching is efficient. We also explore the workers' (firms') side optimality of stable matchings. A stable matching is worker (firm) optimal if there is no other stable matching that all workers (firms) find at least as desirable. Since the set of stable matchings does not form complete lattice, there are generally multiple worker or firm optimal stable matchings. However, without the extremes of stable matchings, we see that there still be trade-offs between the workers' side and the firms' side, so if all workers strictly prefer some stable matching than the other stable matching, then all firms strictly prefer the latter than the former.

Although we directly use the aggregate choice function, we regard a model in which each firm makes decision independently. There we introduce two types of firms' choices which are both more flexible than ones in the standard model. The first model considers that firms have state-dependent choice functions. We say that the status of entire applications *state*, and there are multiple states. Each firm's choice function is defined over not only applications to that firm but also states. Thus each firm's choice is consistent in the same state, but not necessarily in the other state. This can be interpreted as a case that states represent economic status like boom and recession, and firms behave differently in different states. The second model considers that there is a coordinator function for each firm. The coordinator function recommends each firm which sets of workers be suitable for given applications. Then each firm chooses the best workers among them. In both cases, it is possible to guarantee the existence of stable matchings for all workers' preference profiles under suitable conditions for those choice functions, which are generally weaker than substitutability for each firm. It should be emphasized that those choice functions does not depend on the others' choices.

Finally, we conclude the present paper by two remarks: a necessary condition for the existence of stable matchings for all workers' preference profiles and the relation to the previous literature.

[1] The lattice here is described by a product of two application spaces (all pre-matchings in the literature) and a partial order created by a combination of a workers' preference relations for the former pre-matching space and a firms' aggregate choice function in the reverse order for the latter space.

1.1 Motivating examples

We propose simple examples of interdependent choices for which there still exists a stable matching for all workers' preference profiles. Those examples are clearly not covered in the previous literature, but they are important in practice.

Example 1. Suppose that there are two branches, x and y, in a firm. Branches x and y correspond to two different geographical areas. Now a firm is considering to launch an IT accounting section at one of the two, and if possible, the firm would prefer to do so at branch x. An IT accounting section needs exactly two skilled workers. Suppose also that there are two workers i and j, an IT engineer and accountant, respectively. Each of them is able to apply to either x or y, or both, or neither. An application set is said to be a summary of their applications. Then the branches' choices C are interdependent and described by the following aggregate choice function. For each application set (matrix) $A \in \{0,1\}^{2\times 2}$, the first (second) row corresponds to i (j), and the left (right) column corresponds to branch x (y).

$$C\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}, C\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix}, C\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$C\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}, C\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}, C\begin{pmatrix}0&1\\1&1\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix}, C\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}, C\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}.$$

Note that the aggregate choice function C cannot be written as the aggregate of two of independent choice functions, one for x and one for y. The choices of x and y are interdependent. Furthermore, C exhibits complementarity. It is the most important feature of our model which has been never described in the previous literature. It is easy to verify that a stable matching exists for all workers' preference profiles.

The example above introduces complementarity among workers in each branch, and the next example introduces externalities between branches.

Example 2. Suppose that workers and branches are the same as above. Now branches need any worker, and worker i is more suitable than j at branch x and vice versa. Suppose further that both workers at suitable branches yields the highest profit, assigning the two workers at the same branch yields more profit than assigning only one worker at the branch, and the profit made by two workers at any branch is larger than each worker at unsuitable branch. Such a complex but possible situation can be described by means of the following interdependent choice function. The existence of stable matching is still guaranteed for all workers' preference profiles.

$$C\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\ C\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix},\ C\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix},\ C\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ C\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$C\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\1&1\end{pmatrix},\ C\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix},\ C\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ C\begin{pmatrix}0&1\\1&1\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix},\ C\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

We are also able to treat a situation that each firm is endowed with a choice function but the aggregate choice function generated from them is interdependent. Suppose that each firm potentially has several choice functions, which may be assumed to be substitutable if we needed. It may be a case that firm's choice is affected by a social state. That is, a firm demands more selective workers when the firm is relatively popular (many applications for the firm and a few for the other firms), and just cares the number of workers when it is relatively less popular (a few applications for the firm and many for the other firms). Depending on social states (application sets, in this case), the firm may use a different choice function. Then the aggregate choice function will be interdependent. We discuss, for more detail, this kind of firms' choices in Discussion section.

Finally note that an aggregate choice function selects not necessarily a matching. As in the second example, when i applies both x and y, both accept i (and then i chooses the better branch). Our formulation allows an aggregate choice function to choose a subset of an application set, as does the aggregate choice function generated from independent choice functions.

2 Model

Let I and X denote the finite sets of workers and firms, respectively. Let $A = (a_{ix})$ be a generic $|I|$-by-$|X|$ zero-one matrix. If situations are obvious, we sometimes call A a set of applications or offers. We write $\mathbf{1}$ for the matrix of all ones. Each worker is matched with at most one firm, and has a linear (complete, transitive and antisymmetric) ordinal preferences over the set of firms and being unmatched. We describe the preferences of worker i as an $|X|$-dimensional vector, $p_i = (p_{ix})_{x\in X}$, where $p_{ix} > p_{iy}$ means that worker i prefers firm x to y. By assumption, either $p_{ix} > p_{iy}$ or $p_{iy} > p_{ix}$ for any two distinct firms $x, y \in X$. We normalize the value of the outside option (being unmatched) to 0. We say that firm x is **acceptable** for worker i if $p_{ix} > 0$. Let $P = (p_i)_{i\in I}$ denote the preference profile of workers. Given a preference profile P and a matrix A, the induced choice function of workers, $C_I : \{0,1\}^{I\times X} \to \{0,1\}^{I\times X}$, chooses the most preferable acceptable firm from A according to P, that is,

$$[C_I(A)]_{ix} = \begin{cases} 1 & \text{if } p_{ix} > 0 \text{ and } x \in \arg\max_{y\in X}\{p_{iy}a_{iy} | a_{iy} = 1\} \\ 0 & \text{otherwise} \end{cases}.$$

Firms' preferences are allowed to be interdependent and can be described by an aggregate choice function, $C_X : \{0,1\}^{I\times X} \to \{0,1\}^{I\times X}$, satisfying

$$C_X(A) \leq A.$$[2]

For convenience, we define two rejection functions R_I and R_X corresponding to C_I and C_X, respectively; For any A,

$$R_I(A) = A - C_I(A) \text{ and } R_X(A) = A - C_X(A).$$

A matching is a $|I|$-by-$|X|$ zero-one matrix, $M = (\mu_{ix})$, such that each row contains at most one 1 elements. $\mu_{ix} = 1$ means that worker i is assigned to firm x at matching M, and $\mu_{ix} = 0$ otherwise.

A matching M is said to be **individually rational** if it holds that $C_I(M) = C_X(M) = M$. Let $\mathcal{M}$ be the function from the set of application sets to the power set of the set of matchings defined by letting

$$\mathcal{M}(A) = \left\{ M \leq A \,\middle|\, \sum_x \mu_{ix} = \min\left\{1, \sum_x a_{ix}\right\}, \forall i \in I \right\}.$$

In words, $\mathcal{M}(A)$ is the set of all maximal matchings that can be generated from matrix A. Note that if A is itself a matching, $\mathcal{M}(A)$ is a singleton set. Let $\mathcal{I}(A)$ be the set of workers applying to some firm under matrix A, formally,

$$\mathcal{I}(A) = \{i \in I \mid \sum_x a_{ix} \geq 1\}.$$

A **blocking deviation** at a matching M is a non-zero matrix $A \leq \mathbf{1} - M$ such that there is a matching $M' \in \mathcal{M}(C_X(M + A))$ such that

1. For every $i \in \mathcal{I}(A)$,

$$[C_I(M \vee M')]_i = \mu'_i.$$[3]

[2] For simplicity, we sometimes omit X from C_X when it is obvious, especially when writing down all of choices in examples.

[3] The notation $\vee$ means "join," that is, $A \vee A' = (\max\{a_{ix}, a'_{ix}\})$. Later, we also use the notation $\wedge$ for "meet," that is, $A \wedge A' = (\min\{a_{ix}, a'_{ix}\})$.

2. For at least one worker $j \in \mathcal{I}(A)$,

$$\mu_j \neq \mu'_j.$$

A matching is (set-wise) **stable** if it is individually rational and there is no blocking deviation. Note that our stability still allows a situation that some worker who deviates from a matching applies to new firms and resigns the current matched firm. In the standard model, since each firm's choice function is defined only over applicants who coming to it, whether resigning the current matched firm or not does not matter. Therefore, our set-wise stability reduces to a usual notion of stability in the standard model. In our setting, it may matter, however, the notion of stability which blocks such deviations lowers significantly the possibilities of the existence of such stable matchings, which cannot let an aggregate choice function describe complementarities, externalities nor state-dependent choices, subject to the existence of stable matchings. [4]

A matching M is preferred to $M' \neq M$ by all workers if $C_I(M \vee M') = M$. A matching M is strictly (revealed) preferred to $M' \neq M$ by all firms if $M \in \mathcal{M}(C_X(M \vee M'))$ and $M' \notin \mathcal{M}(C_X(M \vee M'))$. Stable matching M is **worker optimal stable** if there is no other stable matching that is preferred to M by all workers. Similarly, a stable matching M is **firm optimal stable** if there is no other stable matching that is (revealed) preferred to M by all firms. Finally, a matching M is **efficient** if there is no other matching M' such that $C_I(M \vee M') = M'$ and $M' \in \mathcal{M}(C_X(M \vee M'))$.

2.1 Sufficient condition

As we noted, it is already known that even in models with independent choices, a stable matching does not necessarily exist for some workers' preference profile. Our starting point is the following observation.

Observation 1. There does not necessarily exist a stable matching for some workers' preference profile.

Therefore, we impose the following two conditions on an aggregate choice function.

2.1.1 Independence of rejected additional applications

The first condition requires that, given two sets of applications one of which contains the other, rejected applications for the large set be never accepted for the smaller set whenever the applications chosen from the larger set are all also chosen from the smaller set. It should be noted that

[4]In a special case of our setting, the notion of stability including workers' resigns reduces to strong stability in Kamada and Kojima (2017). They show that the existence of strong stable matchings for all workers' preferences requires that all choice functions be independent and be free from constraints. We observe that such independency is not necessary for the existence of stable matchings in our setting, but still it radically lowers flexibility for interdependent choices.

there is a room that the chosen applications in the smaller set may be different from one in the larger set when it includes an application which is not in the chosen application in the larger set.

Definition 1. An aggregate choice function C_X exhibits **independence of rejected additional applications** (IRAA) if for any A and A' with $A \leq A'$

$$C_X(A) \geq C_X(A') \Rightarrow C_X(A) = C_X(A').$$

Consistency in decision theory (Aizerman and Malishevski (1981); Moulin (1985)) is a very close notion to IRAA. Indeed, consistency implies IRAA, but the converse is not true.[5].

2.1.2 Interdependent substitutability

The second condition requires that it follow *locally* that, given two application sets one of which contains the other, the chosen applications in the larger set must be included in the smaller set whenever they be available in the smaller set. It should be noted that we only require such relations locally but not universally, by contrast to the previous literature.

Definition 2. An aggregate choice function C_X exhibits **interdependent substitutability** (IS) if for any A and $M \in \mathcal{M}(C_X(A))$,

$$M \leq A' \leq A \Rightarrow R_X(A') \leq R_X(A).$$[6]

Firms' choice functions are usually assumed to be independent, and substitutability plays a critical role for the existence of stable matchings in the literature. To see how interdependent substitutability is weaker than substitutability, we introduce a formal definition of substitutability. Let C_x be the independent choice function of firm x such that $C_x : \{0,1\}^I \to \{0,1\}^I$ and for any $a_x \in \{0,1\}^I$, $C_x(a_x) \leq a_x$, and R_x be the corresponding rejection function for C_x. Then C_x is substitutable if for any $a'_x \leq a_x$,

$$R_x(a'_x) \leq R_x(a_x).$$

Hence if all firms' independent choice functions satisfy substitutability, then the aggregate rejection function satisfies that for any $A' \leq A$,

$$R_X(A') \leq R_X(A).$$

[5]Formally, choice function C is consistent if $C(A') \leq A \leq A'$ implies $C(A) = C(A')$. This is also called irrelevance of rejected contracts in matching with contracts (Aygün and Sönmez (2013)). See also Blair (1988) or Alkan (2002).

[6]We can instead write the IS condition as follows: for any A and $M \in \mathcal{M}(C_X(A))$,

$$M \leq A' \leq A \Rightarrow C_X(A) \wedge A' \leq C_X(A').$$

See Appendix for more detail.

It is obvious that, in terms of an aggregate choice, interdependent substitutability is weaker than substitutability.

3 Results

The main contribution of the present paper is to identify sufficient conditions for the existence of stable matchings for all workers' preference profiles. The two conditions IRAA and IS are essential. In addition, we look into the set of stable matchings, and identify which properties are taken over and which are not from a matching model with independent choice functions.

3.1 Existence

When firms' choices are independent, there are mainly two ways to show the existence of stable matchings. One is algorithmic, and the other is based on a fixed point argument. The algorithmic method uses the Gale and Shapley's deferred acceptance algorithm (Gale and Shapley (1962)). The fixed point method often uses Tarski's fixed point theorem (Tarski (1955), Adachi (2000), Hatfield and Milgrom (2005)).[7] Here we generalize Gale and Shapley's firm-proposing deferred acceptance algorithm as follows:

Generalized firm-proposing deferred acceptance algorithm

STEP 0: Set A^0.

A^0 is a $|I|$-by-$|X|$ zero-one matrix where ix-element is 1 if $p_{ix} > 0$ and 0 otherwise.

STEP 1: $A^1 = A^0 - R_I(C_X(A^0))$.

$\vdots$

STEP t: $A^t = A^{t-1} - R_I(C_X(A^{t-1}))$.

$\vdots$

Termination: $A^T = A^{T-1}$.

The algorithm terminates once no worker rejects a firm. $\star$

When the algorithm terminates, we obtain a matching M such that $M = C_I(C_X(A^{T-1})) = C_X(A^{T-1})$. By construction, $A^t < A^{t-1}$ for all $t \leq T-1$. Thus the algorithm ends in finitely many steps.

We now state our main result.

[7] As we discuss below, when firms' choices are interdependent, the set of stable matchings often has a non-lattice structure. Thus Tarski's fixed point theorem is not immediately applicable in our setteing.

Theorem 1. *If an aggregate choice function C_X exhibits IRAA and IS, then there exists a stable matching for all workers' preference profiles.*

Our proof depends on interdependent substitutability. For a given matching to be stable, it is sufficient to show that there is no profitable deviation from that matching. In the previous literature, substitutability of all firms' independent choice functions ensures it by monotonicity of the aggregate rejection function. It however seems too much since for a given matching, potential deviations from the matching are limited. Hence the aggregate rejection function need not be monotonic universally. Instead, partial monotonicity, which interdependent substitutability guarantees, is enough.

It is usually the case that the firm-proposing deferred acceptance algorithm returns a firm optimal stable matching. However, it is not the case for our generalized firm-proposing deferred acceptance algorithm.

Observation 2. The matching obtained by the algorithm is not necessarily a firm optimal stable matching.

As is noted, it is also possible to generalize the worker-proposing deferred acceptance algorithm. However the generalized algorithm is too complex. Since even rejected workers from some application set may be accepted from a different application set, this structure of the aggregate choice function makes things difficult. Suppose that at the first step the workers' application set is such that only the most preferable firm of each worker is listed. If this application set itself is chosen, then we are done. But if at least one worker is not chosen, then we need to consider a large variety of alternative application set. In general, we need to consider all combinations of application sets. Hence we face easily intractable computations. Contrary to the original deferred acceptance algorithms, the computational complexity of the worker-proposing and firm-proposing deferred acceptance algorithms are asymmetric.

3.2 Welfare

This subsection compares our model to one in the previous models in light of properties of stable matchings. Stability is closely related to welfare. In a model with independent substitutable choice functions, the set of stable matchings coincides with the core. Thus a stable matching is efficient. By contrast, it is not easy to define the core in our setting and in fact our stability notion allows a possible deviation that some worker resigns a match partner in the current matching and proposes new applications. However, the following proposition tells that any stable matching is efficient.

Proposition 1. *Any stable matching is efficient.*

Secondly, we consider the extremes of the set of stable matchings, namely, optimal stable matchings for the one side. When firms' choice functions are independent and they satisfy substitutability, there exists a unique worker or firm optimal stable matching. But in our setting, we have the following.

Observation 3. There are generally multiple worker or firm optimal stable matchings.

Multiple worker optimal stable matchings have been observed when firms' independent choice functions allow indifferences (thus a choice function is a correspondence) but in our setting an aggregate choice function is still single-valued.[8] Furthermore, we observe that there may even be multiple firm optimal stable matchings. Nevertheless, we still have trade-offs between the workers' side and the firms' side among a subset of stable matchings.

Proposition 2. *Suppose that M and M' are stable matchings. If workers prefer M to M', then firms (revealed) prefer M' to M.*

4 Discussion

In this section, for clear understanding, we discuss, as a special case of our model, a model in which each firm has own choice function. It is usually a case that each firm makes decision independently. It is also a case that its choice depends not only on applicants who are coming to it but also entire applications. For example, some firm is more selective if workers apply only to the firm, and it needs to fill its seats even with unqualified workers if workers apply to multiple firms. For another example, firms are aggressive to employ workers under booms and they hesitate to employ workers under recessions. In any case, the aim of this section proposes possibilities of rich classes of modelings even when firms are independent decision makers. It should be emphasized that choice functions below do not depend on others' choice.

4.1 State-dependent choices

We say that the status of entire applications *state*. Let there be a set of N number of signals in the market, $S = \{s_1, s_2, \cdots, s_N\}$, which is a partition of the set of all $|I|$-by-$|X|$ zero-one matrices. Given a state A occurs, each firm observes a signal $s(A) \in S$, and then chooses a subset of its applicants. Suppose that if $A \in s$ then $s(A) = s$. Each firm x has a state-dependent choice function, $C_x : \{0,1\}^I \times S \to \{0,1\}^I$, such that for any A,

$$C_x(a_x|s(A)) \leq a_x.$$

It is obvious that when there is only one state, this model reduces to the standard matching model.

Example 3. There are two workers i and j, and two firms x and y. The set of states is $S = \{s_1, s_2\}$, where $A \in s_1$ if and only if at least one worker does not apply to any firm under A. Suppose that

[8] See Erdil and Ergin (2008) and Abdulkadiroğlu et al. (2009) for many-to-one matching problems with indifferences, and Erdil and Kumano (2019) for explicit choice correspondences.

each firm employs no worker when observes s_1, and it employs any workers when observes s_2. Then each state-dependent choice function does not exhibit substitutability since an application in s_1 is rejected though all workers are accepted by all firms under **1**. Nevertheless the aggregate choice function exhibits IS (and IRAA), so there exists a stable matching for all workers' preference profiles.

4.2 Constraints

We next consider a matching model with constraints. The important point is that constraints here are described not necessarily by a specific restriction such as upper or lower bounds of quotas as in Kamada and Kojima (2015) but by an abstract function over application sets. On one hand, as is the standard setting, each firm x has a linear order over the set of subsets of workers. On the other hand, each firm x is endowed with a function f_x which associates an entire application set to a set of subsets of applicants to firm x. Given an application set A, a firm x faces the set of subsets of its applicants, $f_x(A)$, and then chooses the best group of applicants among $f_x(A)$.

Example 4. Let there be two workers i and j, and two firms x and y. Each firm has a linear order:

$$\{i,j\} \succ_x \{i\} \succ_x \{j\} \succ_x \emptyset \text{ and } \{i,j\} \succ_y \{j\} \succ_y \{i\} \succ_y \emptyset \;.$$

Firms have the following functions f_x and f_y, respectively.

$$f_x(A) = \begin{cases} \{\{i\}\} & \text{if } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \{\{j\}\} & \text{if } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ \{\emptyset\} & \text{if } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \{\{i\},\{j\},\emptyset\} & \text{if } A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ 2^I & \text{if } A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \end{cases}$$

$$f_y(A) = \begin{cases} \{\{i\}\} & \text{if } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ \{\{j\}\} & \text{if } A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \{\emptyset\} & \text{if } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \{\{i\}, \{j\}, \emptyset\} & \text{if } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ 2^I & \text{if } A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{cases}$$

In the above setting, a profile of firms' choices forms the aggregate choice function in the second example.

Note that this formulation straightforwardly implies that already existing matching models with constraints such as Kamada and Kojima (2015) are special cases.

5 Concluding remarks

We conclude the present paper by referring necessity for the existence of stable matchings and the relation to the previous literature.

5.1 Necessity

Neither of the two conditions IRAA and IS is necessary for the existence of stable matchings. But the aggregate choice functions which guarantees the existence of stable matchings for all workers' preference profiles and violates either IRAA or IS seem to be less important. The case where IRAA violates happens when we do not have enough freedom of choices. That is, since we directly treat an aggregate choice function, if its range is very narrow like lots of zeros then violation of IRAA does not matter, see the observation in Appendix B for an example. The case where IS violates happens when workers can have cyclic chances to match. That is, when a worker by herself change what firms offer by her applying behavior, even though such aggregate choice function violates IS there sometimes exists a stable matching for all workers' preference profiles, see the observation in Appendix B for an example. Moreover, we imagine that such a condition looks tedious. Therefore, we propose rather simple and interpretational conditions.

It is worth noting that IRAA and IS imply *path independence* in our setting.[9] The converse does not hold. Thus path independence is a candidate as a necessary and sufficient condition.

[9] A proof is in Appendix A, Lemma 2. Moulin (1985) show that consistency and substitutability are equivalent to path independence in a model with independent choice functions.

However, there does not exist a stable matching for some preference profile when an aggregate choice function only satisfies path independence. We leave the full characterization of situations guaranteeing the existence of stable matchings for future research.

5.2 Related literature

The literature has considered flexibilities of firms' choices in two directions, firm's flexible choice from workers like complementarities, and flexible choices among firms like externalities. Though our model is further possible to describe the mixture of those flexibilities by an aggregate choice function, it is worth noting how a model with independent choice functions treats those flexibilities.

For each firm's flexible choice from workers, the notion of contracts to a many-to-one matching model is introduced. This direction is initiated by Kelso and Crawford (1982) and Hatfield and Milgrom (2005). In the environment, there is a weaker sufficient condition for the existence of stable matchings than substitutability in a sense that the notion of substitutability is extended to a model with contracts (Hatfield and Kojima (2010); Hatfield et al. (2017))[10].

For flexible choices among firms, Sasaki and Toda (1996) allow each firm to have its preferences not on the set of workers as usual but on the set of matchings to represent a sort of externality. Pycia and Yenmez (2019) analyzes a two-sided matching market with externalities. They allow agents to have preferences over entire matchings. In their setting, each agent is endowed with a choice function that depends not only on an application set to the agent, but also a reference set that can be interpreted as the agent's expectation for the resulting matching of the other agents. Given such an expectation, each agent chooses the best partners from the applicants according to the preferences. Since choices of our setting are not conditional on a reference set, it does not include their class of choices even without transfers. On the other hand, their choices do not take into account complementarity as we discussed in the first example, and thus our choices are also not fully described by them. Focusing on specific externalities, Echenique and Yenmez (2007); Pycia (2012) considers peer effects in school choice, Dutta and Massó (1997)); Kojima et al. (2013); Ashlagi et al. (2014) consider a matching market with couples.

A matching model with constraints (Kamada and Kojima (2015); Kamada and Kojima (2017); Kamada and Kojima (2018)) are also a kind of models for flexible choices among firms. They exogenously restrict the outcome space, for example two firms are able to accept more than five workers each but it is only feasible by an exogenous restriction such as policies when workers accepted at two firms are seven in total. In such a case, even though each firm has an independent choice function, the aggregate choice function works as if it is interdependent. Indeed our aggregate choice function subsumes those models and, more importantly, ours is more flexible.

[10]Those weaker conditions reduces to substitutability when the model with contracts reduces to a usual many-to-one matching model.

References

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2009): "Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 99, 1954–78.

Adachi, H. (2000): "On a characterization of stable matchings," *Economics Letters*, 68, 43 – 49.

Aizerman, M. and A. Malishevski (1981): "General theory of best variants choice: Some aspects," *IEEE Transactions on Automatic Control*, 26, 1030–1040.

Alkan, A. (2002): "A Class of Multipartner Matching Markets with a Strong Lattice Structure," *Economic Theory*, 19, 737–746.

Ashlagi, I., M. Braverman, and A. Hassidim (2014): "Stability in Large Matching Markets with Complementarities," *Operations Research*, 62, 713–732.

Aygün, O. and T. Sönmez (2013): "Matching with Contracts: Comment," *American Economic Review*, 103, 2050–51.

Blair, C. (1988): "The Lattice Structure of the Set of Stable Matchings with Multiple Partners," *Mathematics of Operations Research*, 13, 619–628.

Dutta, B. and J. Massó (1997): "Stability of Matchings When Individuals Have Preferences over Colleagues," *Journal of Economic Theory*, 75, 464 – 475.

Echenique, F. and M. B. Yenmez (2007): "A solution to matching with preferences over colleagues," *Games and Economic Behavior*, 59, 46 – 71.

Erdil, A. and H. Ergin (2008): "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice," *American Economic Review*, 98, 669–89.

Erdil, A. and T. Kumano (2019): "Efficiency and stability under substitutable priorities with ties," *Journal of Economic Theory*, 184, 104950.

Gale, D. and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 69, 9–15.

Hatfield, J. W. and F. Kojima (2010): "Substitutes and stability for matching with contracts," *Journal of Economic Theory*, 145, 1704 – 1723.

Hatfield, J. W., S. D. Kominers, and A. Westkamp (2017): "Stable and Strategy-Proof Matching with Flexible Allotments," *American Economic Review*, 107, 214–19.

Hatfield, J. W. and P. R. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

Kamada, Y. and F. Kojima (2015): "Efficient Matching under Distributional Constraints: Theory and Applications," *American Economic Review*, 105, 67–99.

——— (2017): "Stability concepts in matching under distributional constraints," *Journal of Economic Theory*, 168, 107–142.

——— (2018): "Stability and Strategy-Proofness for Matching with Constraints: A Necessary and Sufficient Condition ," *Theoretical Economics*, 13, 761–794.

Kelso, A. S. and V. P. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.

Kojima, F., P. A. Pathak, and A. E. Roth (2013): "Matching with Couples: Stability and Incentives in Large Markets*," *The Quarterly Journal of Economics*, 128, 1585–1632.

Moulin, H. (1985): "Choice Functions Over a Finite Set: A Summary," *Social Choice and Welfare*, 2, 147–160.

Pycia, M. (2012): "Stability and Preference Alignment in Matching and Coalition Formation," *Econometrica*, 80, 323–362.

Pycia, M. and M. B. Yenmez (2019): "Matching with Externalities," *working paper*.

Roth, A. E. (1985): "The college admissions problem is not equivalent to the marriage problem," *Journal of Economic Theory*, 36, 277 – 288.

Sasaki, H. and M. Toda (1996): "Two-Sided Matching Problems with Externalities," *Journal of Economic Theory*, 70, 93 – 108.

Tarski, A. (1955): "A lattice-theoretical fixpoint theorem and its applications." *Pacific J. Math.*, 5, 285–309.

A Proofs

Properties of interdependent substitutability

Claim 1. IS is equivalent to the condition that for any A and $M \in \mathcal{M}(C_X(A))$,

$$M \leq A' \leq A \Rightarrow C_X(A) \wedge A' \leq C_X(A').$$

Proof. Suppose that A and $M \in \mathcal{M}(C_X(A))$ are arbitrary. Let A' satisfy $M \leq A' \leq A$. Suppose that C_X exhibits IS. Then $R_X(A') \leq R_X(A)$. Since $C_X(A) \wedge A' \leq C_X(A)$,

$$A' - C_X(A') \leq A - C_X(A) \leq A - C_X(A) \wedge A'.$$

Further, since $C_X(A) \wedge A' \leq A'$, we have

$$A' - C_X(A') \leq A' - C_X(A) \wedge A'.$$

Hence $C_X(A) \wedge A' \leq C_X(A')$.

Suppose in contrast that $C_X(A) \wedge A' \leq C_X(A')$. Since $A' \leq A$ and $C_X(A) \leq A$,

$$\begin{aligned} A - C_X(A) &= A' - C_X(A) \wedge A' + [(A - A') - C_X(A) \wedge (A - A')] \\ &\geq A' - C_X(A) \wedge A'. \end{aligned}$$

Note that $C_X(A) \wedge A' \leq A'$. By assumption,

$$A' - C_X(A) \wedge A' \geq A' - C_X(A').$$

Combining the above two inequalities, we have $A - C_X(A) \geq A' - C_X(A')$. □

Then we see that if for any A and $M \in \mathcal{M}(C_X(A))$, then $C_X(M) = M$.

Proof of Theorem 1

We begin with a lemma.

Lemma 1. *Each worker finds her assignment at each step is at least as desirable as his assignment at the previous step.*

Proof. By construction, at each step t,

$$C_I(C_X(A^{t-1})) \leq A^t = A^{t-1} - R_I(C_X(A^{t-1})) \leq A^{t-1}.$$

Note that since $A^{t-1} \leq A^0$ and A^0 is an acceptable application matrix, $C_I(C_X(A^{t-1})) \in \mathcal{M}(C_X(A^{t-1}))$. (IS) implies that $C_I(C_X(A^{t-1})) \leq C_X(A^t)$. □

Let M be the matching obtained by the algorithm above. Note that a worker is never made an offer by a firm that she finds unacceptable. Then $C_X(A^{T-1}) = M$ implies that $C_X(M) = M$. Hence, M is individually rational. It remains to show that there is no blocking deviation at M. Suppose by contradiction that A is a blocking deviation and each $M' \in \mathcal{M}(C_X(M + A))$ be a matching such that

1. For every $i \in \mathcal{I}(A)$,
$$[C_I(M \vee M')]_i = \mu'_i.$$
2. For at least one worker $j \in \mathcal{I}(A)$,
$$\mu_j \neq \mu'_j.$$

We show first that $M' \leq A^{T-1}$. Note that $A^{T-1} = A^0 - \sum_{t=1}^{T} R_I(C_X(A^{t-1}))$. Let a worker $i \in \mathcal{I}(A)$. Since the inequality is satisfied for any worker who is unmatched at M', we assume that worker i is matched with some firm x at M'. Since worker i is at least as well off at M' as at M, firm x is acceptable to worker i. Thus $\mu'_i \leq a_i^0$. Suppose in the algorithm that worker i rejects firm x at step s, that is, $[R_I(C_X(A^{s-1}))]_{ix} = 1$. Then there must be another firm y such that $[C_I(C_X(A^{s-1}))]_{iy} = 1$. By Lemma 1, worker i is matched with a more preferable firm than x at M, a contradiction. Hence $\mu'_i \leq a_i^{T-1}$. Whereas a worker $i' \notin \mathcal{I}(A)$ applies at $M + A$ only to the matched partner at M. This implies $\mu'_{i'} \leq \mu_{i'}$. Since $M = C_X(A^{T-1}) \leq A^{T-1}$, we have that $\mu'_{i'} \leq a_{i'}^{T-1}$.

Now we look at the choice of firms from $M \vee M'$. Since $M \leq A^{T-1}$ and $M' \leq A^{T-1}$, $M \vee M' \leq A^{T-1}$. By IS,
$$M \leq C_X(M \vee M').$$
Moreover, since $M \vee M' \leq M + A$,
$$M' \leq C_X(M \vee M').$$
Therefore, $C_X(M \vee M') = M \vee M'$. But this implies that
$$C_X(M \vee M') = M \vee M' > M = C_X(A^{T-1}),$$
a violation of IRAA. □

Proof of Proposition 1

Let M be a stable matching. Suppose by contradiction that there exists $M' \neq M$ such that $C_I(M \vee M') = M'$ and $M' \in \mathcal{M}(C_X(M \vee M'))$. Let $A = (M \vee M') - M \geq 0$. Then $C_X(M + A) = C_X(M \vee M')$. Note that for all $i \in \mathcal{I}(A)$, $\mu_i \neq \mu'_i$. Therefore A is a deviation at M, a contradiction. □

Proof of Proposition 2

We begin with a lemma.

Lemma 2. *If an aggregate choice function C_X exhibits IRAA and IS, then it is path independent, that is, $C_X(C_X(A \vee A') \vee A) = C_X(A \vee A')$ for any A and A'.*

Proof. Let A and A' be any matrices. Since $C_X(A \vee A') \leq A \vee A'$, we have

$$C_X(A \vee A') \leq C_X(A \vee A') \vee A \leq A \vee A'.$$

By IS, $C_X(A \vee A') \leq C_X(C_X(A \vee A') \vee A)$. Then by IRAA this inequality holds as an equality. □

Let M and $M' \neq M$ be stable matchings, and $C_I(M \vee M') = M$. Suppose by contradiction that M' is not preferred to M by all firms. Then either

$$M \in \mathcal{M}(C_X(M \vee M'))$$

or

$$M \notin \mathcal{M}(C_X(M \vee M')) \text{ and } M' \notin \mathcal{M}(C_X(M \vee M')).$$

The former implies that M' is not efficient, contradicts to Proposition 1. We assume the latter. Let $A' = (C_X(M \vee M') \vee M') - M' \geq 0$. By Lemma 2,

$$\begin{aligned} C_X(M' + A') &= C_X(M' + (C_X(M \vee M') \vee M') - M') \\ &= C_X(C_X(M \vee M') \vee M') \\ &= C_X(M \vee M'). \end{aligned}$$

Notice by construction that if an ix-element of A' is 1, then $[C_X(M \vee M')]_{ix} = 1$. This means that all workers in $\mathcal{I}(A')$ could obtain the same matching at M by additional application A' to M', which contradicts to the supposition that M' is stable. □

B Examples

Observation 2

We illustrate an example where the generalized firm-proposing deferred acceptance algorithm does not necessarily find a firm optimal stable matching. Suppose that there are two workers and two firms. The aggregate choice function C_X is given by

$$C\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, C\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, C\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\1&1\end{pmatrix},\ C\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix},\ C\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix},\ C\begin{pmatrix}0&1\\1&1\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ C\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

We show first that the aggregate choice function C_X exhibits IRAA and IS. Let $A = \begin{pmatrix}0&1\\1&1\end{pmatrix}$. Since firms accept any offers except for A, it suffices to check the choice around A. Since firms choose no matrix larger than $C_X(A) = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ from a matrix less than A, C_X exhibits IRAA. Next, matrix $\mathbf{1}$ is the unique matrix larger than A, but A is not larger than $C_X(\mathbf{1}) = \begin{pmatrix}1&0\\1&1\end{pmatrix}$. Thus C_X exhibits IS.

Let P be the preference profile such that

$$P = \begin{pmatrix}1&2\\1&2\end{pmatrix}.$$

Since each worker prefers to match any firm than being unmatched under C_X, any stable matching should have one 1 in each row. For $\begin{pmatrix}1&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, there is a blocking deviation $\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Thus we have two stable matchings

$$M = \begin{pmatrix}0&1\\0&1\end{pmatrix} \text{ and } M' = \begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The algorithm finds M. On the other hand,

$$C_X(M \vee M') = C_X(A) = \begin{pmatrix}0&1\\1&0\end{pmatrix} = M'.$$

Hence M is not firm optimal stable.

Observation 3

We show that there can be multiple worker or firm optimal stable matchings. For workers' optimality, our first Motivating examples is one for which both firms are acceptable for both workers and their most preferred firms are different. For firms' optimality, we need at least three workers and two firms. For simplicity, we shorten the aggregate choice function C_X as follows:

$$C\begin{pmatrix} 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ 1 & 0 \\ 1 & \cdot \end{pmatrix}, C\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \\ 1 & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, C\begin{pmatrix} 1 & 0 \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \cdot & 1 \end{pmatrix}, C\begin{pmatrix} 0 & 1 \\ \cdot & 1 \\ 1 & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \cdot & 1 \\ 1 & 0 \end{pmatrix}$$

where a dot means the element can be both 0 and 1. And if none of the following matchings are available in an application, firms will not accept any worker from the application:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this example, firms always accept the former two matchings whenever they are available in an application. Each of the latter two matchings can be chosen from an application only when at least one of the former two matchings is not available in the application. By construction, a larger matrix than the one chosen from a larger application is never chosen from a smaller application. Thus C_X exhibits IRAA. Moreover, once a matching above has been chosen from an application, it will not be rejected from a smaller application as long as it is available. This implies that C_X also exhibits IS.

Let P be a preference profile such that

$$P = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Then the former two matchings are blocked by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively. Thus we have two stable matchings

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } M' = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$C_X(M \vee M') = C_X\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Neither M nor M' is chosen from $M \vee M'$, and therefore both the matchings are firm optimal stable.

Path independence

We see in Lemma 2 that IRAA and IS imply path independence. On the other hand, path independence implies IRAA but not IS. The following shows that only path independence is not a sufficient condition.

Observation 4. Path independence is not sufficient for the existence of stable matching.

Let there be two workers and two firms. The aggregate choice function is

$$C\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$C\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\ C\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix},$$

$$C\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ C\begin{pmatrix}0&1\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ C\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\ C\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Note that the the aggregate choice function is path independent. However it does not exhibit IS since worker i's application to firm y is not accepted by itself. It is easy to see that there is no stable matching when $p_{iy} > p_{ix} > 0$.

Observation 5.1

(1) Violating IRAA but there still exists a stable matching for all workers' preference profiles.

$$C\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\ C\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\ C\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\ C\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

(2) Violating IS but there still exists a stable matching for all workers' preference profiles.

$$C\begin{pmatrix}1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\end{pmatrix},\ C\begin{pmatrix}0&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\end{pmatrix},\ C\begin{pmatrix}0&0&1\end{pmatrix}=\begin{pmatrix}0&0&1\end{pmatrix},$$

$$C\begin{pmatrix}1&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\end{pmatrix},\ C\begin{pmatrix}0&1&1\end{pmatrix}=\begin{pmatrix}0&1&0\end{pmatrix},\ C\begin{pmatrix}1&0&1\end{pmatrix}=\begin{pmatrix}0&0&1\end{pmatrix},$$

$$C\begin{pmatrix}0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\end{pmatrix},\ C\begin{pmatrix}1&1&1\end{pmatrix}=\begin{pmatrix}1&1&1\end{pmatrix}.$$