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"A Dynamic Mechanism Design for Scheduling with Different Use Lengths"

Ryuji Sano

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A Dynamic Mechanism Design for Scheduling with Different Use Lengths*

Ryuji Sano†
Institute of Economic Research, Kyoto University

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Abstract

This paper considers a dynamic allocation problem in which many perishable goods are allocated at each period. Agents want to keep winning goods for more than one period to make profits. We consider efficient and optimal mechanisms when the seller offers simple long-term contracts. The dynamic VCG mechanism achieves efficient allocations. The expected revenue is maximized by the virtual welfare maximization. In the single unit case, price discounts for long-stay agents can be optimal under certain distributions. Both the efficient and optimal mechanisms are implemented by a simple "handicap auction" in the binary length case.

Keywords: dynamic mechanism design, online mechanism, optimal auction

JEL classification: C73, D44, D82

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†Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan. Telephone: +81-75-7537122. E-mail: sano@kier.kyoto-u.ac.jp

1 Introduction

This paper considers a dynamic allocation problem where a seller allocates many perishable objects at each period. A motivating example is an on-line scheduling problem. Suppose that there are a number of facilities such as city halls, hotel rooms, or shared computer servers, and time slots to use such facilities are allocated at each time. Potential users randomly arrive over time and often want to use the facilities for a long time. For example, a conference or an exhibition would be held at a hotel or a convention center for several days or a week. People would like to stay at a hotel for several nights, or a computer job would need a long processing time to complete on a server. A similar environment would also be considered in electricity markets, in which an electricity supplier or a power plant signs a big contract with firms, factories, and retailers over time. People need the object for different periods to be satisfied, and the necessary duration is in general private information of each person along with the valuation.

The seller faces tradeoff between revenue from current long-stay buyers and that from potential future buyers. The seller often needs to sign long-term contracts for buyers and reserve future slots for them in advance. Although a long-term contract might be contingent on future events, such a complex contract is often difficult in practical situations. Only simple incomplete contracts that are not contingent on future events are available in many real situations. In such a case, how can the seller achieve the efficient outcome? And, how can she maximize her expected revenue?

This paper applies a standard mechanism design to this problem. We characterize incentive compatible mechanisms, and provide an efficient mechanism and a revenue-maximizing mechanism in our domain of mechanisms. Given a period length, incentive compatibility requires that the allocation policy for an agent is increasing in his valuation, and that the payment is determined by the envelope theorem. In addition, to ensure reporting true lengths, the allocation policy needs to satisfy a weak notion of monotonicity on lengths. We show that these conditions are also sufficient for incentive compatibility.

Standard techniques provide the efficient mechanism and the revenue-maximizing mechanism. The efficient mechanism is provided as a straightforward extension of the Vickrey-Clarke-Groves (VCG) mechanism. Each agent pays the expected externality to the other current and future agents. The revenue-maximizing (optimal) mechanism is provided as in Myerson (1981) and Riley and Samuelson (1981). The optimal allocation policy maximizes the discounted virtual valuations under a monotone hazard rate condition.

We then consider two specific cases. First, we consider a single unit case. With certain distribution conditions, the optimal allocation policy is distorted in such a way that long-stay agents are more favored than in the efficient allocation policy. In addition, the optimal posted prices exhibit volume discounts for long-stay agents, as is sometimes observed in the real world. However, such long-stay discounts are not robust. The existence of long-stay discounts or premium depends on the type distribution and population dynamics.

Second, we consider the case where agents stay at most for two periods. We show that the concavity of the maximum social welfare function in terms of current supply units. As a result, the efficient allocation policy is implemented by simple multi-unit uniform price auctions with proper handicaps to long-stay agents. Analogously, revenue maximization is done by multi-unit uniform price auctions with reserve prices and handicaps.

1.1 Related Literature

In recent years, a number of studies consider dynamic auction design.[1] Dynamic auction design in an environment where agents strategically arrive and depart is often called online mechanism design, and has been investigated in the fields of computer science and operations research (Lavi and Nisan, 2000).

Parkes and Singh (2003), Mierendorff (2009), and Pai and Vohra (2013) consider an allocation problem of durable goods, such as air ticket sales and hotel reservations.

[1] See Bergemann and Said (2011) and Parkes (2007) for a review.

Buyers arrive over time, stay for several periods, and participate in auctions during their stay. They buy a good at most once in their stay. Parkes and Singh (2003) extend the VCG mechanism to the case where buyers strategically choose their arrival and departure times. Mierendorff (2009) and Pai and Vohra (2013) investigate the optimal mechanism.

Hajiaghayi et al. (2005) and Parkes (2007) consider a perishable goods case such as scheduling of facilities in the presence of strategic arrivals and departures. They consider incentive compatible mechanisms and investigate the efficiency of an allocation policy. In the literature of machine job scheduling, Porter (2004) considers a dynamic mechanism with heterogeneous job lengths. Kittsteiner and Moldovanu (2005) study queueing mechanism using auctions under stochastic arrivals of jobs with heterogeneous waiting cost and processing time.

Dizdar et al. (2011) consider a problem closely related to ours. They consider a durable goods sale (knapsack problem) to a sequence of short-lived buyers who have multiple-unit demands. They characterize the implementability of deterministic allocation rule with two-dimensional type of value and demand volume. Then, they examine revenue maximization and sufficient conditions for the regularity of the problem. For a difference from their characterization result, we allow random allocation rule to some extent whereas they limit attention on deterministic rules and strategy-proofness.

For other models with random arrivals of agents, Vulcano et al. (2002) and Gershkov and Moldovanu (2009, 2010) consider the efficient or optimal durable goods sales in which agents are impatient and short-lived. Board and Skrzypacz (2010) also consider durable goods sales in which agents are patient and stay until the deadline of the sale. Said (2012) considers a perishable goods case. Buyers arrive at random and stay through the next period with a positive probability common among buyers. Said shows that the outcomes in the efficient and optimal mechanisms can be achieved by repeated ascending auctions in a perfect Bayesian equilibrium.

Some studies examine a similar dynamic allocation problem with dynamic information. Bergemann and Valimaki (2010) consider a dynamic allocation problem with fixed agents whose types fluctuate over time. They formulate an efficient mechanism by extending the VCG mechanism. Athey and Segal (2013) consider a similar situation and provide an efficient budget-balancing mechanism. Pavan et al. (2014) characterize incentive compatibility and provide revenue equivalence in a dynamic information model.

The remainder of the paper is organized as follows. In section 2, we provide a model of the time-slot scheduling problem. We characterize incentive compatibility and show a revenue equivalence. In section 3, we consider the efficient mechanism design. We show that truthful reporting is a dominant strategy in the dynamic VCG mechanism. In section 4, we consider the revenue maximization for the seller. We show that the optimal allocation policy maximizes virtual welfare under the monotone hazard rate condition. In section 5, we focus on a single unit case. We show that with a certain condition, the optimal allocation is distorted in such a way that long-stay agents are more favored. In addition, the optimal posted prices exhibit volume discounts for long-stay agents. In section 6, we examine the allocation policy in the case where agents stay at most for two periods. The efficient and optimal outcomes are implemented by a simple handicap auction. Section 7 concludes the paper.

2 Model

We consider an environment with independent and private values in a discrete-time model. K identical objects, for example, time slots of a city hall or facilities, are supplied at each period $t = 1, \ldots, T$. For simplicity, we consider infinite horizon: $T = \infty$. Objects are non-storable and perish at the end of each period. At each period, a finite number of agents enter the mechanism. The set of entrants at t is denoted by N^t, and the number of entrants $|N^t|$ is an i.i.d. random variable at each period. Each agent at each period is ex ante homogeneous and wants to own at most one unit of the object at a period. The set of agents having entered by t is denoted by $\mathcal{N}^t \equiv \bigcup_{s\leq t} N^s$. An allocation at t is denoted by $a^t = (a_i^t)_{i\in\mathcal{N}^t}$. An allocation $a_i^t \in [0,1]$ for i denotes the probability of obtaining the object at t. An allocation a^t is said to be *feasible at* t if $\sum_i a_i^t \leq K$ and $a_i^t = 0$ for any i who is not in the mechanism at t. Let $\bar{a}_i^t \in \{0,1\}$ be the realized allocation for i at t.

Agents and the seller discount future payoffs by a common factor $\delta \in (0,1)$. Each agent i has his private information $\theta_i \equiv (V_i, l_i) \in [0, \bar{V}] \times \{1, \ldots, L\} \equiv \Theta_i$. Agent i of type $\theta_i = (V_i, l_i)$ stays in a mechanism for at least l_i periods. Given a deterministic sequence of allocations and payments, the utility of $i \in N^t$ evaluated at t is given by

$$u_i = \begin{cases} V_i - \sum_{s=t}^{\infty} \delta^{s-t} p_i^s & \text{if } \bar{a}_i^s = 1 \text{ for } \forall s \in \{t, \ldots, t+l_i-1\}, \\ -\sum_{s=t}^{\infty} \delta^{s-t} p_i^s & \text{otherwise,} \end{cases} \tag{1}$$

where p_i^s denotes i's payment at s. Agent i earns a total profit V_i only if he owns the object for l_i periods of time.[2] Otherwise, agent i earns zero. We call V_i, *valuation type* and l_i, *length type*. Each agent's type θ_i is independently drawn from an identical distribution F on Θ_i. Let $F(\cdot|l_i)$ be the cumulative distribution function conditional on l_i. Given any l_i, $F(\cdot|l_i)$ has a density function $f(\cdot|l_i) > 0$ for all V_i.[3]

The seller offers a long-term contract to each agent. Each agent signs only one contract when he enters a mechanism.[4] There is no renegotiation and a contract is never revised or interrupted before its expiration. A contract for i at t, z_i^t, specifies a sequence of allocations and payments.

We restrict the domain of contracts to simple contracts that depend only on the current history. We assume that a *contract* is defined as $z_i^t \equiv (a_i, m_i, P_i)$, where $a_i \in [0,1]$ specifies a (random) goods allocation, m_i denotes a contract term, and $P_i = \sum_{s\geq t} \delta^{s-t} p_i^s$ is total payment. Contract term m_i is assumed to be determinis-

[2] Note that total profit V_i is evaluated at t. An alternative definition of the (expected) utility is

$$u_i = a_i^t \cdot a_i^{t+1} \cdot \cdots \cdot a_i^{t+l_i-1} V_i - \sum_{s\geq t} \delta^{s-t} p_i^s.$$

[3] We may allow upper bound of valuations to depend on length types. Then, the conditional density $f(\cdot|l_i) > 0$ is assumed in the domain of values for each l_i.

[4] If an agent rejects a contract, then he leaves the mechanism and gets payoff 0.

tic. Moreover, the admissible sequences of realized allocations $(\bar{a}_i^t, \dots, \bar{a}_i^{t+m_i-1})$ are limited to $(1, \dots, 1)$ and $(0, \dots, 0)$. An allocation a_i is the probability assigned to the sequence $(1, \dots, 1)$. In other words, agent i obtains the goods with a probability of a_i at his entry period t, and the realized allocation $\bar{a}_i^t$ at t is allocated with probability 1 in the later periods. We take care of the total payment P_i instead of the sequence of payments because there is a degree of freedom. The set of admissible contracts for i is denoted by $Z_i \equiv [0,1] \times \{1, \dots, L\} \times \mathbb{R}$. The set of bundles of contracts at t is denoted by $Z^t \equiv \times_{i \in N^t} Z_i$. For contract $z_i^t = (\tilde{a}_i, \tilde{m}_i, \tilde{P}_i)$, we use the following notations for the corresponding components: $a_i(z_i^t) = \tilde{a}_i$, $m_i(z_i^t) = \tilde{m}_i$, and $P_i(z_i^t) = \tilde{P}_i$. The realized allocation from z_i^t is denoted by $\bar{a}_i(z_i^t)$.

2.1 Dynamic Direct Mechanisms

We assume that each agent does not manipulate the arrival time. However, they may manipulate their period lengths. We also assume that each agent does not observe past history or the number of current agents N^t when making a report. This assumption would be natural in many practical situations such as facility scheduling and hotel room assignments. The assumption can also be interpreted as the case where the seller hides past events so that agents' incentive constraints are the weakest. One might consider the case in which agents observe some information about past events. In such a case, an incentive constraint is necessary for every history. We will see that the results of the paper hold even in such cases.

We apply the revelation principle and limit our attention to direct mechanisms. Each agent reports the type to the seller at his entry. Then, the seller offers a contract to each current entrant just once.

At each t, each entrant $i \in N^t$ makes a report $\gamma_i^t = (\hat{V}_i, \hat{l}_i) \in \Theta_i$. The profile of types at t is denoted by $\theta^t \equiv (\theta_i)_{i \in N^t}$. The seller makes a contract for each $i \in N^t$ based on the vector of reports $\gamma^t \in \Theta^t \equiv \prod_{i \in N^t} \Theta_i$ and history up to t,

$$h_t = (N^1; \gamma^1, z^1, N^2; \dots; \gamma^{t-1}, z^{t-1}, N^t).$$

Let H^t be the set of possible history at t. A mechanism is denoted by $\{z^t\}_{t=1}^{\infty}$, where

$$z^t : \Theta^t \times H^t \to Z^t.$$

A mechanism is *feasible* if $z_i^t = \emptyset$ for all $i \notin N^t$ and if $\sum_{i \in N^t} a_i(z_i^t) \leq K_t$, where K_t denotes the number of units available for the current entrants.[5]

For a profile of reports $\gamma^t = (\gamma_j^t)_{j \in N^t}$ at t, an entrant i at t earns payoff

$$u_i(\gamma^t, \theta_i, h_t) = a_i(z_i^t(\gamma^t, h_t))\mathbb{I}_{\{m_i(z_i^t) \geq l_i\}} V_i - P_i(z_i^t(\gamma^t, h_t)), \tag{2}$$

where $\mathbb{I}$ denotes the indicator function. For the verifiability of contracts, we assume that agents observe the history after the enforcement of their contracts. Let $U_i(\theta^t, h_t) \equiv u_i((\theta_i, \theta_{-i}^t), \theta_i, h_t)$, which indicates i's payoff when every entrant at t reports true information.

A bidder's strategy is a mapping $\gamma_i : \Theta_i \to \Theta_i$. Given that the others report their true types, agent i's interim expected payoff is

$$\begin{aligned} \pi_i(\gamma_i^t, \theta_i) &= \mathbb{E}[u_i(\gamma_i^t, \theta_{-i}^t, \theta_i, h_t)] \\ &= \alpha_i(\gamma_i^t, l_i) V_i - q_i(\gamma_i^t), \end{aligned} \tag{3}$$

where

$$\alpha_i(\gamma_i^t, l_i) = \mathbb{E}\big[a_i(z_i^t(\gamma_i^t, \theta_{-i}^t, h_t))\mathbb{I}_{\{m_i(z_i^t) \geq l_i\}}\big] \tag{4}$$

and

$$q_i(\gamma_i^t) = \mathbb{E}\big[P_i(z_i^t(\gamma_i^t, \theta_{-i}^t, h_t))\big]. \tag{5}$$

Let $\Pi_i(\theta_i) \equiv \pi_i(\theta_i, \theta_i)$, which denotes the expected payoff when i reports his true information. Note that agent i's winning probability α_i depends on i's true length type l_i through the indicator function. Abusing notations, let $\alpha_i(V_i, l_i) \equiv \alpha_i((V_i, l_i), l_i)$, which indicates the winning probability when reporting the true length type.

Incentive compatibility and individual rationality are defined in a standard manner.

[5]See section 3 for a formal definition of K_t.

Definition 1 A dynamic direct mechanism is *(Bayesian) incentive compatible* if for all i, all t, all θ_i, and all γ_i^t,

$$\Pi_i(\theta_i) \geq \pi_i(\gamma_i^t, \theta_i).$$

A mechanism is *individually rational* if for all i, all t, and all θ_i, $\Pi_i(\theta_i) \geq 0$.

Definition 2 A dynamic direct mechanism is *dominant strategy incentive compatible* if for all i, all t, all h_t, all θ, and all γ_i^t,

$$U_i(\theta^t, h_t) \geq u_i((\gamma_i^t, \theta_{-i}^t), \theta_i, h_t).$$

Note that by the ex post availability of history, dominant strategy incentive compatibility is a little stronger than the standard definition in the sense that incentive compatibility is imposed for every history.

2.2 Characterization of Incentive Compatibility

We characterize incentive compatibility and show the revenue equivalence. We limit our attention to mechanisms such that contract terms are equal to the agents' reported lengths.

Definition 3 A dynamic direct mechanism is *exact* if it satisfies $m_i(z_i(\theta^t, h_t)) = l_i$ for all t, all h_t, all $i \in N^t$, and all θ^t.[6]

In an exact mechanism, every contract term is determined by the reported length. This restriction would be natural and does not lose generality. It is easy to verify that for any mechanism, there exists an equivalent exact mechanism that gives the same payoffs and revenue.

Lemma 1 *For any mechanism $\{z^t\}$, there exists an equivalent exact mechanism that gives the same payoffs for agents and revenue for the seller.*

Proof. All proofs are in Appendix.

[6]The term "exactness" follows Lehmann et al. (2002) in the literature of multi-object auctions.

Hereafter, we focus on exact mechanisms. Incentive compatibility is characterized by two separated monotonicity constraints. Similar results are found in Dizdar et al. (2011), Pai and Vohra (2013) and Mierendorff (2009) in different settings.

Proposition 1 *An exact dynamic direct mechanism is incentive compatible if and only if for all* i*,*

1. $\alpha_i(V_i, l_i)$ *is weakly increasing in* V_i *for every* l_i*,*

2. $\int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu$ *is weakly decreasing in* l_i *for every* V_i*, and*

3. *there exists a constant* $\underline{\Pi}_i$*, and*

$$\Pi_i(\theta_i) = \underline{\Pi}_i + \int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu \tag{6}$$

   for all V_i *and all* l_i*.*

Preceding studies introduce a stronger notion of *monotonicity* of allocation policy, which requires that allocation for an agent is monotone in both valuation and length. We introduce a similar concept in our model. Given a mechanism $\{z^t\}$, the *allocation policy* is denoted by $a^t : \Theta^t \times H_t \to [0,1]^{N^t}$ and

$$a_i^t(\theta^t, h_t) \equiv a_i(z_i^t(\theta^t, h_t)).$$

Definition 4 An allocation policy is said to be *monotone* if for all i, $a_i^t(\theta^t, h_t)$ is weakly increasing in V_i and weakly decreasing in l_i.

Note that incentive compatibility does not require the allocation policy to be monotone in length. Proposition 1 implies that any monotone allocation policy is implementable.[7]

Corollary 1 *If an allocation policy is monotone, then there exists a payment scheme that induces incentive compatibility.*

[7]Bikhchandani et al. (2006) show that in a multi-dimensional model, a deterministic allocation rule is implemented in dominant strategy if and only if it is *weakly monotone*.

3 Efficient Mechanism

In this section, we establish an efficient incentive compatible mechanism, that is an extension of the well-known VCG mechanism. To formulate the social optimization problem, we introduce the *residual periods of* z_i^s *at* $t \geq s$, which is denoted by $\chi(t, z_i^s)$ and

$$\chi(t, z_i^s) \equiv \max\{m_i(z_i^s) + s - t, 0\}.$$

In addition, *the residual periods of objects at* t is K-dimensional vector $x^t = (x_k^t)_{k=1}^K$, where x_k^t is the k-th highest order number of $\chi(t, z_i^s)$ of all i and all $s < t$ such that $\bar{a}_i(z_i^s) = 1$. Note that $\#\{k | x_k^t \geq 1\}$ units of the object are occupied at t by some incumbent agents. The supply at t is denoted by $K_t = K - \#\{k | x_k^t \geq 1\}$. Let X be the set of x^t; $X = \{x \in \mathbb{Z}_+^K | \mathbf{0} \leq x \leq (L-1, \ldots, L-1), x_k \leq x_{k-1}\}$.

3.1 Social Welfare

In what follows, we formulate the socially optimal allocation policy. Because we assume i.i.d. populations and type distributions, the state of the world at t is (θ^t, x^t). It is easy to verify that the efficient allocation policy $\{a^{t*}\}_{t=0}^\infty$ is deterministic. The allocative state of the next period, x^{t+1}, is determined by the current allocation a^t, profile of current length types $l^t = (l_i)_{i \in N^t}$, and current state x^t. Let G be the state transition function: $x^{t+1} = G(a^t, l^t, x^t)$. The socially optimal welfare at t, $W(\theta^t, x^t)$, is written in a recursive form as

$$\begin{aligned} W(\theta^t, x^t) = \max_{a^t} & \sum_{i \in N^t} a_i^t V_i + \delta \mathbb{E}\big[W(\theta^{t+1}, x^{t+1})\big] \\ \text{s.t.} \quad & a_i^t \in \{0, 1\}, \\ & \sum_{i \in N^t} a_i^t \leq K_t, \\ & x^{t+1} = G(a^t, l^t, x^t). \end{aligned} \tag{7}$$

The solution of (7) is the efficient allocation policy $a^*(\theta^t, x^t)$. It is easy to verify that the efficient allocation policy $a^*(\theta^t, x^t)$ is monotone.

3.2 The Dynamic VCG Mechanism

The efficient allocation policy is implemented in weakly dominant strategy via the dynamic VCG mechanism. Let $W_{-i}(\theta^t, x^t)$ be the maximized social welfare when agent i is excluded. The efficient allocation without $i \in N^t$ is denoted by $\hat{a}^t_{-i}$. As defined in the static VCG mechanism, we introduce the *marginal contribution* of i, defined as

$$C_i(\theta^t, x^t) \equiv W(\theta^t, x^t) - W_{-i}(\theta^t, x^t).$$

Let $x^{t+1*} \equiv G(a^{t*}, l^t, x^t)$, which indicates the allocative state at $t+1$ given the efficient allocation at t. Similarly, let $\hat{x}^{t+1}_{-i} \equiv G(\hat{a}^t_{-i}, l^t, x^t)$, which denotes the allocative state at $t+1$ given the efficient allocation $\hat{a}^t_{-i}$ when i is excluded at t. Note that agent i is inactive after $t+1$. Hence, we have

$$W_{-i}(\theta^t, x^t) = \sum_{j \in N^t \setminus \{i\}} \hat{a}^t_{-i,j} V_j + \delta \mathbb{E}\big[W(\theta^{t+1}, \hat{x}^{t+1}_{-i})\big].$$

The payment scheme for the dynamic VCG mechanism is defined so that each agent earns his marginal contribution under the current state. Hence, (total) monetary transfer $P^*_i(\theta^t, x^t)$ in the dynamic VCG mechanism is defined as follows:

$$P^*_i(\theta^t, x^t) = \sum_{j \in N^t \setminus \{i\}} (\hat{a}^t_{-i,j} - a^{t*}_j) V_j + \delta(W(\hat{x}^{t+1}_{-i}) - W(x^{t+1*})), \tag{8}$$

where $W(x) \equiv \mathbb{E}[W(\theta^{t+1}, x)]$.

Definition 5 An exact dynamic mechanism $\{z^*\} = \{(a^*, P^*)\}_{t=0}^{\infty}$ is said to be the *dynamic VCG mechanism* if a^* is the efficient allocation policy and the payment is determined by (8).

Theorem 1 *The dynamic VCG mechanism $\{z^*\}$ is dominant strategy incentive compatible and individually rational. The equilibrium payoff of $i \in N^t$ is $U_i(\theta^t, h_t) = C_i(\theta^t, x^t)$.*

Similar mechanisms that extend the VCG mechanism to dynamic environments are provided by Parkes and Singh (2003), Bergemann and Valimaki (2010), and Athey and Segal (2013).

4 Revenue Maximization

Revenue maximization is done by a standard manner by Myerson (1981). We introduce *virtual valuation*, which is denoted by $\phi(\theta_i)$, and

$$\phi(\theta_i) \equiv V_i - \frac{1 - F(V_i|l_i)}{f(V_i|l_i)}. \tag{9}$$

From Proposition 1 and standard calculations, the expected revenue from agent $i \in N^t$, $\mathbb{E}[q_i(\theta_i)]$, is given by

$$\mathbb{E}[q_i(\theta_i)] = -\underline{\Pi}_i + \int_{\theta_i} \alpha_i(\theta_i)\phi(\theta_i)f(\theta_i)\mathrm{d}\theta_i.$$

The revenue maximization problem is written as

$$\begin{aligned}
\max_{\{a^s\}_{s=t}^{\infty}} \ & \mathbb{E}\Big[\sum_{s=t}^{\infty}\delta^{s-t}\big[-\sum_{i\in N^s}\underline{\Pi}_i + \sum_{i\in N^s} a_i^s\phi(\theta_i)\big]\Big] \\
\text{s.t. } & \alpha_i(\theta_i) \text{ is weakly increasing in } V_i, \\
& \int_0^V \alpha_i(\nu, l_i)\mathrm{d}\nu \text{ is weakly decreasing in } l_i, \\
& \underline{\Pi}_i \geq 0, \\
& a_i^s \in [0,1], \\
& \sum_{i\in N^s} a_i^s \leq K_s.
\end{aligned} \tag{10}$$

Obviously, $\underline{\Pi}_i = 0$ for all i at the optimum. We consider a relaxed problem in a recursive form below:

$$\begin{aligned}
R(\theta^t, x^t) = \max_{a^t} & \sum_{i\in N^t} a_i^t\phi(\theta_i) + \delta\mathbb{E}\big[R(\theta^{t+1}, x^{t+1})\big] \\
\text{s.t. } & a_i^t \in [0,1], \\
& \sum_{i\in N^t} a_i^t \leq K_t, \\
& x^{t+1} = G(a^t, l^t, x^t),
\end{aligned} \tag{11}$$

where $R(\theta^t, x^t)$ denotes the optimal virtual welfare function. This problem is the same as the social optimization problem (7), except that the valuations are replaced with virtual valuations. The solution a^{**} of the virtual welfare maximization problem (11) is weakly increasing in ϕ.

We need a regularity condition so that the solution of the relaxed problem (11) also solves the original problem (10). A sufficient condition is that the virtual valuation ϕ is increasing in V_i and weakly decreasing in l_i.

Assumption 1 The conditional hazard rate $\lambda_{l_i}(V_i) \equiv \frac{f(V_i|l_i)}{1-F(V_i|l_i)}$ is weakly increasing in V_i and weakly decreasing in l_i.[8]

In addition to the standard increasing hazard rate condition, Assumption 1 requires the conditional distributions $F(\cdot|l_i)$ are ordered in terms of hazard rate dominance. Although the hazard rate dominance is known to be stronger than the first order stochastic dominance, this would likely be the case in real situations. The revenue maximization problem is regular when the hazard rate condition above is satisfied. Dizdar et al. (2011) examine weaker sufficient conditions for the regularity.

Theorem 2 *Under Assumption 1, the allocation policy a^{**} derived from the relaxed problem (11) is monotone and maximizes the expected revenue for the seller.*

The optimal allocation policy a^{**} is in general very different from the efficient allocation policy a^* even when the type distribution is identical. In the standard static optimal auction design, the optimal allocation policy is efficient in the sense that the agent awarded the object has the highest value. However, in our model, the virtual value ϕ depends on both valuation and length, and it generates asymmetry between the different length types. In addition, the social welfare and the optimal virtual welfare functions are different, so that the optimal policies a^* and a^{**} are different. In addition, difference in continuation values gives different indifference curves for the seller, which is discussed in the next section. In the following sections, we consider the efficient and optimal mechanisms in some special cases.

[8] Pai and Vohra (2013) impose a similar hazard rate condition. For the monotonicity on V_i, it is sufficient to assume that $\phi(\theta_i)$ is increasing in V_i.

5 Single Unit Case

5.1 Efficient Mechanism

We consider the case of $K = 1$. An allocative state x^t indicates the remaining periods of the currently enforced contract, and $0 \leq x^t \leq L - 1$. For any t such that $x^t \geq 1$, the object is allocated to an incumbent agent and no entrant at t gets the object: $a_i^* = 0$ for all $i \in N^t$. Thus, for $x^t = l \geq 1$,

$$W(\theta^t, l) = \delta^l \mathbb{E}\big[W(\theta^{t+l}, 0)\big] \equiv \delta^l \bar{W}.$$

An allocation problem is considered only when $x^t = 0$. The social optimization problem for $x^t = 0$ is described as

$$W(\theta^t, 0) = \max_{i \in N^t \cup \{0\}} V_i + \delta^{l_i} \bar{W}, \tag{12}$$

where agent 0 is a dummy agent (or the seller), whose type is $\theta_0 = (0, 1)$. The agent who maximizes the value $V_i + \delta^{l_i} \bar{W}$ wins the object, or the seller keeps the object and waits for the next period.

Suppose that agent i wins an object at t and that $j \in N^t \cup \{0\}$ is the second-highest agent. Then i's payment in the dynamic VCG mechanism is

$$P_i^*(\theta^t) = V_j + (\delta^{l_j} - \delta^{l_i}) \bar{W}.$$

5.2 Optimal Mechanism

The optimal allocation policy is analogous to the efficient allocation policy. An allocation problem is considered only when $x^t = 0$. Let $\bar{R} \equiv \mathbb{E}[R(\theta^t, 0)]$. Then, the Bellman equation is

$$R(\theta^t, 0) = \max_{i \in N^t \cup \{0\}} \phi(\theta_i) + \delta^{l_i} \bar{R}, \tag{13}$$

where 0 denotes a dummy agent whose *reservation type* is $\theta_0 = (\underline{V}, 1)$. The reservation value $\underline{V} \in (0, \bar{V})$ is determined from $\phi(\underline{V}, 1) = 0$.

An incentive compatible mechanism is such that winning agents pay the critical value for winning given the others' types. Suppose agent i wins at t and agent j is the second highest: $j \in \arg\max_{j' \in N^t_{-i} \cup \{0\}} \phi(\theta_{j'}) + \delta^{l_{j'}} \bar{R}$. In addition, let $\phi^{-1}(\cdot|l_i)$ be the inverse function of $\phi(\cdot|l_i)$ given l_i. Then, agent i pays the total amount of

$$P_i^{**}(\theta^t) = \phi^{-1}\big(\phi(\theta_j) + (\delta^{l_j} - \delta^{l_i})\bar{R}|l_i\big).$$

An interesting question is how the efficient and the optimal allocation policies are different. To investigate this, let us consider the seller's indifference curves given arbitrary reference type $\hat{\theta} = (\hat{V}, \hat{l})$. The indifference curve under the optimal allocation policy, $V^o(l;\hat{\theta})$, describes the valuation that the seller evaluates equally to $\hat{\theta}$:

$$V^o(l;\hat{\theta}) - \frac{1}{\lambda_l(V^o(l;\hat{\theta}))} = \hat{V} + (\delta^{\hat{l}} - \delta^l)\bar{R} - \frac{1}{\lambda_{\hat{l}}(\hat{V})}. \tag{14}$$

Similarly, the agent with length l is evaluated equally to $\hat{\theta}$ in the efficient mechanism when the valuation type satisfies

$$V^e(l;\hat{\theta}) \equiv \hat{V} + (\delta^{\hat{l}} - \delta^l)\bar{W}. \tag{15}$$

The relationship between two indifference curves is generally unclear because the indifference curve under the optimal allocation policy critically depends on type distribution. Here, we consider a special case in which valuation V_i and length l_i are independently distributed. Then, we have $\lambda_l(\cdot) = \lambda(\cdot)$ for all l. Under this specification, the indifference curve under the optimal allocation policy is flatter than that under the efficient allocation policy.

Proposition 2 *Suppose that the valuation and length types are independently distributed and $\lambda(\cdot)$ is non-decreasing. Then, agents with a long length are more "favored" under the optimal mechanism than under the efficient mechanism; if for $l_i \geq \hat{l}$, the seller prefers θ_i to $\hat{\theta}$ under the efficient mechanism, then she prefers θ_i to $\hat{\theta}$ under the optimal mechanism too.*

5.3 Long-Stay Discount vs. Premium

In what follows, we examine the seller's pricing strategy as the optimal mechanism. We now assume that $|N^t| \leq 1$ for all t. At most one agent enters the mechanism in a period with an arrival rate $\eta \in (0,1]$. Both the incentive compatible efficient and optimal mechanisms are posted prices: the seller sets a proper price for each period length.

In the real world situations such as for hotel rooms, the seller often offers a discounted price for long-stay agents. However, the seller has to give up the potential revenue from future agents by assigning slots to a long-stay agent. Hence, it is uncertain whether the long-stay discount is supported by optimal pricing.

First, consider the efficient mechanism as a benchmark. It is easy to verify that the efficient total price for each length, $P^*(l)$, is given by $P^*(l) = (\delta - \delta^l)\bar{W}$. Thus, the average (per period) price $p^*(l)$ for an l-period stay is given by

$$p^*(l) = \frac{\delta - \delta^l}{1 - \delta^l}\bar{w},$$ [9]

where $\bar{w} = (1-\delta)\bar{W}$ indicates average social welfare. Since $\frac{\delta-\delta^l}{1-\delta^l}$ is increasing in l, the per period price $p^*(l)$ is increasing in length in the efficient mechanism. Thus, long-stay discounts do not exist but a long-stay premium necessarily exists.

Proposition 3 *Suppose that $K = 1$ and $|N^t| \leq 1$. Under the efficient pricing, the per period price is increasing in period length.*

Next, consider the optimal mechanism. We show that long-stay discounts are optimal under a certain situation. Suppose again that the total value and length type are independently distributed: $\lambda_l(\cdot) = \lambda(\cdot)$ for all l. As observed in the previous section, long-stay agents are more favored in the optimal mechanism than in the efficient mechanism. The total payment $P^{**}(l)$ for an l-period stay in the optimal mechanism is given from $\phi(P^{**}(l)) = (\delta - \delta^l)\bar{R}$. The following theorem shows that the long-stay discount exists for two periods. Moreover, the average price is decreasing in period lengths with an additional condition. Let $\bar{r} \equiv (1-\delta)\bar{R}$, which indicates the average revenue per period.

[9] An average price $p(l)$ for a total price $P(l)$ is defined as $p(l) = \frac{1-\delta}{1-\delta^l}P(l)$.

Theorem 3 *Suppose that* $K = 1$*,* $|N^t| \leq 1$*, and valuation and length types are independently distributed. The long-stay discount exists for two periods:* $p^{**}(2) < p^{**}(1)$*. In addition, if there exists* $\bar{l} \geq 2$ *such that* $\lambda(P^{**}(\bar{l}))\bar{r} \leq 1$*, then the average price* $p^{**}(l+1) < p^{**}(l)$ *for all* $l \leq \bar{l}$*.*

Note that the average revenue $\bar{r}$ is endogenously determined by the optimization problem (13) and depends on the type distribution. However, $\bar{r}$ has a degree of freedom by the parameter of population dynamics η.

Although long-stay discount exists in the above specification, it does not in another specification. Consider the case where the per period value v_i and length type l_i are independently drawn. Assume also that the value per period v_i is drawn from a density function $f > 0$ on $[0, 1]$. The distribution of v_i satisfies the monotone hazard rate condition, so that Assumption 1 is satisfied. Then, we have the virtual valuation by a simple calculation

$$\phi(\theta_i) = \frac{1 - \delta^{l_i}}{1 - \delta}\tilde{\phi}(v_i),$$

where $\tilde{\phi}(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$. Hence, the average price is given by

$$p^{**}(l) = \tilde{\phi}^{-1}\Big(\frac{\delta - \delta^l}{1 - \delta^l}\bar{r}\Big), \tag{16}$$

which implies the long-stay premium.

When long-stay premium exists, a long-stay agent would not like to accept a long-term contract but would prefer to repeat a short-term contract in each period, which is not considered in our model. The result would change if agents are allowed to enter into a new contract after the expiration of a contract. We need to strengthen incentive compatibility to prevent such a deviation. Nevertheless, the long-stay premium might still exist even if agents can repeat short-term contracts because long-stay agents accepting a short-term contract face the risk of losing in a future auction.

6 Binary Length Types and the Multi-Unit Handicap Auction

In this section, we consider the case of $K \geq 2$ and $L = 2$: Agents stay for at most two periods. We can simply let the allocative state x^t be the number of objects reserved for incumbent agents: $x^t \in \{0, \dots, K\}$. For simplicity of notations, a type of agent i is denoted by v_i for $\theta_i = (v_i, 1)$ and V_i for $\theta_i = (V_i, 2)$. In addition, let $v^{(j)}$ indicate the j-th highest order statistics among short-stay agents ($l_i = 1$) at the current period. Let $V^{(j)}$ be the j-th highest order statistics among long-stay agents ($l_i = 2$).

6.1 Efficient Mechanism

It is obvious that only the highest agents among the same length win the objects in the efficient allocation policy. That is, if agent i with v_i wins, then any agent i' with $v_{i'} > v_i$ wins in the efficient allocation. This holds for long-stay agents too. In addition, social welfare improves by allocating a unit to a short-stay agent rather than keeping it unallocated. Hence, all the units are allocated to someone by adding a sufficient number of dummy agents with $v_i = 0$ in each period. Let k^t be the number of units allocated to long-stay agents at t. Thus, we have $x^t = k^{t-1}$ and $K_t = K - k^{t-1}$. The Bellman equation for the efficient allocation policy is

$$W(\theta^t, k^{t-1}) = \max_{k^t \in \{0,\dots,K_t\}} \sum_{j=1}^{k^t} V^{(j)} + \sum_{j=1}^{K_t - k^t} v^{(j)} + \delta W(k^t). \tag{17}$$

$W(k)$ is decreasing in k. To have the efficient allocation policy, we first show that $W(k)$ is "concave;" i.e., $W(k) - W(k-1)$ is decreasing (increasing in absolute value) in k. The intuition is simple. Given supply units $K_t = K - k$ at the current period, the difference $W(k-1) - W(k)$ indicates the marginal value for an additional supply of the $(K_t + 1)$-th unit at the current period. As the supply decreases (i.e., k increases), the market at the current period becomes more competitive and an agent with a high value might lose the auction. Thus, the marginal welfare for the additional unit is increasing in k. Let $w(\theta^t, k) \equiv -(W(\theta^t, k) - W(\theta^t, k-1))$.

Proposition 4 *For any θ^t, $W(\theta^t, k)$ is concave in k; i.e., $w(\theta^t, k)$ is increasing in k.*

By Proposition 4, the efficient allocation policy is described in a simple optimization. In addition, the efficient allocation policy is implemented by a multi-unit uniform price auction with heterogeneous handicaps to long-stay agents. In the multi-unit handicap auction below, winners are determined by just selecting the highest bids, those of long-stay agents are handicapped depending on their rank order. The VCG payment is determined by the highest rejected bid.

Let $w(k) \equiv E[w(\theta^t, k)]$.

Theorem 4 *Suppose $L = 2$. The following multi-unit handicap auction is efficient and dominant strategy incentive compatible:*

1. *Each agent reports his type v_i or V_i.*

2. *Sort the reported values for each length. For the k-th highest order bid $V^{(k)}$ for a long-stay agent, define*

$$\hat{V}^{(k)} \equiv V^{(k)} - \delta w(k). \tag{18}$$

3. *The highest K_t bids from $\mathcal{B} \equiv \{v^{(1)}, v^{(2)}, \dots, \hat{V}^{(1)}, \hat{V}^{(2)}, \dots\}$ are selected for winning bids.*

4. *For the number of units allocated to long-stay agents k^*, each winning short-stay agent pays $\max\{v^{(K_t - k^* + 1)}, \hat{V}^{(k^*+1)}\}$. Each winning long-stay agent pays $\max\{V^{(k^*+1)}, v^{(K_t - k^* + 1)} + \delta w(k^*)\}$.*

6.2 Optimal Mechanism

The optimal mechanism for the seller is analogous to the efficient handicap auction. We obtain the optimal allocation policy by replacing the values and social welfare with the virtual values and virtual welfare. Let $\phi^{(j)}$ and $\Phi^{(j)}$ be the j-th highest order virtual values among short- and long-stay agents in a period, respectively. That is, $\phi^{(j)} = \phi(v^{(j)}, 1)$ and $\Phi^{(j)} = \phi(V^{(j)}, 2)$. The Bellman equation for the revenue maximization is

$$R(\theta^t, k^{t-1}) = \max_{k^t \in \{0,\ldots,K_t\}} \sum_{j=1}^{k^t} \Phi^{(j)} + \sum_{j=1}^{K_t - k^t} \phi^{(j)} + \delta E[R(\theta^{t+1}, k^t)], \tag{19}$$

where sufficiently many reservation types $\underline{\theta} = (\underline{v}, 1)$ satisfying $\phi(\underline{v}, 1) = 0$ are included in the above expression. Let

$$r(k) \equiv -E[R(\theta^t, k) - R(\theta^t, k-1)],$$

which is increasing in k by Proposition 4. The multi-unit auction with handicaps $r(k)$ is optimal for the seller. Proof is omitted because it is the same as the efficient mechanism Theorem 4.

Theorem 5 *Suppose $L = 2$ and Assumption 1. The multi-unit handicap auction below is dominant strategy incentive compatible and maximizes the seller's expected revenue:*

1. *Each agent reports his type v_i or V_i.*
2. *Sort the reported values for each length. For the k-th highest order virtual value $\Phi^{(k)}$ for a long-stay agent, define*

$$\hat{\Phi}^{(k)} \equiv \Phi^{(k)} - \delta r(k). \tag{20}$$

3. *The highest K_t bids from $\mathcal{B} \equiv \{\phi^{(1)}, \phi^{(2)}, \ldots, \hat{\Phi}^{(1)}, \hat{\Phi}^{(2)}, \ldots\}$ are selected for winning bids.*
4. *For the number of units allocated to long-stay agents k^*, each winning short-stay agent pays* $\max\{v^{(K_t - k^* + 1)}, \phi^{-1}(\hat{\Phi}^{(k^*+1)} | l = 1)\}$. *Each winning long-stay agent pays* $\max\{V^{(k^*+1)}, \phi^{-1}(\phi^{(K_t - k^* + 1)} + \delta r(k^*) | l = 2)\}$.

7 Conclusion

We formulate a model of a dynamic allocation problem in which agents want to obtain an object for periods of time. We characterize incentive compatibility and construct the efficient mechanism and optimal mechanism in a domain where the seller offers simple long-term contracts. The dynamic VCG mechanism achieves efficiency in a dominant strategy equilibrium. With a monotone hazard rate condition, the optimal allocation policy maximizes virtual welfare. In a single unit case, long-stay discount is not observed in the efficient posted prices, whereas it can be optimal for the seller under some specifications. When agents stay at most for two periods, both the efficient and optimal allocations are implemented by simple multi-unit auctions and properly handicapping long-stay agents.

Several avenues are open for future research. First, the efficient or optimal allocation policy needs to be investigated further in detail. Although we focus on simple incomplete contracts, it is difficult to specify the allocation policy when there are multiple units. Second, it would be interesting and important to consider the case in which agents often make new contracts after expiration, as formulated by Athey and Segal (2013), Bergemann and Valimaki (2010), Pavan et al. (2014). We would need to consider a dynamic structure for both the seller and agents in order to examine various practical situations.

A Proofs

A.1 Proof of Lemma 1

For any mechanism $\{z^t\}$, define an exact mechanism $\{\hat{z}^t\}$ as for every t, every i, and every θ_i, $m_i(\hat{z}_i^t) = l_i$, $P_i(\hat{z}_i^t) = P_i(z_i^t)$, and

$$a_i(\hat{z}_i^t) = \begin{cases} a_i(z_i^t) & \text{if } m_i(z_i^t) \geq l_i \\ 0 & \text{if } m_i(z_i^t) < l_i \end{cases}.$$

It is obvious that the payoff function under $\{\hat{z}^t\}$ is the same as that under $\{z^t\}$. In addition, mechanism $\{\hat{z}^t\}$ satisfies $\sum_{i\in N^t} a_i(\hat{z}_i^t) \leq \sum_{i\in N^t} a_i(z_i^t)$, thus it is feasible. ■

A.2 Proof of Proposition 1

(Only if part.) Suppose a mechanism is incentive compatible. Then, we have

$$\alpha_i(V_i, l_i)V_i - q_i(V_i, l_i) \geq \alpha_i(V_i', l_i)V_i - q_i(V_i', l_i),$$

hence,

$$(\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i \geq q_i(V_i, l_i) - q_i(V_i', l_i).$$

Similarly, we have

$$(\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i' \leq q_i(V_i, l_i) - q_i(V_i', l_i).$$

Therefore,

$$(\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i' \leq (\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i.$$

Therefore, $V_i' < V_i$ implies $\alpha_i(V_i', l_i) \leq \alpha_i(V_i, l_i)$.

From the standard argument of the envelope theorem (Milgrom and Segal, 2002), if $V_i \in \arg\max_{\nu\in[0,\bar{V}]} \alpha_i(\nu, l_i)V_i - q_i(\nu, l_i)$, then

$$\frac{\partial \Pi_i(V_i, l_i)}{\partial V_i} = \alpha_i(V_i, l_i)$$

almost everywhere, and

$$\Pi_i(V_i, l_i) - \Pi_i(V_i', l_i) = \int_{V_i'}^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu. \tag{21}$$

Suppose $l_i' > l_i$. Then, $\alpha_i((V_i', l_i'), l_i) = \mathbb{E}[a_i(V_i', l_i', \theta_{-i}^t, h_t)\mathbb{I}_{\{m_i(z_i^t)=l_i'\geq l_i\}}] = \alpha_i(V_i', l_i')$. Then incentive compatibility implies for all V_i,

$$\begin{aligned}\Pi_i(V_i, l_i) &\geq \alpha_i((V_i, l_i'), l_i)V_i - q_i(V_i, l_i') \\ &= \alpha_i(V_i, l_i')V_i - q_i(V_i, l_i') \\ &= \Pi_i(V_i, l_i').\end{aligned} \tag{22}$$

From the envelope formula (21), (22) yields

$$\Pi_i(0, l_i) + \int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu \geq \Pi_i(0, l_i') + \int_0^{V_i} \alpha_i(\nu, l_i')\mathrm{d}\nu. \tag{23}$$

For $V_i = 0$, it also holds that $\Pi_i(0, l_i) = -q_i(0, l_i)$. Incentive compatibility requires $-q_i(0, l_i) \geq -q_i(0, l_i')$ for any l_i and l_i', thus that $-q_i(0, l_i)$ does not depend on l_i. Hence, $\Pi_i(0, l_i) = \underline{\Pi}_i$ for all l_i.

(If part.) Since α_i is monotone, each agent's expected payoff $\pi_i((\nu, l_i), (V_i, l_i))$ satisfies the single crossing condition on (V_i, ν). Given l_i, a standard argument as in Myerson (1981) implies for all V_i',

$$\alpha_i(V_i, l_i)V_i - q_i(V_i, l_i) \geq \alpha_i(V_i', l_i)V_i - q_i(V_i', l_i).$$

Suppose $l_i' > l_i$. Note that $\alpha_i((V_i, l_i'), l_i) = \alpha_i(V_i, l_i')$. Hence, for any V_i',

$$\begin{aligned}\alpha_i((V_i', l_i'), l_i)V_i - q_i(V_i', l_i') &= \alpha_i(V_i', l_i')V_i - q_i(V_i', l_i') \\ &\leq \Pi_i(V_i, l_i') \\ &\leq \Pi_i(V_i, l_i).\end{aligned}$$

The last inequality holds from the conditions 2 and 3.

The envelope condition (6) implies

$$\begin{aligned}-q_i(V_i, l_i) &= \underline{\Pi}_i + \int_0^{V_i} [\alpha_i(\nu, l_i) - \alpha_i(V_i, l_i)]\mathrm{d}\nu \\ &\leq \Pi_i(0, l_i),\end{aligned}$$

where inequality comes from the monotonicity of α_i. Then, for all V_i' and all $l_i' < l_i$,

$$\begin{aligned}\alpha_i((V_i', l_i'), l_i)V_i - q_i(V_i', l_i') &= -q_i(V_i', l_i') \\ &\leq \Pi_i(0, l_i') \\ &= \Pi_i(0, l_i) \\ &\leq \Pi_i(V_i, l_i).\end{aligned}$$

The second equality comes from condition 3, and the last inequality is from the monotonicity and (6). ■

A.3 Proof of Theorem 1

Suppose $i \in N^t$. For any θ_i and reports of the others θ^t_{-i}, i's ex post payoff given a report γ_i is

$$
\begin{aligned}
u_i((\gamma_i, \theta^t_{-i}), \theta_i, h_t) &= a^*_i(\gamma_i, \theta^t_{-i}, x^t)V_i - P^*_i(\gamma_i, \theta^t_{-i}, x^t) \\
&= \sum_{N^t} a^*_j(\gamma_i, \theta^t_{-i}, x^t)V_j + \delta W(G(a^*(\gamma_i, \theta^t_{-i}, x^t), (\gamma_{il}, l^t_{-i}), x^t)) - W_{-i}(\theta^t, x^t) \\
&\leq \sum_{N^t} a^*_j(\theta^t, x^t)V_j + \delta W(G(a^*(\theta^t, x^t), l^t, x^t)) - W_{-i}(\theta^t, x^t) \\
&= U_i(\theta^t, h_t).
\end{aligned}
\tag{24}
$$

Therefore, truth-telling is optimal, and the associated payoff is $W(\theta^t, x^t) - W_{-i}(\theta^t, x^t) = C_i(\theta^t, x^t)$. Since $W(\theta^t, x^t) \geq W_{-i}(\theta^t, x^t)$ by definition, $\{z^*\}$ is individually rational. ■

A.4 Proof of Theorem 2

It suffices to show that the solution of (11) is monotone. By Assumption 1, the virtual value ϕ weakly increases by shortening l_i. Hence, a^{**} is weakly decreasing in l_i because a^{**} is weakly increasing in ϕ. Thus, the solution of (11) is monotone in terms of $(\phi(\theta_i), l_i)$. Since ϕ is increasing in V_i, the solution is monotone. ■

A.5 Proof of Proposition 2

Suppose $l_i \geq \hat{l}$. From the regularity and the non-decreasing hazard rate, $V^o(l_i, \hat{\theta}) > \hat{V}$ and $\lambda(V^o(l_i, \hat{\theta})) \geq \lambda(\hat{V})$. Note that the expected revenue for the seller $\bar{R}$ is strictly less than the expected social welfare $\bar{W}$; $\bar{R} < \bar{W}$. Therefore, we have

$$
\frac{1}{\lambda(V^o(l_i; \hat{\theta}))} - \frac{1}{\lambda(\hat{V})} \leq 0 \leq (\delta^{\hat{l}} - \delta^{l_i})(\bar{W} - \bar{R}), \tag{25}
$$

thus $V^o(l_i; \hat{\theta}) \leq V^e(l_i; \hat{\theta})$. ■

A.6 Proof of Theorem 3

The average price $p^{**}(l)$ satisfies

$$p^{**}(l) = \frac{\delta - \delta^l}{1 - \delta^l}\bar{r} + \frac{1-\delta}{1-\delta^l} \cdot \frac{1}{\lambda(P^{**}(l))}. \tag{26}$$

From Assumption 1 and $P^{**}(l) < P^{**}(l+1)$, we have $\lambda(P^{**}(l)) \leq \lambda(P^{**}(l+1))$. Hence,

$$\begin{aligned} p^{**}(l+1) - p^{**}(l) &= \frac{\delta^l(1-\delta)^2}{(1-\delta^l)(1-\delta^{l+1})}\bar{r} + \frac{1-\delta}{(1-\delta^l)(1-\delta^{l+1})}\left(\frac{1-\delta^l}{\lambda(P^{**}(l+1))} - \frac{1-\delta^{l+1}}{\lambda(P^{**}(l))}\right) \\ &\leq \frac{\delta^l(1-\delta)^2}{(1-\delta^l)(1-\delta^{l+1})}\left(\bar{r} - \frac{1}{\lambda(P^{**}(l))}\right). \end{aligned}$$

Therefore, we have $p^{**}(l+1) < p^{**}(l)$ for some $\bar{l}$ and $l \leq \bar{l}$ when

$$\lambda(P^{**}(\bar{l}))\bar{r} \leq 1. \tag{27}$$

Note that $p^{**}(1) = P^{**}(1)$ and $\lambda(P^{**}(1))p^{**}(1) = 1$. The seller earns revenue $p^{**}(1)$ when agent i with $V_i \geq p^{**}(1)$ and $l_i = 1$ enters. Further, $l_i = 1$ is the best length type. Therefore, it is obvious that $\bar{r} < p^{**}(1)$ regardless of the arrival rate η.

■

A.7 Proof of Proposition 4

For the current state (θ^t, k), let $k^*(\theta^t, k)$ be the efficient allocation policy that determines the number of units allocated to long-stay agents. To show the proposition, we first show the following lemma.

Lemma 2 *For any θ^t, $k^*(\theta^t, k)$ is non-increasing in k.*

Proof. Suppose for contradiction that there exists k and for some θ^t, $k^*(\theta^t, k) > k^*(\theta^t, k-1)$. Let $k^* \equiv k^*(\theta^t, k)$ and $k^{**} \equiv k^*(\theta^t, k-1)$. Since $k^{**} < K-k$, we have[10]

$$\begin{aligned} W(\theta^t, k) &= \sum_{j=1}^{k^*} V^{(j)} + \sum_{j=1}^{K-k-k^*} v^{(j)} + \delta W(k^*) \\ &> \sum_{j=1}^{k^{**}} V^{(j)} + \sum_{j=1}^{K-k-k^{**}} v^{(j)} + \delta W(k^{**}), \end{aligned}$$

and hence,

$$\sum_{j=k^{**}+1}^{k^*} V^{(j)} - \sum_{j=K-k-k^*+1}^{K-k-k^{**}} v^{(j)} + \delta(W(k^*) - W(k^{**})) > 0. \tag{28}$$

Therefore, we have

$$\begin{aligned} W(\theta^t, k-1) - W(\theta^t, k) &= \sum_{j=1}^{k^{**}} V^{(j)} + \sum_{j=1}^{K-k-k^{**}} v^{(j)} + \delta W(k^{**}) - \sum_{j=1}^{k^*} V^{(j)} - \sum_{j=1}^{K-k-k^*} v^{(j)} - \delta W(k^*) \\ &= - \sum_{j=k^{**}+1}^{k^*} V^{(j)} + \sum_{j=K-k-k^*+1}^{K-k-k^{**}} v^{(j)} - \delta(W(k^*) - W(k^{**})) + v^{(K-k-k^{**}+1)} \\ &< v^{(K-k-k^{**}+1)}. \end{aligned} \tag{29}$$

However, it is obvious that

$$\begin{aligned} W(\theta^t, k-1) - W(\theta^t, k) &\geq v^{(K-k-k^*+1)} \\ &\geq v^{(K-k-k^{**}+1)}, \end{aligned}$$

which is a contradiction to (29). □

Thus, $k^* \leq k^{**}$. Next, we show $k^{**} \in \{k^*, k^*+1\}$.

Lemma 3 *If $W(k)$ is concave, $k^{**} \in \{k^*, k^*+1\}$.*

Proof. Suppose $W(k)$ is concave. Suppose for contradiction that for some k and θ^t, $k^* + 2 \leq k^{**}$.

[10]If equality holds, k^{**} is also an efficient allocation given (θ^t, k), and we can construct a non-increasing allocation policy.

Since $k^* + 1 \le k^{**} - 1 \le K - k$,

$$
\begin{aligned}
W(\theta^t, k) &= \sum_{j=1}^{k^*} V^{(j)} + \sum_{j=1}^{K-k-k^*} v^{(j)} + \delta W(k^*) \\
&> \sum_{j=1}^{k^*+1} V^{(j)} + \sum_{j=1}^{K-k-k^*-1} v^{(j)} + \delta W(k^*+1),
\end{aligned}
$$

and hence,

$$
-V^{(k^*+1)} + v^{(K-k-k^*)} + \delta w(k^*+1) > 0. \tag{30}
$$

Since $W(k)$ is concave, we have

$$
\begin{aligned}
0 &< -V^{(k^*+1)} + v^{(K-k-k^*)} + \delta w(k^*+1) \\
&\le -V^{(k^*+2)} + v^{(K-k-k^*-1)} + \delta w(k^*+2) \\
&\le \dots .
\end{aligned} \tag{31}
$$

Therefore, we have

$$
\begin{aligned}
& W(\theta^t, k-1) - \big[\sum_{j=1}^{k^*+1} V^{(j)} + \sum_{j=1}^{K-k-k^*} v^{(j)} + \delta W(k^*+1)\big] \\
&= \sum_{j=k^*+2}^{k^{**}} V^{(j)} - \sum_{j=K-k-k^{**}+2}^{K-k-k^*} v^{(j)} - \delta(W(k^*+1) - W(k^{**})) \\
&\le \sum_{j=k^*+1}^{k^{**}-1} V^{(j)} - \sum_{j=K-k-k^{**}+2}^{K-k-k^*} v^{(j)} - \delta(w(k^*+2) + w(k^*+3) + \cdots + w(k^{**})) \\
&\le \sum_{j=k^*+1}^{k^{**}-1} V^{(j)} - \sum_{j=K-k-k^{**}+2}^{K-k-k^*} v^{(j)} - \delta(w(k^*+1) + w(k^*+2) + \cdots + w(k^{**}-1)) \\
&< 0.
\end{aligned}
$$

The second inequality comes from the concavity of $W(k)$. This contradicts the optimality of $W(\theta^t, k-1)$. □

Finally, we show the proposition. It suffice to show that if any continuation value $\hat{W}(\theta^t, k)$ is concave in k, then the mapping is also concave. Let $k^{***} \equiv k^*(\theta^t, k-2)$.

Case 1: $k^{**} = k^*$ or $k^{***} = k^{**}$.

Note that

$$
w(\theta^t, k) \ge v^{(K-k-k^*+1)}.
$$

Since $k^* \le K-k$ and $k^{***} \le k^*+1 \le K-k+1$,

$$\begin{aligned}
&w(\theta^t, k-1) \\
&\le \sum_{j=1}^{k^{***}} V^{(j)} + \sum_{j=1}^{K-k-k^{***}+2} v^{(j)} + \delta W(k^{***}) - \sum_{j=1}^{k^{***}} V^{(j)} - \sum_{j=1}^{K-k-k^{***}+1} v^{(j)} - \delta W(k^{***}) \\
&= v^{(K-k-k^{***}+2)} \\
&\le v^{(K-k-k^*+1)} \\
&\le w(\theta^t, k).
\end{aligned}$$

Case 2: $k^{**} = k^*+1$ and $k^{***} = k^{**}+1 = k^*+2$.

Note that

$$W(\theta^t, k-1) - W(\theta^t, k) = V^{(k^*+1)} + \delta(\hat{W}(k^*+1) - \hat{W}(k^*))$$

and

$$W(\theta^t, k-2) - W(\theta^t, k-1) = V^{(k^*+2)} + \delta(\hat{W}(k^*+2) - \hat{W}(k^*+1)).$$

Hence,

$$W(\theta^t, k-1) - W(\theta^t, k) \ge W(\theta^t, k-2) - W(\theta^t, k-1).$$

■

A.8 Proof of Theorem 4

Suppose $K_t > 0$. The social welfare when k units are allocated to long-stay agents is

$$\begin{aligned}
&\sum_{j=1}^{k} V^{(j)} + \sum_{i=1}^{K_t-k} v^{(i)} + \delta W(k) \\
=&\sum_{j=1}^{k} \big(V^{(j)} - \delta w(j)\big) + \sum_{i=1}^{K_t-k} v^{(i)} + \delta\Big(W(k) + \sum_{j=1}^{k} w(j)\Big) \\
=&\sum_{j=1}^{k} \hat{V}^{(j)} + \sum_{i=1}^{K_t-k} v^{(i)} + \delta W(0).
\end{aligned}$$

Since $W(0)$ is just a constant term, the social welfare maximization is equivalent to

$$\max_{k\in\{0,\ldots,K_t\}} \sum_{j=1}^{k} \hat{V}^{(j)} + \sum_{i=1}^{K_t-k} v^{(i)}. \tag{32}$$

By Proposition 4, $w(k)$ is increasing in k and thus $\hat{V}^{(k)}$ is decreasingly ordered. Hence, the solution of (32) is simply to select the highest K_t bids from

$$\mathcal{B} \equiv \{v^{(1)}, v^{(2)}, \dots, \hat{V}^{(1)}, \hat{V}^{(2)}, \dots\}.$$

Note that because the efficient allocation policy when agent i is excluded is given in the same manner, the externality that agent i gives to the other agents is determined by the highest rejected bid or the $(K_t + 1)$-th highest bid of $\mathcal{B}$. Let k^* and $b^{(j)}$ be the solution of (32) and j-th highest bid of $\mathcal{B}$, respectively. Thus, k^* units are allocated to long-stay agents and $K_t - k^*$ units to short-stay agents. When a winner i is a short-stay agent, then his payment in the dynamic VCG mechanism is $b^{(K_t+1)} = \max\{v^{(K_t-k^*+1)}, \hat{V}^{(k^*+1)}\}$. When a winner i is a long-stay agent, then his total payment in the dynamic VCG mechanism is $V^{(k^*+1)}$ if $b^{(K_t+1)} = \hat{V}^{(k^*+1)}$. If $b^{(K_t+1)} = v^{(K_t-k^*+1)}$, the total payment is $v^{(K_t-k^*+1)} + w(k^*)$. ■

References

[1] Athey, S., and I. Segal (2013), "An Efficient Dynamic Mechanism," *Econometrica*, 81, 2463-2485.

[2] Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan, and A. Sen (2006), "Weak Monotonicity Characterizes Deterministic Dominant-Strategy Implementation," *Econometrica*, 74, 1109-1132.

[3] Bergemann, D., and M. Said (2011), "Dynamic Auctions: A Survey," *Wiley Encyclopedia of Operations Research and Management Science.*

[4] Bergemann, D., and J. Valimaki (2010), "The Dynamic Pivot Mechanism," *Econometrica*, 78, 771-789.

[5] Board, S., and A. Skrzypacz (2010), "Revenue Management with Forward-Looking Buyers," Unpublished manuscript, Stanford University.

[6] Dizdar, D., A. Gershkov, and B. Moldovanu (2011), "Revenue Maximization in the Dynamic Knapsack Problem," *Theoretical Economics*, 6, 157-184.

[7] Gershkov, A., and B. Moldovanu (2009), "Dynamic Revenue Maximization with Heterogeneous Objects: A Mechanism Design Approach," *American Economic Journal: Microeconomics*, 1, 168-198.

[8] Gershkov, A., and B. Moldovanu (2010), "Efficient Sequential Assignment with Incomplete Information," *Games and Economic Behavior*, 68, 144-154.

[9] Hajiaghayi, M., R. Kleinberg, M. Mahdian, and D.C. Parkes (2005), "Online Auctions with Re-Usable Goods," *Proceedings of the 6th ACM Conference on Electronic Commerce (EC '05)*, 165-174.

[10] Kittsteiner, T., and B. Moldovanu (2005), "Priority Auctions and Queue Disciplines That Depend on Processing Time," *Management Science*, 51, 236-248.

[11] Lavi, R., and N. Nisan (2000), "Competitive Analysis of Incentive Compatible Online Auctions," *Proceedings of the 2nd Conference on Electronic Commerce (EC '00)*, 233-241.

[12] Lehmann, D., L.I. O'Callaghan, and Y. Shoham (2002), "Truthful Revelation in Approximately Efficient Combinatorial Auctions," *Journal of ACM*, 49, 577-602.

[13] Mierendorff, K. (2009), "Optimal Dynamic Mechanism Design with Deadlines," Unpublished manuscript, University of Bonn.

[14] Milgrom, P., and I. Segal (2002), "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583-601.

[15] Myerson, R. (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

[16] Pai, M., and R. Vohra (2013), "Optimal Dynamic Auctions and Simple Index Rules," *Mathematics of Operations Research*, 38, 682-697.

[17] Parkes, D.C. (2007), "Online Mechanisms," in N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani (eds.), *Algorithmic Game Theory*, Cambridge University Press, New York.

[18] Parkes, D.C., and S. Singh (2003), "An MDP-Based Approach to Online Mechanism Design," *Proceedings of the 17th Annual Conference on Neural Information Processing Systems (NIPS '03)*, 791-798.

[19] Pavan, A., I. Segal, and J. Toikka (2014), "Dynamic Mechanism Design: A Myersonian Approach," *Econometrica*, 82, 601-653.

[20] Porter, R. (2004), "Mechanism Design for Online Real-Time Scheduling," *Proceedings of the 5th ACM Conference on Electronic Commerce (EC '04)*, 61-70.

[21] Riley, J., and W. Samuelson (1981), "Optimal Auctions," *American Economic Review*, 71, 381-392.

[22] Said, M. (2012), "Auctions with Dynamic Populations: Efficiency and Revenue Maximization," *Journal of Economic Theory*, 147, 2419-2438.

[23] Vulcano, G., G. van Ryzin, and C. Maglaras (2002), "Optimal Dynamic Auctions for Revenue Management," *Management Science*, 48, 1388-1407.