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"Dynamic Inconsistency in Pension Fund Management"

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# Dynamic Inconsistency in Pension Fund Management

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**Abstract**

We formulate the pension fund's problem of choosing optimal pension schemes in an infinite, discrete-time setting as a sequence of Nash bargaining problems in which the members (contributors) of the fund are the bargainers and the disagreement points are determined by the utility levels they can attain by quitting and receiving lump-sum payments from the fund. We show that if the members are heterogeneous in their subjective time discount rates, then the sequence of the Nash bargaining solutions, obtained at each point in time, leads to an inefficient allocation of consumption processes, thereby indicating a source of dynamic inconsistency in pension fund management. Based on a set of micro data, we show the welfare loss of dynamic inconsistency can be as high as 14% of the members' total wealth, and the dynamically inconsistent choices of pension schemes tend to favor myopic members.

**JEL Classification Codes**: D51, D53, D61, D81, D91, E43, G12.

**Keywords**: Pension fund, Discount factor, discount rate, dynamic inconsistency, Nash bargaining solution.

## 1 Introduction

In any aging society, good management of a pension system is of paramount importance, and yet there seems no sound theoretical model in which we can discuss its efficacy. The purpose of this paper is to provide such a model. Although many social and economic factors are apparently relevant in pension fund management, there are three aspects, from the viewpoint of microeconomic theory, of pension fund management that deserve special attention.

First, the members (subscribers, contributors) of the pension fund wish to smooth consumption but have limited access to financial markets. In other words, they wish to save before

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retirement and maintain similar consumption levels after retirement, but the pension fund, as an institutional investor, typically has a better trading technology with lower transaction costs. Second, the amount of contributions that the pension fund collects from its members may depend on the members' incomes but not on their preferences. For example, the fund may collect a fixed proportion of the members' monthly salaries, but must not collects amounts that are contingent on their unobservable characteristics, such as subjective time discount rates, for legal or informational reasons. Third, the fund may scrap the scheme that is currently in place and implement a new one. From each member's perspective, this means that what he ends up with contributing to and receiving from the fund after retirement may well be different from what he is supposed to do so at the time of joining it.

The model of this paper has the following characteristics to take these three aspects into consideration. First, there is a group of members who have no income, other than pension benefits, after retirement, and no access to financial markets, so that they cannot save their earnings to the post-retirement period. The fund, on the other hand, has full access to financial markets so that it can borrow and lend at the market rate without incurring any transaction costs. Its mission is, therefore, to save on behalf of its members. However, since the fund is not allowed to differentiate members by their preferences, any two members with equal incomes must necessarily make the same contribution and, thus, enjoy the same consumption process. This is true even when the two have different discount rates and, thus, prefer different levels of contributions. This constraint renders the optimal pension-scheme choice problem a collective choice problem, which we formulate as a Nash bargaining problem among the members. In other words, the fund chooses a pension scheme that maximizes the so-called Nash product (Nash (1950)), the product of the utility levels that the members attain by staying in the fund in excess of those which they attain by quitting and receiving lump-sum payments from the fund. We interpret the Nash product as representing the rules or codes of conduct that the fund is supposed to obey, possibly as a result of laws and regulations. Finally, even after the fund chooses a scheme by solving the Nash bargaining solution, the fund has another opportunity to choose, again, a scheme by solving the Nash bargaining solution once some time has elapsed. Since the members now have different future income processes (shifted backwards by one period) and have accumulated a larger sum of contributions at the fund, the new solution typically calls for levels of contributions that are different from what was planned in the old solution. This is the type of dynamic inconsistency in pension fund management that we analyze in this paper. It arises from the fact that the fund is unable to commit itself to any particular pension scheme.

Since the model of this paper builds on the members' incomes and preferences and fully specifies what they can or cannot receive by staying in or quitting the fund, we could ask many interesting and important questions regarding pension fund management. Of these, we shall focus on how the heterogeneity in the members' subjective time discount rates renders the fund's choice of pension schemes biased towards some members and dynamically inconsistent. The first thing we should notice then is the impact of discount rates on the disagreement points. Since each member has no earning after retirement, the consumption process from quitting the fund involves higher consumption levels on early periods and lower (in fact, zero) consumption levels on late periods. Such a process would be more appreciated by the more myopic members (the members having higher discount rates). This, at first sight, seems to suggest that the more myopic members are better treated at the Nash bargaining solution. This line of thoughts is, however, flawed because it misses the role of prices, or interest rates, in the bargaining solution of Nash (1950). Indeed, our first main result (Proposition 5) shows that for every profile of the members' preferences, there is a price process under which all members are equally treated. Based on this result, our second result (Proposition 6) shows that if the members have differing discount rates, then, for each member, there is a price process under which that member is better treated than any other member. An intriguing aspect of the second result is that besides the most myopic member and the most patient one, any member, having a moderate discount rate, may well be best treated if we impose no restrictions on equilibrium interest rates. These results points to the need to impose restrictions on interest rates to identify the nature of optimal pension schemes.

As for the nature and extent of dynamic inconsistency, our results are empirical. Based on the micro data from the Preference Parameters Study of Osaka University's 21st Century COE Program 'Behavioral Macrodynamics Based on Surveys and Experiments' and its Global COE project 'Human Behavior and Socioeconomic Dynamics', we will give (in Section 9) some estimates of the welfare losses, in terms of the fraction of the total wealth that could be foregone were the dynamic consistency guaranteed. The fraction is far from negligible, and can be as large as 14%. We will also see that the dynamic inconsistency tends to favor myopic members, that is, the contribution that the fund actually collects from each member on a later earning period turns out to be smaller than the contribution that it plans, on an earlier period, to collect on the later earning period.

We should also mention two things we do not investigate in our model. First, we do not deal with the dynamic inconsistency of the members. We assume, instead, that they are dynamically consistent and stationary, that is, they always retain the same ranking between future consumption processes as time elapses and use the same utility functions regardless of the period on which they make decisions. While experimental evidences, earlier one of which are mentioned in Loewenstein and Prelec (1992), often suggests otherwise, the assumption of dynamically consistent members has the virtue of allowing us to pin down any dynamic inconsistency in the fund's choice to the way its choice is made, particularly the lack of commitment. Second, we do not deal with transfers of purchasing power among members. We assume, instead, that in any pension scheme, the discounted present value of the benefits each member receives from the fund after retirement is equal to that of the contributions he pays in to the fund before retirement. By assuming away the transfers among members, we are excluding, say, schemes that are financed by the pay-as-you-go method, which nowadays tend to favor older generations at the sacrifice of younger ones, and for which welfare consequences are quantifiable with monetary units. We shall not deal with them, simply because we intend to concentrate on the the conflict of interest arising from heterogeneous preferences, which is a theoretically challenging research topic.

Since this paper lies at the intersection of finance and bargaining theory and touches on behavioral economics, it is related to many existing works. Of these, the one we should mention here is Jackson and Yariv (2014). They considered the problem of aggregating profiles of individuals' stationary time-additive intertemporal utility functions that are defined on the set of common consumption processes into a single social utility function defined on the same set of common consumption processes. They proved (Theorem 2) that if the individuals are heterogeneous in their discount rates, then the dictatorial rules are the only aggregation rules that satisfy unanimity and always leads to dynamically consistent social utility functions. The Nash products are no exception to their theorem: they satisfy unanimity but does not generate dynamically consistent objective functions for the fund. The type of dynamic (in-)consistency that we analyze in this paper is, however, different from the notion of dynamic (in-)consistency that they took up: the fund's objective function, albeit dynamically inconsistent in the sense of Jackson and Yariv (2014), changes over time, because the disagreement points change as a consequence of reduced time to retirement and increased contributions accumulated at the fund. We are interested in the extent of dynamic inconsistency resulting from changing objective functions, and we even propose, in Section 6, a measure of such inconsistency that relies one the objective function that may be dynamically inconsistent in the sense of Jackson and Yariv (2014).

This paper is organized as follows. In Section 2, we lay out the environment the fund is faced with. In Section 3, we define pension schemes that the fund can implement if it can commit itself to the scheme it chooses at the beginning. In Section 4, we define the resulting consumption processes if the fund revises the existing scheme on each period. In Section 5, we define objective functions that the fund may have. In Section 6, we define state-dependent objective functions, a particular case of which is the Nash product. In Section 7, we deal with an easy, but important, case in which the members of the fund are identical in their incomes and preferences. In Section 8, we present the two main results of this paper, regarding whether the members are symmetrically or asymmetrically treated. In Section 9, we give an empirical analysis of dynamic inconsistency. In Section 10, we conclude the paper by indicating some possible extensions of the model of this paper.

# 2 Model

## 2.1 Commodities and prices

The time span of consumption is $\{0, 1, 2, \dots\}$, which we denote by $T$. There is only one type of goods at each date. Denote by $L$ the set of all sequences, or real-valued functions defined on $T$. We write each $c \in L$ as $(c(0), c(1), c(2), \dots)$, $(c(t))_{t \in T}$, and so on. For each $c \in L$ and $t \in T$, write $c^t = (c(t), c(t+1), c(t+2), \dots) \in L$. Denote by $L_+$ the set of all elements of $L$ of which all the entries are nonnegative, and by $L_{++}$ the set of all members of $L$ of which all the entries are strictly positive. Denote by $\ell_\infty$ the set of all elements of $L$ that are bounded and write $C = \ell_\infty \cap L_+$. We take $\ell_\infty$ as the commodity space and $C$ as the consumption set. Denote by $\ell_1$ the set of all elements of $L$ that are absolutely summable. Write $P = \ell_1 \cap L_{++}$. We take $P$ as the price space. Each element $p$ of $\ell_1$ defines a continuous linear functional on $\ell_\infty$ via $\sum_{t=0}^{\infty} p(t)c(t)$ for every $c \in \ell_\infty$. Although the topological dual of the commodity space $\ell_\infty$ is larger than $\ell_1$, Prescott and Lucas (1972) showed that if the consumers' utility functions discount future felicities, as the additively separable utility functions, which we assume later, would do, then for every equilibrium price in the topological dual of $C$, there is a price process in $P$ that constitute an equilibrium with the same consumption processes. In this sense, restricting prices on $P$ comes with no loss of generality. For each $(c, p) \in C \times P$, write $p \cdot c = \sum_{t=0}^{\infty} p(t)c(t)$.

Let $p \in P$ be a process of prices of the goods available at each date. The price, at date $t$ of the discount bond that matures at date $t+1$ is then equal to $p(t+1)/p(t)$. The (continuously compounded) discount rate at date $t$ is equal to $\ln p(t) - \ln p(t+1)$. For the price process $p \in P$, define the (continuously compounded) interest rate process $r \in L$ by $r(t) = \ln p(t) - \ln p(t+1)$, that is,

$$\exp(-r(t)) = \frac{p(t+1)}{p(t)}$$

for every $t$. The implicit assumption in this benchmark model is that the pension fund has full access to bond markets but the members has none.

### 2.2 Members of the pension fund

Let $I$ be the set of (names of) members (subscribers, contributors) of the pension fund. We assume throughout this paper that $I$ is finite. Occasionally, we let $I = \{1, 2, \dots, I\}$. Let $\rho_i > 0$. Let $u_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$. Assume that $u_i(0) = 0$ and that $u_i$ is twice continuously differentiable and satisfies $u_i'' < 0 < u_i'$ and the Inada condition on $\boldsymbol{R}_{++}$. Then the utility function $U_i$ of member $i$ over consumption processes $c_i \in C_+$ is defined by

$$U_i(c_i) = \sum_{t=0}^{\infty} \exp(-\rho_i t) u_i(c_i(t)).$$

That is, we assume that each member $i$ has time-additive expected utility functions over consumption processes, with the continuously compounded discount rates $\rho_i$. The important properties embedded in this definition of $U_i$ are dynamic consistency and stationarity. Then $U_i(c_i) \geq 0$ and $U_i(c_i) < \infty$ because $u_i$ is concave, $\sum_t \exp(-\rho_i t) < \infty$, and $c_i$ is bounded. Since the domain of $u_i$ coincides with $\boldsymbol{R}_+$, we are excluding the utility functions having constant coefficients of relative risk aversion at least as large as one.

In the subsequence analysis, it is useful or even necessary to use monotone transformations of $U_i$, rather than $U_i$ itself. Of these, the first one is $\tilde{U}_i$, which is defined by $\tilde{U}_i = (1 - \exp(-\rho_i))U_i$. This has the advantage that the coefficients multiplied to the $u_i(c_i(t))$ add up to one. The second one is $\hat{U}_i$, which is defined by $\hat{U}_i = u_i^{-1} \circ \tilde{U}_i$. The advantage of this transformation is that the utility level $\hat{U}_i(c_i)$ is the (unique) level of consumption that member $i$ finds, if received in perpetuity, equally desirable to $c_i$. It may not be concave, but we will use the function $u_i$

such that $\hat{U}_i$ is guaranteed to be concave.

Each member $i$ has an initial endowments, or income stream, which is denoted by $e_i \in C$. Let $t_i \in T$ be the period on which member $i$ has just retired and starts receiving pension benefits. A typical situation we have in mind is where $e_i(t) > 0$ for every $t < t_i$ and $e_i(t) = 0$ for every $t \geq t_i$. The members would then like to smooth his consumption over time (before and after retirement), but they cannot do so because they have no access to asset markets. Since the pension fund has full access to asset markets, it is its role to provide the members with more smoothed consumption on behalf of its members. Throughout this paper, we implicitly assume that the fund knows (observes) $e_i$. In addition to $e_i$, member $i$ has an accumulated contribution, denoted by $a_i \in \boldsymbol{R}_+$ that is held by the fund on period zero. This accumulated contribution is the monetary value of the contributions he has made during the time span up to period zero, which are not explicitly modeled here, and is to be eventually be paid back to him. We allow for the case where $t_i = 0$, which means that member $i$ has contributed to the fund and retired, and only receives benefit from the fund in the time span explicitly modeled here.

To summarize, member $i$ is characterized by his felicity function $u_i$, continuously compounded discount rate $\rho_i$, income process $e_i$, accumulated contribution $a_i$, and time to retirement $t_i$. Together with the price process $p$, the state of the pension fund on period zero is summarized as a profile $\left((e_i, a_i, t_i)_{i \in I}, p\right) \in (C \times \boldsymbol{R}_+ \times T)^I \times P$. This profile, however, evolves as time passes, in the way that can be formulated as follows. Write $S = (C \times \boldsymbol{R}_+ \times T)^I \times P$ and define $S^T$ as the set of all sequences in $S$ (mappings from $T$ into $S$). We thus write $(s(t))_{t \in T} = (s(0), s(1), s(2), \dots) \in S^T$ with $s(t) \in S$ for every $t$. Each element of $S$ specifies the state of a given period, while each element of $S^T$ specifies the state of the entire time span. We shall thus refer to each element of $S$ as a state and each element of $S^T$ as a state process. A typical element of $S$ we consider is of the form $\left(\left(e_i^t, k_i(t), (t_i - t)^+\right)_{i=1,2,\dots,I}, p^t\right)$. Here, $e_i^t = (e_i(t), e_i(t+1), e_i(t+2), \dots)$ and this is member $i$'s income process from period $t$ onwards; $(t_i - t)^+ = \max\{t_i - t, 0\}$ and this is his time, on period $t$, to retirement; $p^t = (p(t), p(t+1), p(t+2), \dots)$ and this is the price process from period $t$ onwards; and $k_i(t)$ is his deposit kept in the fund on (the beginning of) period $t$, to be calculated in the next section. In the subsequent analysis, we shall occasionally use $e_i$, $a_i$, $t_i$ to denote generic elements of $C$, $\boldsymbol{R}_+$, and $t_i \in T$, not necessarily those characterizing member $i$, but this will cause no confusion.

# 3 Pension schemes

In this section, we formulate an anonymous pension scheme, which is applied to all members equally and, as such, any two members of the same income process and the same retirement age would necessarily have the same consumption process. We also assume that the contribution that each member makes to the fund on each period depends on the income he obtains on that period, but not the income on the other periods. We then formally analyze the allocation of consumption processes attained by such a pension scheme, assuming that the fund does not revise the scheme once it is determined on period zero.

This situation can be formulated as follows. Let $h : \boldsymbol{R}_+ \times T \to \boldsymbol{R}$ and assume that $0 \leq h(x,t) \leq x$ for every $(x,t) \in \boldsymbol{R}_+ \times T$. Denote by $H$ the set of all functions $h : \boldsymbol{R}_+ \times T \to \boldsymbol{R}$ that satisfy this condition, and call each element of $H$ an anonymous pension scheme, or, simply, a pension scheme. We interpret $h(x,t)$ as the contribution that each member $i$ makes to the fund on period $t < t_i$ when his income on that period is equal to $x$. Then consumer $i$ consumes $e_i(t) - h(e_i(t),t)$ on period $t$. Up to (but not including) period $t_i$, he has accumulated the contributions

$$a_i + \sum_{\tau=0}^{t_i} p(\tau) h(e_i(\tau),\tau)$$

in the nominal term (that is, the monetary value under the price process $p$). In this scheme, he receives the benefit $b_i \in \boldsymbol{R}_+$ constantly on each period from period $t_i$ onwards such that

$$\sum_{\tau=t_i}^{\infty} p(\tau) b_i = a_i + \sum_{\tau=0}^{t_i-1} p(\tau) h(e_i(\tau),\tau),$$

that is,

$$b_i = \frac{a_i + \displaystyle\sum_{\tau=0}^{t_i-1} p(\tau) h(e_i(\tau),\tau)}{\displaystyle\sum_{\tau=t_i}^{\infty} p(\tau)}. \tag{1}$$

The resulting consumption process $c_i$ is given by

$$c_i(t) = \begin{cases} e_i(t) - h(e_i(t),t) & \text{if } t < t_i, \\ e_i(t) + b_i & \text{if } t \geq t_i, \end{cases} \tag{2}$$

where $b_i$ is defined by (1). Then $c_i \in C$. It is convenient to give a name to the allocation thus generated, making explicit reference to the initial state $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right)$.

**Definition 1** Let $h \in H$, $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right) \in S$, and $c = (c_1, c_2, \ldots, c_I) \in C^I$. We say that the allocation $c$ of consumption processes is *generated by the scheme $h$ without revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right)$ if for every $i$, $c_i$ satisfies (1) and (2).

Note that

$$p \cdot c_i = p \cdot e_i + a_i \tag{3}$$

for every $i$. To grasp the nature of the consumption process attained by a pension scheme, denote by $c_i^h$ the consumption process defined by (2) and by $c_i^*$ the solution to the standard utility maximization problem of maximizing $U_i(c_i)$ subject to (3), if it exists, which is what member $i$ would choose were he to have full access to financial markets. Then $c_i^h(t) > 0$ (unless $b_i = 0$) and $c_i^*(t)$ (by the Inada condition) for every $t \geq t_i$. Since $e_i(t) = 0$ for every $t \geq t_i$, $e_i(t) - c_i^h(t) < 0$ and $e_i(t) - c_i^*(t) < 0$ for every $t \geq t_i$. Thus, both $c_i^h$ and $c_i^*$ exhibit demand for consumption smoothing. The difference lies in the possible signs of the savings $e_i(t) - c_i^h(t)$ and $e_i(t) - c_i^*(t)$ on periods $t < t_i$ prior to retirement. The former must not be negative, because the pension contribution must not be negative, while the latter may be negative, because member $i$ can borrow money on period $t$ at financial markets, to which he has full access. Now consider another member $j$ and define $c_j^h$ and $c_j^*$ in the same way as we defined $c_i^h$ and $c_i^*$ for member $i$. We can then see another type of difference. If $e_i = e_j$ and $t_i = t_j$, that is, if the two members earn the same income on every period and retire on the same period, then $c_i^h = c_j^h$, that is, they must necessarily enjoy the same consumption process. In general, however, $c_i^* \neq c_j^*$ unless $U_i = U_j$. That is, if their discount rates or felicity functions are different, then they would typically choose different consumption processes were they to have full access to financial markets. Such a difference does not exist in the consumption processes generated by pension schemes, because the amount of contributions specified by pension schemes may depend on incomes but not on preferences.

It will turn out to be useful to define a process that represents the deposits member $i$ has in the fund, and write the generated consumption process in terms of the deposit process. Then $k_i^h(t)$ is the deposit for member $i$, the difference between the contributions that member $i$ has accumulated and the benefits he has received so far up to (but not including) period $t$, measured in the nominal term. As a convention, which is consistent with the definition of $a_i$, we let $k_i^h(0) = a_i$.

**Definition 2** Let $h \in H$, $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right) \in S$, $c = (c_1, c_2, \ldots, c_I) \in C^I$, $k = (k_1, k_2, \ldots, k_I) \in C^I$, and $(s(t))_{t \in T} \in S^T$. We say that the allocations $c$ and $k$ of consumption and deposit processes and the state process $(s(t))_{t \in T}$ are *generated by the scheme $h$ without revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right)$ if for every $i$, $k_i(0) = a_i$ and

$$c_i(t) = \begin{cases} e_i(t) - h(e_i(t), t) & \text{if } t < t_i, \\ e_i(t) + \dfrac{k_i(t)}{\sum_{\tau=t}^{\infty} p(\tau)} & \text{if } t \geq t_i, \end{cases} \tag{4}$$

$$k_i(t+1) - k_i(t) = p(t)\left(e_i(t) - c_i(t)\right). \tag{5}$$

$$s(t) = \left(\left(e_i^t, k_i(t), (t_i - t)^+\right)_{i=1,2,\ldots,I}, p\right)$$

for every $t \in T$.

In the sequel, the allocations of consumption and deposit processes in Definition 2 are denote by $c^h$ and $k^h$. The following lemma shows that Definitions 1 and 2 are in fact equivalent.

**Lemma 1** *Let $h \in H$, $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right) \in S$, and $c = (c_1, c_2, \ldots, c_I) \in C^I$. The allocation $c$ of consumption processes is generated by the scheme $h$ without revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right)$ if and only if there is a unique $k = (k_1, k_2, \ldots, k_I) \in C^I$ such that the allocations $c$ and $k$ of consumption and deposit processes are generated by the scheme $h$ without revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right)$.*

We denote $c$ and $k$ in this lemma by $c^h$ and $k^h$ to make their dependence on $h$ explicitly, although this notation keeps their dependence on state $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$ implicit.

**Proof of Lemma 1** It suffices to prove that if $b_i$ is defined by (1) for every $i$, and the allocations $c$ and $k$ of consumption and deposit processes are generated by the scheme $h$ without revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$, then

$$b_i = \frac{k_i(t)}{\sum_{\tau=t}^{\infty} p(\tau)}$$

for every $t \geq t_i$. By induction, $k_i(t_i) = a_i + \sum_{\tau=0}^{t_i-1} p(\tau)h(e_i(\tau), \tau)$. Thus, the above equality holds when $t = t_i$. Assume that it holds for $t$ and prove it for $t+1$. Then,

$$\frac{k_i(t+1)}{\sum_{\tau=t+1}^{\infty} p(\tau)} = \frac{k_i(t) - p(t)b_i}{\sum_{\tau=t}^{\infty} p(\tau) - p(t)} = \frac{\left(\sum_{\tau=t}^{\infty} p(\tau)\right) b_i - p(t)b_i}{\sum_{\tau=t}^{\infty} p(\tau) - p(t)} = b_i.$$

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The advantage of Definition 2 over Definition 1 is the following inductive (recursive) property. The proof is easy, which we omit.

**Lemma 2** *Let $h \in H$, $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right) \in S$, $c = (c_1, c_2, \dots, c_I) \in C^I$, and $k = (k_1, k_2, \dots, k_I) \in C^I$. Then the allocations $c$ and $k$ of consumption and deposit processes are generated by the scheme $h$ without revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$ if and only if the allocations $c^t$ and $k^t$ of consumption and deposit processes are generated by the scheme $h^t$ without revision from the initial state $\left((e_i^t, a_i^t, (t_i - t)^+)_{i=1,2,\dots,I}, p^t\right)$ for every $t$.*

An important implication of this lemma is that

$$p^t \cdot e_i^t + k_i(t) = p^t \cdot c_i^t, \tag{6}$$

which shows that the market value of current and future consumptions is always equal to the sum of the market value of current and future incomes and the deposit kept in the fund. It generalizes the equality (3) on period zero to the analogous one on every period $t$.

## 4 Pension scheme with revision

To implement a pension scheme $h \in H$, the fund needs to commit itself to the scheme over the entire time span. However, there are some cases where the fund cannot commit itself to any scheme and, instead, revises the scheme it has chosen before. In this section, we define the consumption processes would actually be attained in such a situation, in a way analogous to Definition 2. Denote the set of all sequences in $H$ (mappings of $T$ into $H$) by $H^T$.

**Definition 3** Let $(h_t)_{t\in T} \in H^T$, $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right) \in S$, $c = (c_1, c_2, \dots, c_I) \in C^I$, $k = (k_1, k_2, \dots, k_I) \in C^I$, and $(s(t))_{t\in T} \in S^T$. We say that the allocations $c$ and $k$ of consumption and deposit processes and the state process $(s(t))_{t\in T}$ are *generated by the sequence of pension schemes,* $(h_t)_{t\in T}$*, with revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$ if for every $i$, $k_i(0) = a_i$ and

$$c_i(t) = \begin{cases} e_i(t) - h_t(e_i(t), 0) & \text{if } t < t_i, \\ e_i(t) + \dfrac{k_i(t)}{\displaystyle\sum_{\tau=t}^{\infty} p(\tau)} & \text{if } t \geq t_i, \end{cases} \tag{7}$$

$$k_i(t+1) - k_i(t) = p(t)\left(e_i(t) - c_i(t)\right). \tag{8}$$

$$s(t) = \left(\left(e_i^t, k_i(t), (t_i - t)^+\right)_{i=1,2,\dots,I}, p^t\right)$$

for every $t \in T$.

According to this definition, on each period $t$, the fund intends to implement a scheme $h_t \in H$ and does indeed implement it on that period, but once the next period $t+1$ comes, he scraps his own decision on the previous period, intends to implement another scheme $h_{t+1} \in H$, and does indeed implement it on period $t+1$. Note that $c_i(t)$ does not depend on $h_\tau$ if $t_i \leq \tau \leq t$, that is, any member's post-retirement consumption is not affected by any subsequent change in pension schemes. The proof of the next lemma is easy, which we omit.

**Lemma 3** *Let* $(h_t)_{t\in T} \in H^T$*,* $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right) \in S$*,* $c = (c_1, c_2, \dots, c_I) \in C^I$*, and* $k = (k_1, k_2, \dots, k_I) \in C^I$*. Suppose that the allocations* $c$ *and* $k$ *of consumption and deposit processes are generated by the sequence of pension schemes,* $(h_t)_{t\in T}$*, with revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$*. Then, for every* $t \in T$*, the allocations* $c^t = (c_1^t, c_2^t, \dots, c_I^t)$ *and* $k^t = (k_1^t, k_2^t, \dots, k_I^t)$ *of consumption and deposit processes are generated by the sequence of pension schemes,* $(h_\tau)_{\tau\geq t}$*, with revision from the initial state* $\left(\left(e_i^t, k_i(t), (t_i - t)^+\right)_{i=1,2,\dots,I}, p^t\right)$*.*

The next proposition shows that the resulting consumption allocation can also be generated by a pension scheme to which the fund commit itself.

**Proposition 1** *Let* $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right) \in S$ *and* $(h_t)_{t\in T} \in H^T$*. Let* $c$ *and* $k$ *be the allocations of consumption and deposit processes generated by* $(h_t)_{t\in T}$ *with revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$*. Define* $h \in H$ *by* $h(x,t) = h_t(x, 0)$ *for every* $(x,t) \in \boldsymbol{R}_+ \times T$*. Let* $c^h$ *and* $k^h$ *be the allocations of consumption and deposit processes generated by* $h$ *with revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$*. Then* $c = c^h$ *and* $k = k^h$*.*

There are two important implications of this proposition. First, $p \cdot c_i = a_i + p \cdot e_i$, because $p \cdot c_i^h = a_i + p \cdot e_i$. Second, to make any welfare comparison, it suffices to compare allocations that can be generated by pension schemes that are implemented throughout the entire time span.

**Proof of Proposition 1** For every $t < t_i$, by (7), (4), and the definition of $h$, $c_i(t) = c_i^h(t)$. Thus, by (8) and (5), $k_i(t+1) - k_i(t) = k_i^h(t+1) - k_i^h(t)$. Since $k_i(0) = a_i = k_i^h(0)$, $k_i(t) = k_i^h(t)$ for every $t \leq t_i$. As for $t \geq t_i$, we can show, by induction using (8) and (5), that $k_i(t) = k_i^h(t)$. Thus, by (7) and (2), $c_i(t) = c_i^h(t)$ for every $t > t_i$. ///

# 5 Pension fund's objective functions

In this section, we formulate the pension fund's objective functions and postulate that the fund chooses pension schemes to maximize its objective function. Recall that $U_i(C)$ is the range of member $i$'s utility function. Denote by $\mathscr{W}$ the set of the functions $W : \prod_i U_i(C) \to \boldsymbol{R}$. The welfare evaluation of the allocation $c = (c_1, c_2, \ldots, c_I) \in C^I$ of consumption processes is given by $W(U_1(c_1), U_2(c_2), \ldots, U_I(c_I))$. We regard $\mathscr{W}$ as the set of the objective function that the fund may have and also of the social welfare function with respect to which we assess the desirability of consumption allocations.[1] Here are some examples of elements of $\mathscr{W}$.

**Example 1** 1. There is a $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_I) \in \boldsymbol{R}_+^I$ such that $W(r) = \sum_i \lambda_i m_i$ for every $m = (m_1, m_2, \ldots, m_I) \in \boldsymbol{R}^I$. Restrictive as it may seem, this functional form covers many consumption allocations of interest. Let's say a pension scheme $h^*$ is efficient if there is no other pension scheme $h$ such that $U_i(c_i^h) \geq U_i(c_i^{h^*})$ for every $i$ and $U_i(c_i^h) > U_i(c_i^{h^*})$ for some $i$. Write

$$M = \left( \left\{ \left( U_1\left(c_1^h\right), U_2\left(c_2^h\right), \ldots, U_I\left(c_I^h\right) \right) \in \boldsymbol{R}^I \mid h \in H \right\} - \boldsymbol{R}_+^I \right) \bigcap \boldsymbol{R}_+^I,$$

then an equivalent definition is that

$$M \bigcap \left( \left\{ \left( U_1\left(c_1^{h^*}\right), U_2\left(c_2^{h^*}\right), \ldots, U_I\left(c_I^{h^*}\right) \right) \right\} + \left( \boldsymbol{R}_+^L \setminus \{0\} \right) \right) = \varnothing.$$

Since $H$ is a convex subset of the set of all real-valued functions defined on $\boldsymbol{R}_+ \times T$, the mapping $h \mapsto c^h$ is affine, and the $U_i$ are concave, $M$ is a convex subset of $\boldsymbol{R}^I$. Hence, by the separation hyperplane theorem, for every efficient scheme $h^*$, there is a $\lambda \in \boldsymbol{R}_+^I \setminus \{0\}$ such that $h^*$ is a solution to the problem of maximizing $\sum_i \lambda_i U_i(c_i^h)$ subject to $h \in H$. Therefore, every efficient allocation is a solution to the problem of maximizing an objective function of the form $W(m) = \sum_i \lambda_i m_i$. Note, however, that $W$ is strictly increasing if and only if $\lambda \in \boldsymbol{R}_{++}^I$. This welfare function is often used to establish the existence of a competitive equilibrium in general equilibrium theory via Negishi's (1960) approach. In that approach, the coefficients $\lambda_i$ are determined so that the market value

[1] More generally, we could formulate the fund's objective in terms of a binary relation on $\boldsymbol{R}^I$. By doing so, we could accommodate, for example, the leximin ordering, on which the solution to the Nash bargaining problem by Imai (1983) is based.

of the consumption vector that each consumer receives at the Pareto efficient allocation corresponding to the $\lambda_i$ is equal to that of his initial endowment vector. In our modoel, since the market value of the benefit each member receives after retirement is always equal to that of the contributions he makes before retirement regardless of the values of the $\lambda_i$, we cannot pin down the values of the $\lambda$ to use to find the optimal pension scheme. For this reason, we will not use $W(m) = \sum_i \lambda_i m_i$ as the fund's objective function.

2. Although every member is assumed to maintain the membership (subscription) of the pension fund throughout the entire time span, let's consider what he would obtain if he quit. If member $i$ quits on period zero, then he would receive $a_i$ (in the nominal term) from the fund. He can consume it immediately, but cannot save it for future consumption, because he has no access to asset markets by assumption. Thus, the consumption process he would attain is $\left(e_i(0) + (p(0))^{-1}a_i, e_i(1), e_i(2), \dots\right)$. Write $\underline{m}_i = U_i\left(e_i(0) + (p(0))^{-1}a_i, e_i(1), e_i(2), \dots\right)$ and define $W$ by letting $W(m) = \prod_i (m_i - \underline{m}_i)$ for every $m = (m_1, m_2, \dots, m_I) \in \boldsymbol{R}^I$. Although this function is strictly increasing on the set $\prod_i (\underline{m}_i, \infty)$ but not on the entire $\boldsymbol{R}^I$, it is an important objective function because it is the solution obtained by Nash (1950) to the bargaining problem, for which the disagreement point is given by the utility levels that the members would obtain if they quit.

3. In the Nash bargaining problem, the convexity of the utility possibility set is justified by interpreting that the bargainers have expected utility functions and bargain over lotteries defined on the set of (deterministic) consequences; and the scale invariance of the Nash bargaining solution is justified by noticing that once we accept this interpretation, every affine transformation of an expected utility function represents the same risk attitudes. In fact, Kihlstrom, Roth, and Schmeidler (1981) showed that if the two-person bargaining problem is modified by replacing a bargainer by another one who is more risk averse, while retaining the same set of consequences, then the other bargainer, who has the same utility function before and after the modification, obtains a higher utility at the Nash bargaining solution after modification. Preceding to them, Aumann and Kurz (1977) introduced the concept of fear of ruin, which measures the unwillingness to take large risks, and proved the the bargainer with a lower fear of ruin obtains a higher utility.

   In our model, if we consider the Nash bargaining solution based on the utility function $U_i$, as we did in part 2, then we are presuming that the risk attitude of member $i$ is represented by $u_i$. This presumption may well turn out to be inappropriate. First, since what the members actually receive are deterministic consumption processes, there is no apparent reason why we should include their risk attitudes as one of the determinants of the optimal pension scheme. Second, even if we should, for whatever reason, take the member's risk attitudes into consideration, there is no reason why we should use $U_i$ (or, equivalently, $u_i$) for member $i$, because his risk attitude may well be different from the attitude that is represented by $u_i$, which is used to represent intertemporal elasticity of substitution. In other words, member $i$ may not have time-additive expected utility functions, and may well have recursive utility functions of the type of Epstein and Zin.

Without knowing what the members' risk attitudes are, it would perhaps be most straightforward to assume that the members are as close to risk-neutrality as is consistent with the preferences represented by the $U_i$. To do so, we use a strictly increasing transformation $\hat{U}_i$ of $U_i$ that is least concave, that is, if $\tilde{U}_i$ is another strictly transformation of $U_i$ and concave, $\psi$ is concave, and $\hat{U}_i = \psi \circ \tilde{U}_i$, then $\psi$ must in fact be affine. A least concave utility function can easily be constructed if the function

$$y \mapsto \frac{\left(u_i'\left(u_i^{-1}(y)\right)\right)^2}{u_i''\left(u_i^{-1}(y)\right)} \tag{9}$$

of $u_i(\boldsymbol{R}_+)$ into $\boldsymbol{R}$ is convex. The proof goes as follows. For each $i$ and $c_i \in C$, define $\hat{U}_i(c_i)$ so that $(1-\exp(-\rho_i))^{-1} u_i(\hat{U}_i(c_i)) = U_i(c_i)$, or $\hat{U}_i(c_i) = u_i^{-1}\left((1-\exp(-\rho_i))U_i(c_i)\right)$. By modifying the analysis in Section 3.16 of Hardy, Littlewood, and Polya (1952), we can show that $\hat{U}_i$ is concave if and only if the function (9) is convex. It is then easy to show that $\hat{U}_i$ is a least concave utility function. Note that by the definition of $\hat{U}_i$, if $b = (1, 1, \dots) \in C$ and $\kappa \in \boldsymbol{R}_+$, then $\hat{U}_i(\kappa b) = \kappa$ for every $i$. Thus, the least concave utility functions $\hat{U}_i$ are all normalized so that all members assign the same utility level to each constant consumption stream. It is also easy to check that the function (9) is convex if $u_i$ exhibits constant absolute risk aversion ($-u_i''(x_i)/u_i'(x_i)$ does not depend on $x_i$) or constant relative risk aversion ($-u_i''(x_i)x_i/u_i'(x_i)$ does not depend on $x_i$). We can then apply the Nash bargaining solution to the $\hat{U}_i$.

4. Define

$$\underline{M} = M \bigcap \left( \prod_{i=1}^{I} [\underline{m}_i, \infty) \right),$$

where $\underline{m}_i$ is defined in part 2. This is compact because $M$ is closed and bounded from above. Denote the maximum value of the $i$-th coordinate by $\overline{m}_i$. This is the maximum utility that member $i$ can receive from a pension scheme for which no member has incentive to quit the fund. Assume that $\overline{m}_i > \underline{m}_i$ for every $i$. Define $W$ by letting

$$W(m) = \min \left\{ \frac{m_1 - \underline{m}_1}{\overline{m}_1 - \underline{m}_1}, \frac{m_2 - \underline{m}_2}{\overline{m}_2 - \underline{m}_2}, \dots, \frac{m_I - \underline{m}_I}{\overline{m}_I - \underline{m}_I} \right\}$$

for every $m = (m_1, m_2, \dots, m_I) \in \boldsymbol{R}^I$. If $I = 2$, then the solution to the problem of maximizing $W(m)$ subject to $m \in \underline{M}$ gives the solution to the Nash bargaining problem by Kalai and Smorodinsky (1975).

If the fund has the objective function $W$ and no plan to revise the scheme it chooses on period 0, then its maximization problem can be formulated as

$$\begin{aligned}
\max_{(c,k,h)\in C^I\times C^I\times H} \quad & W(U_1(c_1),U_2(c_2),\dots,U_I(c_I)), \\
\text{subject to} \quad & c \text{ and } k \text{ are generated by } h \\
& \text{without revision from } s,
\end{aligned} \tag{10}$$

If $(c,k,h)$ is a solution to this problem, then $c = c^h$ and $k = k^h$, that is, $c$ and $k$ are uniquely determined by $h$. Thus, in such a case, we simply say that $h$ is a solution to (11). By suppressing the state $s$ from which the processes $c$ and $k$, we can state this maximization problem more simply as

$$\max_{h\in H} \quad W(U_1(c_1),U_2(c_2),\dots,U_I(c_I)). \tag{11}$$

# 6 Pension fund's state-dependent objective functions

In many cases of interest, the fund's objective function changes from period to period depending on the states to be realized. This is true particularly when the objective functions are imposed on the fund by some other entity, such as the government. To formulate such objective functions, we denote by $\mathscr{W}^S$ the set of all mappings from $S$ into $\mathscr{W}$. Each element of $\mathscr{W}^S$ is regarded a profile of objective functions that depends on states. We shall thus call it simply a state-dependent objective function, and write it as $(W(\cdot\,|\,s))_{s\in S}$, with $W(\cdot\,|\,s)\in\mathscr{W}$ for each $s\in S$.

We have already encountered a state-dependent objective function. Indeed, in part 2 of Example 1, the objective function was defined through the utility levels $\underline{m}_i$ at the disagreement points determined by the consumption processes $\big(e_i(0)+(p(0))^{-1}a_i, e_i(1), e_i(2),\dots\big)$, which depends on the state $s = \Big((e_i,a_i,t_i)_{i=1,2,\dots,I}\,, p\Big)$.

For a state-dependent utility function $(W(\cdot\,|\,s))_{s\in S}$, we define a resulting consumption allocation $c^* = (c_1^*,c_2^*,\dots,c_I^*)\in C^I$, along with the allocation $k^* = (k_1^*,k_2^*,\dots,k_I^*)\in C^I$ of deposit processes, the sequence $(h_t)_{t\in T}\in H^T$ of pension schemes, and the state process $(s(t))_{t\in T}\in S^T$, as follows. To start, we let $k_i^*(0)=a_i$ and $s(0) = \big((e_i,k_i^*(0),t_i)_{i=1,2,\dots,I}\,,p\big)$. Let $t\in T$ and suppose that $k_i^*(t)$ has been determined for every $i$. Then we define $s(t) = \big((e_i^t,k_i^*(t),(t_i-t)^+)_{i=1,2,\dots,I}\,,p^t\big)$. Then let $(c^{h_t},k^{h_t},h_t)$ be a solution to the problem

$$\begin{aligned}
\max_{(c,k,h)\in C^I\times C^I\times H} \quad & W(U_1(c_1),U_2(c_2),\dots,U_I(c_I)\,|\,s(t)), \\
\text{subject to} \quad & c \text{ and } k \text{ are generated by } h \\
& \text{without revision from } s(t).
\end{aligned} \tag{12}$$

Then we let $c^*(t) = c^{h_t}(0)$, $k^*(t{+}1) = k^{h_t}(1)$, and $s(t{+}1) = \big(\big(e_i^{t+1},k_i^*(t+1),(t_i-(t+1))^+\big)\,,p^{t+1}\big)$. Thus, we have inductively defined $c^*$, $k^*$, $(h_t)_{t\in T}$, and $s$. The following definition gives a name to the allocations of consumption and deposit processes thus generated.

**Definition 4** Let $(W(\cdot\,|\,s))_{s\in S}\in\mathscr{W}^S$, $\Big((e_i,a_i,t_i)_{i=1,2,\dots,I}\,,p\Big)\in S$, $c^* = (c_1^*,c_2^*,\dots,c_I^*)\in C^I$, $k^* = (k_1^*,k_2^*,\dots,k_I^*)\in C^I$, and $(h_t)_{t\in T}\in H^T$. We say that the allocations $c^*$ and $k^*$ of consumption and deposit processes and the process $(h_t)_{t\in T}$ of pension schemes are *generated by the state-dependent objective function $(W(\cdot\,|\,s))_{s\in S}$ with revision from the initial state* $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}\,, p\right)$ if $c^*$, $k^*$, and $(h_t)_{t\in T}$ are defined via (12).

According to the above definition, in each state $s(t)$, the fund uses an objective function $W(\cdot\,|\,s(t))$, chooses a pension scheme $h_t$ that maximizes the value of the objective function, and implement the scheme only on that period; on the next period, the fund scraps the scheme it chose on the previous period, uses a new objective function $W(\cdot\,|\,s(t+1))$, chooses a new pension scheme $h_{t+1}$ that maximizes the value of the new objective function, and implement the new scheme only on that period; and this behavior is repeated indefinitely. Along with the generated consumption and deposit processes $c^*$ and $k^*$, a process of objective functions, $(W(\cdot\,|\,s(t)))_{t\in T}$, is also generated.

The behavior is considered as naive and myopic, as it does not take into consideration the fact that the current pension scheme is scrapped on the next period. An alternative formation is that when solving the maximization problem, the fund takes into consideration the fact that it will scrap the scheme and implement the new scheme. We do not employ this formulation because we believe that our formulation has more practical relevance.

The following lemma is straightforward. We omit its proof.

**Lemma 4** *Let $(W(\cdot\,|\,s))_{s\in S} \in \mathscr{W}^S$, $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}\,, p\right) \in S$, $c^* \in C^I$, $k^* \in C^I$, and $(h_t)_{t\in T} \in H^T$. If the allocations $c^*$ and $k^*$ of consumption and deposit processes and the process $(h_t)_{t\in T}$ of pension schemes are generated by the state-dependent objective function $(W(\cdot\,|\,s))_{s\in S}$ with revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}\,, p\right)$, then the allocations $c^*$ and $k^*$ of consumption and deposit processes are generated by the process $(h_t)_{t\in T}$ of pension schemes with revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}\,, p\right)$,*

Since the objective functions $W(\cdot\,|\,s(t))$ keep changing as time elapses, and we are interested in the dynamic inconsistency in the fund's choices that lead to a suboptimal consumption allocation. More specifically, let $t \in T$ and $\tau \in T$ with $t < \tau$, and compare $h_t(\cdot\,, \tau - t)$ and $h_\tau(\cdot\,, 0)$. The former is the pension scheme that is planned on period $t$ on the contributions to be made on period $\tau$. The latter is the pension scheme that is (chosen and) implemented on period $\tau$. If the two are different, then we say that the process of pension schemes is dynamically inconsistent.

We now introduce a measure of dynamic inconsistency for consumption allocations that are generated by a state-dependent objective function. Let $(W(\cdot\,|\,s))_{s\in S} \in \mathscr{W}^S$ and $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}\,, p\right) \in S$, which we denote by $s$, and let $c^* \in C^I$. Suppose that the allocation $c^*$ of consumption processes is generated by the state-dependent objective function $(W(\cdot\,|\,s))_{s\in S}$ with revision from the initial state $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}\,, p\right)$. Then member $i$ attains the utility level $\hat{U}_i\left(c_i^*\right)$. Our measure of inefficiency, or dynamic inconsistency, is defined along the lines of Debreu (1951), as follows. For each $\theta \geq 0$, define $s^\theta \in S$ by $s^\theta = \left((\theta e_i, \theta a_i, t_i)_{i=1,2,\ldots,I}\,, p\right)$. That is, every member's initial income process and initial deposit are scaled down by the common factor $\theta$. Denote by $\Theta_1$ the set of all $\theta \geq 0$ for which there is an $h^\theta \in H$ such that $\hat{U}_i\left(c_i^\theta\right) \geq \hat{U}_i\left(c_i^*\right)$, where

$\left(c_i^\theta\right)_i$ is the allocation of consumption processes generated by $h^\theta$ without revision starting from $s^\theta$. In other words, we let $\theta \in \Theta_1$ if and only if the fund can guarantee the utility levels that the members attain at the allocation $c^*$ even when the initial endowments and deposits are scaled down by factor $\theta$, as long as it can commit itself to implementing whatever scheme it decides on period zero. Define $\theta_1^* = \inf \Theta_1$, and we let $1 - \theta_1^*$ as a measure of dynamic inconsistency. It is the maximum fraction of the resources that can be given up if the fund can commit itself to any scheme it chooses on period zero, while still guaranteeing the utility levels that the members would receive at the allocation generated by the state-dependent objective function.

An alternative measure of dynamic inconsistency can be defined as follows. Denote by $\Theta_2$ the set of all $\theta \geq 0$ for which there is an $h^\theta \in H$ such that $W(U_1(c_1^\theta), U_2(c_2^\theta), \ldots, U_I(c_I^\theta) \mid s) \geq W(U_1(c_1^*), U_2(c_2^*), \ldots, U_I(c_I^*) \mid s)$, where $s = \left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right)$ and $c^\theta$ is the allocation of consumption processes generated by $h^\theta$ without revision starting from $s^\theta = \left((\theta e_i, \theta a_i, t_i)_{i=1,2,\ldots,I}, p\right)$. In other words, we let $\theta \in \Theta_2$ if and only if the fund can guarantee the level that the objective function attains at the allocation $\left(c_i^\theta\right)_i$ even when the initial endowments and deposits are scaled down by factor $\theta$, as long as it can commit itself to implementing whatever scheme it decides on period zero. Note that we use the initial state $s$, not the scaled-down state $s^\theta$, in the objective function $W(\cdot \mid s)$. Define $\theta_2^* = \inf \Theta_2$, and we let $1 - \theta_2^*$ as a measure of dynamic inconsistency. If $W$ is nondecreasing, then $\Theta_1 \subseteq \Theta_2$. Hence $\theta_1^* \geq \theta_2^*$ and $1 - \theta_1^* \leq 1 - \theta_2^*$. That is, the second measure of dynamic inconsistency is no smaller than the first. A more conceptual difference between the two measures is that while the second measure relies on the objective function $W(\cdot \mid s)$, which may itself be dynamically inconsistent in the sense of Jackson and Yariv (2014), the first measure relies only on the utility functions $\hat{U}_i$, which is dynamically consistent.

## 7 Case of homogeneous members

If the members are sufficiently homogeneous, then the choice of the objective function does not affect what the fund should choose. We now give two sufficient conditions for this to be the case. We say that $U_i$ exhibits constant relative risk aversion if so does the felicity function $u_i$, that is, $-u_i''(x_i)x_i/u_i'(x_i)$ does not depend on $x_i$. Since the domain of $u_i$ is assumed to contain 0, the constant must necessarily be less than one. We say that a pension scheme $h$ is linear if $h(\cdot, t) : \boldsymbol{R}_+ \to \boldsymbol{R}$ is linear for every $t \in T$. Denote by $H^{\mathrm{L}}$ the set of all linear pension schemes.

**Proposition 2** *Let* $\left((e_i, a_i, t_i)_{i=1,2,\ldots,I}, p\right) \in S$ *and suppose that* $U_1 = U_2 = \cdots = U_I$ *and* $t_1 = t_2 = \cdots = t_I$. *Suppose also that for each* $i$, *there is a* $\theta_i > 0$ *such that* $\theta_1^{-1}e_1 = \theta_2^{-1}e_2 = \cdots = \theta_I^{-1}e_I$ *and* $\theta_1^{-1}a_1 = \theta_2^{-1}a_2 = \cdots = \theta_I^{-1}a_I$. *If, in addition, either of the following two additional conditions is met, then there is an* $h \in H^{\mathrm{L}}$ *such that for every* $V \in \mathscr{W}$, $h$ *is a solution to* (11). *Moreover, if* $\hat{h}$ *is another solution, then* $c^h = c^{\hat{h}}$ *and* $k^h = k^{\hat{h}}$.

1. $\theta_1 = \theta_2 = \cdots = \theta_I$.

2. *The* $U_i$ *exhibit constant relative risk aversion.*

**Proof of Proposition 2** For each $(e_i, a_i, t_i, p) \in C \times \boldsymbol{R}_+ \times T \times P$, define $D(e_i, a_i, t_i, p)$ as the set of all $c_i \in C$ such that $c_i(t) \leq e_i(t)$ for every $t < t_i$ and

$$c_i(t) = e_i(t) + \frac{a_i + \sum_{\tau=0}^{t_i-1} p(\tau)(e_i(\tau) - c_i(\tau))}{\sum_{\tau=t_i}^{\infty} p(\tau)}$$

for every $t \geq t_i$. Then $D(e_i, a_i, t_i, p)$ is nonempty and convex. Since, for every $c_i \in D(e_i, a_i, t_i, p)$, $c_i(t_i) = c_i(t_i{+}1) = c_i(t_i{+}2) = \cdots$ and these values are uniquely determined by $c_i(0), c_i(1), \ldots, c_i(t_i - 1)$, it can be easily shown that $D(e_i, a_i, t_i, p)$ is compact with respect to the sup-norm topology. Since $U_i$ is continuous with respect to the sup-norm topology, there is a solution to the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i, a_i, t_i, p)$. Since $U_i$ is strictly concave and $D(e_i, a_i, t_i, p)$ is convex, the solution is unique. Moreover, for every $c_i \in C$, there is an $h \in H$ such that $c_i = c_i^h$ starting from state $(e_i, a_i, t_i, p)$ if and only if $c_i \in D(e_i, a_i, t_i, p)$. Furthermore, if $e_i(t) > 0$, denote this value by $\kappa(t)$, then $\kappa(t) \in [0, 1]$. If $e_i(t) = 0$, then let $\kappa(t)$ be any value in $[0, 1]$. Define $h \in H$ by $h(x, t) = (1 - \kappa(t))x$ for every $(x, t) \in \boldsymbol{R}_+ \times T$. Then $h \in H^{\mathrm{L}}$ and $c^* = c^h$. That is, the solution to the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i, a_i, t_i, p)$ can be generated by a linear scheme.

1. Since $e_1 = e_2 = \cdots = e_I$, $a_1 = a_2 = \cdots = a_I$, and $t_1 = t_2 = \cdots = t_I$, $D(e_1, a_1, t_1, p) = D(e_2, a_2, t_2, p) = \cdots = D(e_I, a_I, t_I, p)$. Since $U_1 = U_2 = \cdots = U_I$, the unique solutions $c_i^*$ to the problems of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i, a_i, t_i, p)$ are all identical, and the objective function $V(c)$ in the fund's maximization problem (11) coincides with the $I$-tuple of the individual members' common utility levels $U_i(c_i)$. Let $h \in H^{\mathrm{L}}$ be such that $c_i^h = c_i^*$, then $(c^*, k^h, h)$ is the solution to (12).

2. Consider two members $i$ and $j$, with the unique solutions $c_i^*$ and $c_j^*$ to the problems of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i, a_i, t_i, p)$ and analogously for $j$. We now prove that $\theta_i^{-1} c_i^* = \theta_j^{-1} c_j^*$. To do so, note first that for every $c_j \in C$, $c_j \in D(e_j, a_j, t_j, p)$ if and only if $\theta_i \theta_j^{-1} c_j \in C_i$ because $e_i = \theta_i \theta_j^{-1} e_j$ and $a_i = \theta_i \theta_j^{-1} a_j$. Then, since $U_i = U_j$ and they are homothetic, and since $c_j^*$ is the solution in $D(e_j, a_j, t_j, p)$, $\theta_i \theta_j^{-1} c_j^*$ is the solution in $D(e_i, a_i, t_i, p)$, that is, $c_i^* = \theta_i \theta_j^{-1} c_j^*$.

Take any $i$ and let $h$ be a linear scheme that generates $c_i^*$. We now prove that $c_j^* = c_j^h$ for every $j \neq i$ as well. In fact, since $\theta_1^{-1} e_1 = \theta_2^{-1} e_2 = \cdots = \theta_I^{-1} e_I$ and $\theta_1^{-1} c_1^* = \theta_2^{-1} c_2^* = \cdots = \theta_I^{-1} c_I^*$, for every $t \in T$, if $e_i(t) > 0$ for every $i$, then $c_1^*(t)/e_1(t) = c_2^*(t)/e_2(t) = \cdots = c_I^*(t)/e_I(t)$. Thus $c_j^* = c_j^h$ for every $j \neq i$ as well. We next prove that $h$ is a solution to (12). Let $\hat{h} \in H$, then $c_i^{\hat{h}} \in D(e_i, a_i, t_i, p)$ for every $i$. Hence $U_i(c_i^{\hat{h}}) \leq U_i(c_i^*)$ for every $i$. Thus $V(c^{\hat{h}}) \leq V(c^*)$. Hence $h$ is a solution to (11). Moreover, if $\hat{h}$ is another solution, then $U_i(c_i^{\hat{h}}) = U_i(c_i^*)$ for every $i$. Hence, by the uniqueness of the solution to the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i, a_i, t_i, p)$, $c_i^{\hat{h}} = c_i^*$ for every $i$. ///

The following proposition shows that the sequence of pension schemes is dynamically consistent in the case of homogeneous members.

**Proposition 3** *If either of the two conditions of Proposition 2 is met, then for every state-dependent objective functions, the pension fund is dynamically consistent.*

**Proof of Proposition 3** Let $(V(\cdot \mid s))_{s\in S}$ be a state-dependent objective function, and let the sequence $(h_t)_{t\in T}$ of pension schemes, the allocations $c^*$ and $k^*$ of consumption and deposit processes, and the state process $s^*$ be generated by $(V(\cdot \mid s))_{s\in S}$. Let $\hat{h}$ be a solution to (11). We shall prove that $c^* = c^{\hat{h}}$ by showing by induction on $t$ that $c^*(t) = c^{\hat{h}}(t)$ for every $t$.

For each $(e_i, a_i, t_i, p) \in C \times \boldsymbol{R}_+ \times T \times P$, define $D(e_i, a_i, t_i, p)$ as in the proof of Proposition 2. It follows immediately from the definition of $k^{\hat{h}}$ that $k_1^{\hat{h}}(t) = k_2^{\hat{h}}(t) = \cdots = k_I^{\hat{h}}(t)$ for every $t$ in the first case of Proposition 2. Thus, for every $i$, the objective function $V(\cdot \mid s^{\hat{h}}(t), t)$ is a strictly increasing transformation of $U_i(c_i)$, where $c_i \in D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$. It follows from induction $t$ that $\theta_1^{-1}k_1^{\hat{h}}(t) = \theta_2^{-1}k_2^{\hat{h}}(t) = \cdots = \theta_I^{-1}k_I^{\hat{h}}(t)$ in the second case of Proposition 2. Since the $U_i$ are identical and homogeneous, for every $i$, the objective function $V(\cdot \mid s^{\hat{h}}(t), t)$ is a strictly increasing transformation of $U_i(c_i)$, where $c_i \in D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$. The same can be said of for the $k_i^*(t)$: By inductively applying the proof of Proposition 2, for every $t \in T$, $k_1^*(t) = k_2^*(t) = \cdots = k_I^*(t)$ in the first case of the proposition and $\theta_1^{-1}k_1^*(t) = \theta_2^{-1}k_2^*(t) = \cdots = \theta_I^{-1}k_I^*(t)$ and in the second case of the proposition. Hence, in either case, for every $i$, the objective function $V(\cdot \mid s^*(t), t)$ is a strictly increasing transformation of $U_i(c_i)$, where $c_i \in D(e_i^t, k_i^*(t), (t_i - t)^+, p^t)$. Thus, to consider the problem of maximizing the state-dependent objective function on each period $t$, it suffices to consider the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$ or $c_i \in D(e_i^t, k_i^*(t), (t_i - t)^+, p^t)$. In particular, for every $i$, $c_i^{\hat{h}}$ is obtained by maximizing $U_i(c_i)$ subject to $c_i \in D(e_i, a_i, t_i, p)$, and, for every $t$, $c_i^*(t)$ is obtained by maximizing $U_i(c_i)$ subject to $c_i \in D(e_i^t, k_i^*(t), (t_i - t)^+, p^t)$ and letting $c_i^*(t) = c_i(0)$ for its unique solution $c_i$.

We now prove that for every $t$, $c^{\hat{h},t}$ belongs to $D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$ and solves the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$. Indeed, if such a $c_i$ is chosen on period $t$ onwards, then the consumption process $\left(c_i^{\hat{h}}(0), c_i^{\hat{h}}(1), \ldots, c_i^{\hat{h}}(t-1), c_i\right)$ is obtained. Since $\left(c_i^{\hat{h}}(0), c_i^{\hat{h}}(1), \ldots, c_i^{\hat{h}}(t-1), c_i\right) \in D(e_i, a_i, t_i, p)$, the definition of $\hat{h}$ implies that

$$U_i\left(c_i^{\hat{h}}(0), c_i^{\hat{h}}(1), \ldots, c_i^{\hat{h}}(t-1), c_i\right) \leq U_i(c_i^{\hat{h}}).$$

This implies that $U_i(c_i) \leq U_i\left(c_i^{\hat{h},t}\right)$ because

$$U_i(c_i) = \sum_{\tau=0}^{t-1} \exp(-\rho_i(\tau))u_i(c_i(\tau)) + \exp(-\rho_i t)U_i(c_i^t)$$

for every $c_i$. Thus $c^{\hat{h},t}$ solves the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$.

By construction, for every $t$, $c_i^{h_t}$ is the unique solution to the problem of maximizing $U_i(c_i)$ subject to $c_i \in D(e_i^t, k_i^*(t), (t_i - t)^+, p^t)$. When $t = 0$, this implies that $c_i^{h_0} = c_i^{\hat{h}}$. Thus $c_i^*(0) = c_i^{\hat{t}}(0)$. Hence $k_i^*(1) = k^{\hat{h}}(1)$. Let $t \geq 1$ and suppose that $k_i^*(t) = k_i^{\hat{h}}(t)$. Then $D(e_i^t, k_i^*(t), (t_i - t)^+, p^t) = D(e_i^t, k_i^{\hat{h}}(t), (t_i - t)^+, p^t)$. Hence $c_i^{h_t} = c_i^{\hat{h},t}$. Thus $c_i^*(t) = c_i^{\hat{t}}(t)$. Hence $k_i^*(t+1) = k^{\hat{h}}(t+1)$, and the proof is completed. ///

# 8 Nash bargaining solution

In this section, we consider the case where the pension fund's objective function is the so-called Nash product, the maximization of which leads to the Nash bargaining solution. The main results of this paper are Propositions 4, 5, and 6. Together, they show that depending on the price processes $p$, all members may be equally treated and any particular member may be most favorably treated. The lemmas leading to these propositions are provided before them.

In the following lemma, we give a first-order necessary and sufficient condition for a pension scheme to generate the Nash bargaining solution. To state it, for each member $i$, each scheme $h$, and each function $\eta : \boldsymbol{R}_+ \times T \to \boldsymbol{R}_+$, if $h + \varepsilon\eta \in H$ for every sufficiently small $\varepsilon > 0$, denote by $\pi_i(\eta \,|\, h)$ the (right) derivative of the function $\varepsilon \mapsto \tilde{U}_i\left(c_i^{h+\varepsilon\eta}\right)$ evaluated at $\varepsilon = 0$, where $\tilde{U}_i = (1 - \exp(-\rho_i))U_i$. Since $\tilde{U}_i$ is differentiable and $c_i^{h+\varepsilon\eta}$ is affine in $\varepsilon$, this function is indeed differentiable and

$$
\begin{aligned}
&\pi_i(\eta \,|\, h) \\
&= -\sum_{t<t_i} (1 - \exp(-\rho_i)) \exp(-\rho_i t) u_i'\left(c_i^h(t)\right) \eta(e_i(t), t) \\
&\quad + \exp(-\rho_i t_i) u_i'\left(c_i^h(t_i)\right) \frac{\sum_{t<t_i} p(t)\eta(e_i(t), t)}{\sum_{t \geq t_i} p(t)} \\
&= \sum_{t<t_i} \left( \exp(-\rho_i t_i) u_i'\left(c_i^h(t_i)\right) \frac{p(t)}{\sum_{\tau \geq t_i} p(\tau)} - (1 - \exp(-\rho_i)) \exp(-\rho_i t) u_i'\left(c_i^h(t)\right) \right) \eta(e_i(t), t)
\end{aligned}
\tag{13}
$$

Thus $\pi_i(\eta \,|\, h)$ is linear in $\eta$. Define $\bar{p}_i \in \boldsymbol{R}_{++}^{1+t_i}$ by letting $\bar{p}_i(t) = (1 - \exp(-\rho_i)) \exp(-\rho_i t) u_i'\left(c_i^h(t)\right)$ for every $t$. Then define $\bar{p}_i \in \boldsymbol{R}_{++}^{1+t*}$ by letting $\bar{p}_i(t) = p_i(t)$ for every $t < t_*$ and $\bar{p}_i(t_*) = \sum_{\tau=t_*}^{\infty} p_i(\tau)$. Define $\bar{p} \in \boldsymbol{R}_{++}^{1+t_i}$ analogously. For each $t < t_i$, if $\eta(e_i(t), t) = 1$ and $\eta(e_i(\tau), \tau) = 0$ whenever $\tau < t_i$ and $\tau \neq t$, then

$$
\pi_i(\eta \,|\, h) = \bar{p}_i(t_i) \frac{\bar{p}(t)}{\bar{p}(t_i)} - \bar{p}_i(t). \tag{14}
$$

**Lemma 5** *Suppose that the function* (9) *is concave for every* $i$. *For each* $i$, *let*

$$
\underline{c}_i = e_i + \left( \frac{a_i}{p(0)}, 0, 0, \ldots \right) \in C.
$$

*Let $h^* \in H$ and suppose that $\hat{U}_i(c_i^{h^*}) > \hat{U}_i(\underline{c}_i)$ for every $i$. Then $h^*$ is a solution to the problem of maximizing $\prod_{i=1}^{I}\left(\hat{U}_i(c_i^h) - \hat{U}_i(\underline{c}_i)\right)$ by choosing an $h \in H$ subject to the constraint that $\hat{U}_i(c_i^h) > \hat{U}_i(\underline{c}_i)$ for every $i$ if and only if there is a $\theta = (\theta_1, \theta_2, \ldots, \theta_I) \in \boldsymbol{R}_{++}^I$ such that that*

$$\theta_1 u_1'\left(\hat{U}_1\left(c_1^{h^*}\right)\right)\left(\hat{U}_1(c_1^{h^*}) - \hat{U}_1(\underline{c}_1)\right) = \cdots = \theta_I u_I'\left(\hat{U}_I\left(c_I^{h^*}\right)\right)\left(\hat{U}_I(c_I^{h^*}) - \hat{U}_I(\underline{c}_I)\right). \tag{15}$$

*and*

$$\sum_{i=1}^{I} \theta_i \pi_i\left(h - h^* \mid h^*\right) \leq 0 \tag{16}$$

*for every $h \in H$.*

**Proof of Lemma 5** By the chain rule,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \ln\left(\prod_{i=1}^{I}\left(\hat{U}_i\left(c_i^{h^*+\varepsilon(h-h^*)}\right) - \hat{U}_i(\underline{c}_i)\right)\right)\Bigg|_{\varepsilon=0}\\
=&\sum_{i=1}^{I} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \ln\left(\hat{U}_i\left(c_i^{h^*+\varepsilon(h-h^*)}\right) - \hat{U}_i(\underline{c}_i)\right)\Bigg|_{\varepsilon=0}\\
=&\sum_{i=1}^{I} \frac{\left(u_i^{-1}\right)'\left(\tilde{U}_i\left(c_i^{h^*}\right)\right)}{\hat{U}_i\left(c_i^{h^*}\right) - \hat{U}_i\left(\underline{c}_i\right)} \pi_i(h - h^* \mid h^*)\\
=&\sum_{i=1}^{I} \frac{\pi_i(h - h^* \mid h^*)}{u_i'\left(\hat{U}_i\left(c_i^{h^*}\right)\right)\left(\hat{U}_i\left(c_i^{h}\right) - \hat{U}_i\left(c_i^{h^*}\right)\right)}.
\end{aligned}$$

Thus, if $h^*$ is a solution, then $\prod_{i=1}^{I}\left(\hat{U}_i\left(c_i^{h^*+\varepsilon(h-h^*)}\right) - \hat{U}_i(\underline{c}_i)\right) \leq 0$ for every $h \in H$ and $\varepsilon \geq 0$, and, hence,

$$\sum_{i=1}^{I} \frac{\pi_i(h - h^* \mid h^*)}{u_i'\left(\hat{U}_i\left(c_i^{h^*}\right)\right)\left(\hat{U}_i\left(c_i^{h}\right) - \hat{U}_i\left(c_i^{h^*}\right)\right)} \leq 0.$$

Thus, if we define $\theta = (\theta_1, \theta_2, \ldots, \theta_I) \in \boldsymbol{R}_{++}^I$ by letting $\theta_i = \left(u_i'\left(\hat{U}_i\left(c_1^{h^*}\right)\right)\left(\hat{U}_i(c_i^{h^*}) - \hat{U}_i(\underline{c}_i)\right)\right)^{-1}$ for each $i$, then both (15) and (16) hold.

Suppose that there is no $\theta$ for which both (15) and (16) hold. Define $\theta = (\theta_1, \theta_2, \ldots, \theta_I) \in \boldsymbol{R}_{++}^I$ by letting $\theta_i = \left(u_i'\left(\hat{U}_i\left(c_1^{h^*}\right)\right)\left(\hat{U}_i(c_i^{h^*}) - \hat{U}_i(\underline{c}_i)\right)\right)^{-1}$ for each $i$, then (15) holds. Thus, (16) must not hold, that is, there is an $h \in H$ for which (16) fails to hold. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \ln\left(\prod_{i=1}^{I}\left(\hat{U}_i\left(c_i^{h^*+\varepsilon(h-h^*)}\right) - \hat{U}_i(\underline{c}_i)\right)\right)\Bigg|_{\varepsilon=0} > 0.$$

Hence, for such an $h$ and a sufficiently small $\varepsilon > 0$, $\hat{U}_i\left(c_i^{h^*+\varepsilon(h-h^*)}\right) - \hat{U}_i\left(\underline{c}_i\right)$ for every $i$ and

$$\prod_{i=1}^{I}\left(\hat{U}_i\left(c_i^{h^*+\varepsilon(h-h^*)}\right) - \hat{U}_i\left(\underline{c}_i\right)\right) > \prod_{i=1}^{I}\left(\hat{U}_i\left(c^{h^*}\right) - \hat{U}_i\left(\underline{c}_i\right)\right).$$

Thus $h^*$ is not a solution. ///

If all members have the same income process, then the first-order condition for the Nash bargaining solution has a particularly intuitive form.

**Lemma 6** *Suppose that the function* (9) *is concave for every $i$. Assume that there are a $t_* \geq 1$ and a $e_* \in C$ such that $t_i = t_*$ and $e_i = e_*$ for every $i$. For each $i$, let*

$$\underline{c}_i = e_i + \left(\frac{a_i}{p(0)}, 0, 0, \ldots\right) \in C.$$

*Let $h^* \in H$ and suppose that $\hat{U}_i(c_i^{h^*}) > \hat{U}_i(\underline{c}_i)$ for every $i$. For each $i$, define $p_i \in P$ by letting $p_i(t) = (1-\exp(-\rho_i))\exp(-\rho_i t)u_i'\left(c_i^{h^*}(t)\right)$ for every $t$. Then define $\bar{p}_i \in \boldsymbol{R}_{++}^{1+t*}$ by letting $\bar{p}_i(t) = p_i(t)$ for every $t < t_*$ and $\bar{p}_i(t_*) = \sum_{\tau=t_*}^{\infty} p_i(\tau)$. Define $\bar{p}$ analogously. Then $h^*$ is a solution to the problem of maximizing $\prod_{i=1}^{I}\left(\hat{U}_i(c_i^h) - \hat{U}_i(\underline{c}_i)\right)$ by choosing an $h \in H$ subject to the constraint that $\hat{U}_i(c_i^h) > \hat{U}_i(\underline{c}_i)$ for every $i$ if and only if there is a $\theta = (\theta_1, \theta_2, \ldots, \theta_I) \in \boldsymbol{R}_{++}^I$ such that*

$$\theta_1 u_1'\left(\hat{U}_1\left(c_1^{h^*}\right)\right)\left(\hat{U}_1(c_1^{h^*}) - \hat{U}_1(\underline{c}_1)\right) = \cdots = \theta_I u_I'\left(\hat{U}_I\left(c_I^{h^*}\right)\right)\left(\hat{U}_I(c_I^{h^*}) - \hat{U}_I(\underline{c}_I)\right). \tag{17}$$

*and*

$$\sum_{i=1}^{I}\theta_i\bar{p}_i(t) \geq \bar{p}(t) \text{ for every } t \geq t_*, \tag{18}$$

$$\sum_{i=1}^{I}\theta_i\bar{p}_i(t) = \bar{p}(t) \text{ for every } t \geq t_* \text{ with } h^*(e_*(t), t) > 0, \tag{19}$$

$$\sum_{i=1}^{I}\theta_i\bar{p}_i(t_*) = \bar{p}(t_*). \tag{20}$$

**Proof of Lemma 6** Suppose that $h^*$ is a solution and let $\theta$ be as Lemma 5. By multiplying a scalar to $\theta$ if necessary, we can assume that (20) holds. For each $t < t_i$, define $\eta^t \in H$ by letting $\eta^t(x,\tau) = 1$ if $(x,\tau) = (e_i(t), t)$ $\eta^t(x,\tau) = 0$ otherwise. Since $\sup_{x_i>0} u_i'(x_i) = \infty$, $c_i^{h^*}(t) > 0$ and, hence, $h(e_*(t), t) = e_*(t) - c_i^{h^*}(t) < e_*(t)$. Thus, $h^* + \varepsilon\eta \in H$ for every sufficiently small $\varepsilon > 0$. By (14),

$$\pi_i(\eta^t \mid h^*) = \bar{p}_i(t_*)\frac{\bar{p}(t)}{\bar{p}(t_*)} - \bar{p}_i(t). \tag{21}$$

Thus, by (16) and (20) and (20),

$$0 \geq \sum_i \theta_i \pi_i(\eta^t \mid h^*) = \bar{p}(t) - \sum_i \theta_i \bar{p}_i(t).$$

Hence (18) holds. If, in addition, $h^*(e_*(t), t) > 0$, then $h - \varepsilon\eta \in H$ for every sufficiently small $\varepsilon > 0$. Thus

$$0 \geq \sum_i \theta_i \pi_i(-\eta^t \mid h^*) = -\bar{p}(t) + \sum_i \theta_i \bar{p}_i(t).$$

Thus (19) holds.

Suppose conversely that there is a $\theta$ for which (18), (19), and (20) holds. Let $h \in H$, then

$$h - h^* = \sum_{t<t_i} (h(e(t), t) - h^*(e(t), t))\, \eta^t(e(t), t).$$

Since $\pi(\cdot\,|h^*)$ is linear for each $i$, by (21) and (20),

$$\begin{aligned}
&\sum_i \theta_i \pi_i(h - h^* \mid h^*) \\
=&\sum_i \theta_i \left( \sum_{t<t_i} (h(e(t), t) - h^*(e(t), t))\, \pi_i(\eta^t(e(t), t)) \right) \\
=&\sum_i \theta_i \left( \sum_{t<t_i} (h(e(t), t) - h^*(e(t), t)) \left( \bar{p}_i(t_*) \frac{\bar{p}(t)}{\bar{p}(t_*)} - \bar{p}_i(t) \right) \right) \\
=&\sum_{t<t_i} (h(e(t), t) - h^*(e(t), t)) \left( \sum_i \theta_i \left( \bar{p}_i(t_*) \frac{\bar{p}(t)}{\bar{p}(t_*)} - \bar{p}_i(t) \right) \right) \\
=&\sum_{t<t_i} (h(e(t), t) - h^*(e(t), t)) \left( \bar{p}(t_*) - \sum_i \theta_i \bar{p}_i(t) \right).
\end{aligned}$$

By (18), $\bar{p}(t_*) - \sum_i \theta_i \bar{p}_i(t) \leq 0$. If, in addition, $h(e(t), t) - h^*(e(t), t) < 0$, then $h^*(e(t), t) > 0$ and hence, by (19), $\bar{p}(t_*) - \sum_i \theta_i \bar{p}_i(t) = 0$. Thus (16) holds. ///

The next proposition shows that when all members retirement ages, any consumption process, once scaled up or down to satisfy the budget constraint, can be attained at the Nash bargaining solution, if there is no restriction on the price process under consideration. The proof is based on a fixed-point argument, just like the proof of the existence of a competitive equilibrium.

**Proposition 4** *Let $t_* \in T \setminus \{0\}$ and $b \in C \cap L_{++}$ and suppose that $b(t) = b(t_*)$ for every $t > t_*$. Define $e_* \in C$ by*

$$e_*(t) = \begin{cases} b(t) & \text{if } t < t_*, \\ 0 & \text{if } t \geq t_*. \end{cases}$$

*Then there is a $p \in P$ such that if $e_i = e_*$, $a_i = 0$, and $t_i = t_*$ for every $i$, and $h^*$ is a solution* *to the problem of maximizing* $\prod_{i=1}^{I}\left(\hat{U}_i(c_i^h)-\hat{U}_i(e_*)\right)$ *subject to* $\hat{U}_i\left(c_i^h\right)\geq\hat{U}_i(e_*)$ *for every* $i$ *by choosing an* $h\in H$ *under the price process* $p$*, then*

$$c_i^{h^*}=\frac{p\cdot e_*}{p\cdot b}b$$

*for every* $i$*.*

**Proof of Proposition 4** Our proof consists of four steps.

**Step 1** For each $p\in P$, consider the following maximization problem:

$$\begin{aligned}\max_{c_*\in C}\quad & \prod_i\left(\hat{U}_i(c_*)-\hat{U}_i(e_*)\right)\\ \text{subject to}\quad & c_*(t)=c_*(t_*)\text{ for every }t>t_*,\\ & p\cdot c_*\leq p\cdot e_*,\\ & \hat{U}_i(c_*)\geq\hat{U}_i(e_*)\text{ for every }i.\end{aligned}\tag{22}$$

Define $\bar{p}=(\bar{p}(t))_{t\leq t_*}\in\boldsymbol{R}_{++}^{1+t*}$ by letting $\bar{p}(t)=p(t)$ for every $t<t_*$ and $\bar{p}(t_*)=\sum_{\tau=t_*}^{\infty}p(\tau)$. For each $c_*\in C$, define $\bar{c}_*=(\bar{c}(t))_{t\leq t_*}\in\boldsymbol{R}_{+}^{1+t*}$ by letting $\bar{c}_*(t)=c_*(t)$ for every $t\leq t_*$, and similarly for $\bar{b}$ and $\bar{e}_*$. Then (22) can be rewritten as

$$\begin{aligned}\max_{\bar{c}_*\in\boldsymbol{R}_+^{1+t_*}}\quad & \prod_i\left(\hat{U}_i(\bar{c}_*)-\hat{U}_i(\bar{e}_*)\right)\\ \text{subject to}\quad & \bar{p}\cdot\bar{c}_*\leq\bar{p}\cdot\bar{e}_*,\\ & \hat{U}_i(\bar{c}_*)\geq\hat{U}_i(\bar{e}_*)\text{ for every }i,\end{aligned}\tag{23}$$

where, by a slight abuse of notation, $\hat{U}_i(\bar{c}_*)$ is meant to be $\hat{U}_i(c_*)$ and similarly for $\hat{U}_i(\bar{e}_*)$. Then $\bar{e}_*$ satisfies the constraints and the set of all $\bar{c}_*$'s that satisfy them is convex and compact. Since the objective function is continuous, there is a solution. Moreover, since $u_i'(x)\to\infty$ as $x\to 0$ for every $i$, for every $t<t_*$ and every sufficiently small $\varepsilon>0$, if we define $\bar{c}_*$ by replacing $\bar{e}_*(t)$ by $\bar{e}_*(t)-\varepsilon\bar{p}(t_*)$ and $\bar{e}_*(t_*)$ (which is equal to zero) by $\varepsilon\bar{p}(t)$, and retaining the other coordinates of $\bar{e}_*$, then $\hat{U}_i(\bar{c}_*)>\hat{U}_i(\bar{e}_*)$ for every $i$. Thus, at every solution, $\hat{U}_i(\bar{c}_*)>\hat{U}_i(\bar{e}_*)$ for every $i$. Since it is strictly quasi-concave on the set of all such $\bar{c}_*$'s, the solution is unique. Denote it by $y(\bar{p})$. We have thus defined a function $y:\boldsymbol{R}_{++}^{1+t_*}\to\boldsymbol{R}_{+}^{1+t_*}$.

**Step 2** It is easy to show that $y$ is continuous and homogeneous of degree zero, and that $\bar{p}\cdot y(\bar{p})=\bar{p}\cdot\bar{e}_*$ for every $\bar{p}\in\boldsymbol{R}_{++}^{1+t_*}$. Since the $\hat{U}_i$ are strictly increasing, if $\bar{p}\notin\boldsymbol{R}_{++}^{1+t_*}$, then there is no solution to the problem (23). We now prove by a contradiction argument that if $(\bar{p}^n)_n$ is a sequence on $\boldsymbol{R}_{++}^{1+t_*}$, $\bar{p}\in\boldsymbol{R}_{+}^{1+t_*}\setminus\left(\boldsymbol{R}_{++}^{1+t_*}\cup\{0\}\right)$, and $\bar{p}^n\to\bar{p}$ as $n\to\infty$, then $\|y(\bar{p}^n)\|\to\infty$. If not, then, by taking a subsequence if necessary, we can assume that there is a $\bar{c}_*\in C$ such that $y(\bar{p}^n)\to\bar{c}_*^*$ as $n\to\infty$. Let $\bar{c}_*\in C$ and suppose that $\bar{p}\cdot\bar{c}_*\leq\bar{p}\cdot\bar{e}_*$ and $\hat{U}_i(\bar{c}_*)\geq\hat{U}_i(\bar{e}_*)$ for every $i$. Without loss of generality, we can assume that $\bar{c}_*\in\boldsymbol{R}_{++}^{1+t_*}$ because, if not, then $\bar{c}_*(t)=0$ for some $t$ and we could replace such coordinate can be replaced by a sufficiently small $\varepsilon>0$ to increase the $\hat{U}_i\left(\bar{c}_*\right)$. Then $\bar{p}\cdot\bar{c}_*>0$. If $\hat{U}_i(\bar{c}_*)=\hat{U}_i(\bar{e}_*)$ for every $i$ for some $i$, then $\prod_i \left(\hat{U}_i(\bar{c}_*) - \hat{U}_i(\bar{e}_*)\right) = 0$. Hence $\prod_i \left(\hat{U}_i(\bar{c}^*_*) - \hat{U}_i(\bar{e}_*)\right) \geq \prod_i \left(\hat{U}_i(\bar{c}_*) - \hat{U}_i(\bar{e}_*)\right)$. If $\hat{U}_i(\bar{c}_*) > \hat{U}_i(\bar{e}_*)$ for every $i$ for every $i$, then, for each $n$, define

$$\bar{c}^n_* = \frac{\bar{p}^n \cdot \bar{e}_*}{\bar{p}^n \cdot \bar{c}_*}\bar{c}_*,$$

then, for every sufficiently large $n$, $\bar{p}^n \cdot \bar{c}^n_* = \bar{p}^n \cdot \bar{e}_*$ and $\hat{U}_i(\bar{c}^n_*) > \hat{U}_i(\bar{e}_*)$ for every $i$. Moreover, since $\bar{p} \cdot \bar{c}_* > 0$, $\bar{c}^n_* \to \bar{c}_*$ as $n \to \infty$. Thus

$$\prod_i \left(\hat{U}_i(\bar{c}^n_*) - \hat{U}_i(\bar{e}_*)\right) \leq \prod_i \left(\hat{U}_i(y(\bar{p}^n)) - \hat{U}_i(\bar{e}_*)\right).$$

Hence, by taking the limits of both sides as $n \to \infty$, we obtain

$$\prod_i \left(\hat{U}_i(\bar{c}_*) - \hat{U}_i(\bar{e}_*)\right) \leq \prod_i \left(\hat{U}_i(\bar{c}^*_*) - \hat{U}_i(\bar{e}_*)\right).$$

Thus $\bar{c}^*_*$ is a solution to the problem (23) under $\bar{p} \notin \boldsymbol{R}^{1+t_*}_{++}$. This is a contradiction. Hence $\|y(\bar{p}^n)\| \to \infty$.

**Step 3** Define $\bar{b} \in \boldsymbol{R}^{1+t_*}_{++}$ by letting $\bar{b}(t) = b(t)$ for every $t \leq t_*$. Define $z : \boldsymbol{R}^{1+t_*}_{++} \to \boldsymbol{R}^{1+t_*}$ by letting

$$z(\bar{p}) = y(\bar{p}) - \frac{\bar{p} \cdot \bar{e}_*}{\bar{p} \cdot \bar{b}}\bar{b}$$

for every $\bar{p} \in \boldsymbol{R}^{1+t_*}_{++}$. Then $z$ is continuous and homogenous of degree zero, and satisfies Walras's law, that is, $\bar{p} \cdot z(\bar{p}) = 0$ for every $\bar{p} \in \boldsymbol{R}^{1+t_*}_{++}$. Since $0 < \bar{p} \cdot \bar{e}_* < \bar{p} \cdot \bar{b}$, the set

$$\left\{\frac{\bar{p} \cdot \bar{e}_*}{\bar{p} \cdot \bar{b}} b \mid \bar{p} \in \boldsymbol{R}^{1+t_*}_{++}\right\}$$

is bounded. Thus $z$ is bounded from below and if $(\bar{p}^n)_n$ is a sequence on $\boldsymbol{R}^{1+t_*}_{++}$, $\bar{p} \in \boldsymbol{R}^{1+t_*}_{+} \setminus \left(\boldsymbol{R}^{1+t_*}_{++} \cup \{0\}\right)$, and $\bar{p}^n \to \bar{p}$ as $n \to \infty$, then $\|z(\bar{p}^n)\| \to \infty$. Thus, by Proposition 17.C.1 of Mas-Colell, Whinston, and Green (1995), there is a $\bar{p} \in \boldsymbol{R}^{1+t_*}_{++}$ such that $z(\bar{p}) = 0$, that is,

$$y(\bar{p}) = \frac{\bar{p} \cdot \bar{e}_*}{\bar{p} \cdot \bar{b}}\bar{b}.$$

Let $p \in P$ and assume that $p(t) = \bar{p}(t)$ for every $t < t_*$ and $\sum_{\tau=t_*}^{\infty} p(\tau) = \bar{p}(t_*)$. Let $\bar{c}_* = y(\bar{p})$ and define $c_* \in C$ by letting $c_*(t) = \bar{c}_*(t)$ for every $t \leq t_*$ and $c_*(t) = \bar{c}_*(t_*)$ for every $t > t_*$. Then $c^*$ coincides with

$$\frac{p \cdot e_*}{p \cdot b} b$$

and solves the problem (22).

**Step 4** Let $h^* \in H$ and assume that $h^*(e_*(t), t) = e_*(t) - c_*(t)$ for every $t < t_*$, then

$$c_i^{h^*} = \frac{p \cdot e_*}{p \cdot b} b$$

for every $i$, because $p \cdot c_* = p \cdot e_*$. Let $h \in H$ and suppose that $\hat{U}_i\left(c_i^h\right) \geq \hat{U}_i(e_*)$ for every $i$. Then $c_1^h = c_2^h = \cdots = c_I^h$, and, for every $i$, $p \cdot c_i^h \leq p \cdot e_*$ and $c_i^h(t) = c_i^h(t_*)$ for every $t > t_*$. Since $c_*$ is the solution to the problem (22), $\prod_{i=1}^I \left(\hat{U}_i(c_i^{h^*}) - \hat{U}_i(e_*)\right) \geq \prod_{i=1}^I \left(\hat{U}_i(c_i^h) - \hat{U}_i(e_*)\right)$. Thus $h^*$ is a solution to the problem of maximizing $\prod_{i=1}^I \left(\hat{U}_i(c_i^h) - \hat{U}_i(e_*)\right)$ subject to $\hat{U}_i\left(c_i^h\right) \geq \hat{U}_i(e_*)$ for every $i$ by choosing an $h \in H$ under the price process $p$. ///

The following proposition can be derived from Proposition 4 by letting $b = (1, 1, \dots)$ and noticing that $\hat{U}_i(\kappa b) = \kappa$ for every $\kappa \in \boldsymbol{R}_+$. It claims that all the members may well be equally treated under some price process.

**Proposition 5** *Let $t_* \in T \setminus \{0\}$. Then, there are an $e_* \in C$ and a $p \in P$ such that if $e_i = e_*$, $a_i = 0$, and $t_i = t_*$ for every $i$, and if $h^*$ is a solution to the problem of maximizing $\prod_{i=1}^I \left(\hat{U}_i(c_i^h) - \hat{U}_i(e_*)\right)$ subject to $\hat{U}_i\left(c_i^h\right) \geq \hat{U}_i(e_*)$ for every $i$ by choosing an $h \in H$ under the price process $p$, then $\hat{U}_1\left(c_1^{h^*}\right) = \hat{U}_2\left(c_2^{h^*}\right) = \cdots = \hat{U}_I\left(c_I^{h^*}\right)$.*

The following proposition shows that for every member, there is a price process under which the member is best treated among all the members at the Nash bargaining solution. The proof is based on Farkas's lemma and the aggregation theorem of heterogeneous discount rates along the lines of Weitzman (2001), Gollier and Zeckhauser (2005), and Hara (2008).

**Proposition 6** *Suppose that $\rho_i \neq \rho_j$ whenever $i \neq j$. Let $t_* \in T \setminus \{0\}$. Suppose that $t_* \geq 3$ or $|I| \leq 2$. Then, for every $i$, there are an $e_* \in C$ and a $p \in P$ such that if $e_i = e_*$, $a_i = 0$, and $t_i = t_*$ for every $i$, and if $h^*$ is a solution to the problem of maximizing $\prod_{i=1}^I \left(\hat{U}_i(c_i^h) - \hat{U}_i(e_*)\right)$ subject to $\hat{U}_i\left(c_i^h\right) > \hat{U}_i(e_*)$ for every $i$ by choosing an $h \in H$ under the price process $p$, then $\hat{U}_i\left(c_i^{h^*}\right) > \hat{U}_j\left(c_j^{h^*}\right)$ for every $j \neq i$.*

**Proof of Proposition 6** To start, we note that by replacing each consumption process $c_* \in C$ satisfying $c_*(t) = c_*(t_*)$ for every $t > t_*$ by $\bar{c}_* \in \boldsymbol{R}_+^{1+t_*}$ satisfying $\bar{c}_*(t) = c_*(t)$ for every $t \leq t_*$, we can regard the utility function $\tilde{U}_i$ as defined on $\boldsymbol{R}_+^{1+t_*}$. Then we prove the following fact. Let $\bar{c}_*$ be the solution to the problem (23). For each $i$, define $\bar{p}_i \in \boldsymbol{R}_{++}^{1+t_*}$ by letting $\bar{p}_i(t) = (1 - \exp(-\rho_i)) \exp(-\rho_i t) u_i'(\bar{c}_*(t))$ for every $t < t_*$ and $\bar{p}_i(t_*) = \exp(-\rho_i t_*) u_i'(\bar{c}_*(t_*))$. We then claim that $\bar{p}$ is a strictly positive linear combination of the $\bar{p}_i$. Indeed,

$$
\begin{aligned}
&\nabla \ln \left( \prod_{i=1}^I \left( \hat{U}_i\left(\bar{c^*}_i\right) - \hat{U}_i(\bar{e}_*) \right) \right) \\
&= \sum_{i=1}^I \nabla \ln \left( \hat{U}_i\left(\bar{c}_*\right) - \hat{U}_i(\bar{e}_*) \right) \\
&= \sum_{i=1}^I \frac{(u_i^{-1})' \left( \tilde{U}_i\left(\bar{c}_*\right) \right)}{\hat{U}_i\left(\bar{c}_*\right) - \hat{U}_i\left(\bar{c}_*\right)} \nabla \tilde{U}_i\left(\bar{c}_*\right) \\
&= \sum_{i=1}^I \frac{1}{u_i' \left( \hat{U}_i\left(\bar{c}_*\right) \right) \left( \hat{U}_i\left(c_*\right) - \hat{U}_i\left(\bar{e}_*\right) \right)} \bar{p}_i.
\end{aligned}
$$

Hence this is a strictly positive linear combination of the $\bar{p}_i$. It is, also, a strictly positive multiple of $\bar{p}$, because $\bar{c}_*$ is the solution to (23) and $\hat{U}_i\left(\bar{c}_*\right) > \hat{U}_i\left(\bar{e}_*\right)$ for every $i$. Thus, $\bar{p}$ is a strictly positive linear combination of the $\bar{p}_i$.

Now, let $b = (1, 1, \dots) \in C$. By Proposition 4, there are an $e_* \in C$ and a $p \in P$ such that if $h^*$ is a solution to the problem of maximizing $\prod_{i=1}^{I}\left(\hat{U}_i(c_i^h) - \hat{U}_i(e_*)\right)$ subject to $\hat{U}_i\left(c_i^h\right) \geq \hat{U}_i(e_*)$ for every $i$ by choosing an $h \in H$ under the price process $p$, then

$$c_i^{h^*} = \frac{p \cdot e_*}{p \cdot b} b$$

for every $i$. Denote the vector on the right-hand side by $c_*$. For each $i$, define

$$p_i = \left((1 - \exp(-\rho_i)) \exp(-\rho_i t)\right)_{t \in T} \in P$$

and $\bar{p}_i = \left(\bar{p}_i(t)\right)_{t \leq t_*} \in \boldsymbol{R}_{++}^{1+t_*}$ by letting $\bar{p}_i(t) = (1 - \exp(-\rho_i)) \exp(-\rho_i t)$ for every $t < t_*$ and $\bar{p}_i(t_*) = \exp(-\rho_i t_*)$.

We now prove that for every $i$, there is no $\left(\mu_j\right)_{j \neq i} \in \boldsymbol{R}_+^{I \setminus \{i\}}$ such that $\bar{p}_i = \sum_{j \neq i} \mu_j \bar{p}_j$. This is true if $|I| \leq 2$ because $\rho_i \neq \rho_j$ whenever $i \neq j$. Suppose that $t_* \geq 3$ and there are an $i$ and a $\left(\mu_j\right)_{j \neq i} \in \boldsymbol{R}_+^{I \setminus \{i\}}$ such that $\bar{p}_i = \sum_{j \neq i} \mu_j \bar{p}_j$. Then

$$\begin{aligned}
\sum_{j \neq i} \mu_j (1 - \exp(-\rho_j)) &= 1 - \exp(-\rho_i), \\
\sum_{j \neq i} \mu_j (1 - \exp(-\rho_j)) \exp(-\rho_j) &= (1 - \exp(-\rho_i)) \exp(-\rho_i), \\
\sum_{j \neq i} \mu_j (1 - \exp(-\rho_j)) \exp(-2\rho_j) &= (1 - \exp(-\rho_i)) \exp(-2\rho_i).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{j \neq i} \frac{\mu_j (1 - \exp(-\rho_j))}{1 - \exp(-\rho_i)} \left(\exp(-\rho_j) - \exp(-\rho_i)\right)^2 \\
&= \sum_{j \neq i} \frac{\mu_j (1 - \exp(-\rho_j))}{1 - \exp(-\rho_i)} \left(\exp(-\rho_j)\right)^2 - \left(\sum_{j \neq i} \frac{\mu_j (1 - \exp(-\rho_j))}{1 - \exp(-\rho_i)} \exp(-\rho_j)\right)^2 \\
&= \exp(-2\rho_i) - \left(\exp(-\rho_i)\right)^2 = 0.
\end{aligned}$$

Thus, $\rho_j = \rho_i$ whenever $\mu_j > 0$, but this is a contradiction because there is a $j \neq i$ such that $\mu_j > 0$. Hence there is no $\left(\mu_j\right)_{j \neq i} \in \boldsymbol{R}_+^{I \setminus \{i\}}$ such that $\bar{p}_i = \sum_{j \neq i} \mu_j \bar{p}_j$.

Therefore, by Farkas's lemma, there is a $\bar{v} = (v(t))_{t \leq t_*} \in \boldsymbol{R}^{1+t_*}$ such that $\bar{p}_i \cdot \bar{v} > 0$ and $\bar{p}_j \cdot \bar{v} \leq 0$ for every $j \neq i$. Define $v = (v(t))_{t \in T} \in \ell_\infty$ by letting $v(t) = \bar{v}(t)$ for every $t \leq t_*$ and $v(t) = \bar{v}(t_*)$ for every $t > t_*$. Then $p_j \cdot v = \bar{p}_j \cdot \bar{v}$ for every $j \in I$. For every $\varepsilon \geq 0$ sufficiently close to zero, define $b^\varepsilon = b + \varepsilon v$. Then $b^\varepsilon \in C \cap L_{++}$, $b^\varepsilon(t) = b^\varepsilon(t_*)$ for every $t > t_*$, and

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\hat{U}_j(\kappa b^\varepsilon)\right|_{\varepsilon=0} &= \left(u_j^{-1}\right)'\left(\tilde{U}_j(\kappa b)\right)\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{U}_j(\kappa b^\varepsilon)\right|_{\varepsilon=0} \\ &= \frac{1}{u_j'\left(\hat{U}_j(\kappa b)\right)}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{U}_j(\kappa \bar{b}^\varepsilon)\right|_{\varepsilon=0} \\ &= \frac{1}{u_j'(\kappa)}u_j'(\kappa)p_j \cdot v = p_j \cdot v\end{aligned}$$

for every $j \in I$. Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\hat{U}_i(\kappa b^\varepsilon) - \hat{U}_j(\kappa b^\varepsilon)\right)\right|_{\varepsilon=0} = p_i \cdot v - p_j \cdot v > 0$$

for every $j \neq i$. Since $\hat{U}_i(\kappa b) - \hat{U}_j(\kappa b) = \kappa - \kappa = 0$, there is an $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$, $\hat{U}_i(\kappa b^\varepsilon) > \hat{U}_j(\kappa b^\varepsilon)$ for every $j \neq i$. Moreover, by the continuous differentiability of the $u_j$, such an $\bar{\varepsilon}$ can be taken to be uniform with respect to $\kappa$ over every compact interval of $\boldsymbol{R}_{++}$, that is, for all $\underline{\kappa}$ and $\overline{\kappa}$ satisfying $0 < \underline{\kappa} < \overline{\kappa} < \infty$, there is an $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ and every $\kappa \in [\underline{\kappa}, \overline{\kappa}]$, $\hat{U}_i(\kappa b^\varepsilon) > \hat{U}_j(\kappa b^\varepsilon)$ for every $j \neq i$.

Define $e_*^\varepsilon$, $p^\varepsilon \in P$, and $h^\varepsilon \in H$ as in Proposition 4 except that $b$ is replaced by $b^\varepsilon$. Then $b^\varepsilon \to b$ and $e_*^\varepsilon \to e_*$ as $\varepsilon \to 0$, and $c_i^{h^\varepsilon} = \kappa^\varepsilon b^\varepsilon$ for every $i$, where

$$\kappa^\varepsilon = \frac{p^\varepsilon \cdot e_*^\varepsilon}{p^\varepsilon \cdot b^\varepsilon} = \frac{\bar{p}^\varepsilon \cdot \bar{e}_*^\varepsilon}{\bar{p}^\varepsilon \cdot \bar{b}^\varepsilon}.$$

As we showed at the beginning of this proof, $\bar{p}^\varepsilon$ is a strictly positive combination of the $\bar{p}_i^\varepsilon$. Hence,

$$0 < \frac{1}{2}\min_i \frac{\bar{p}_i \cdot \bar{e}_*}{\bar{p}_i \cdot \bar{b}} < \min_i \frac{\bar{p}_i \cdot \bar{e}_*^\varepsilon}{\bar{p}_i \cdot \bar{b}^\varepsilon} \leq \kappa^\varepsilon \leq 1$$

for every sufficiently small $\varepsilon > 0$. Thus, there is an $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$, $\hat{U}_i(\kappa b^\varepsilon) > \hat{U}_j(\kappa b^\varepsilon)$ for every $j \neq i$. We can thus complete the proof by letting $e_* = e_*^\varepsilon$ and $p = p^\varepsilon$ for any $\varepsilon \in (0, \bar{\varepsilon})$. ///

# 9 Empirical results on dynamically inconsistency

In this section, we report some empirical results on the nature and extent of dynamic inconsistency. To simplify the analysis, we use the following assumptions regarding the initial state $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$.

**Assumption 1** 1. $I = 2$.

2. For every $i$, $t_i = 2$.
3. For every $i$, $a_i = 0$.
4. For every $i$, $e_i(t) = 1$ for every $t = 0, 1$ and $e_i(t) = 0$ for every $t \geq 2$.

5. There is a $\gamma \in (0,1)$ such that $u_i(x_i) = x_i^{1-\gamma}$ for every $i$ and every $x_i > 0$.

6. $\rho_1 > \rho_2$, that is, $\delta_1 < \delta_2$.

7. Let $h_0 \in H$ be a Nash bargaining solution in the state $\left((e_i, a_i, t_i)_{i=1,2,\dots,I}, p\right)$, then $c_i^{h_0}$ is a scalar multiple of $(1,1,\dots) \in C$.

None of these assumptions needs explanation, possibly except for Condition 7. The existence of such a price process $p$ is guaranteed by Proposition 4, where $b = (1,1,\dots)$. Our intension here is to show that even if the pension fund plans to achieve the perfect consumption smoothing, it may still change the scheme on period one.

The assumption is admittedly unrealistic, because it requires that there be only two members and only two periods for them to work, that the earnings be equal between the two members and over the two working periods, and they have the same felicity function that exhibits constant relative risk aversion. Yet, it does make some sense to use empirical data on the heterogeneity of discount rates to assess the extent of dynamic inconsistency, because the method to do so can be extended to more general cases.

Before presenting our empirical results, let us give a brief theoretical analysis of the Nash bargaining solution on period 0. First, since $a_i = 0$ for every $i$, we can assume without loss of generality that $\sum_{t=0}^{\infty} p(t) = 1$. Second, define $\bar{p} \in \boldsymbol{R}^3_{++}$ by $\bar{p}(t) = p(t)$ for every $t = 0,1$ and $\bar{p}(2) = \sum_{t=2}^{\infty} p(t)$. For each $i$, define $\bar{p}_i \in \boldsymbol{R}^3_{++}$ by $\bar{p}_i(t) = (1 - \exp(-\rho_i)) \exp(-\rho_i t)$ for every $t = 0,1$ and $\bar{p}_i(2) = \exp(-2\rho_i)$.

For simplicity, write $c_*^0 = c_i^{h_0}$ for each $i$. Then, by Proposition 4, $c_*^0 = (\bar{p}(0) + \bar{p}(1))b = (1 - \bar{p}(2))b$. In particular, each member saves, in real term, $\bar{p}(2)$ on each of periods 0 and 1. Thus $\hat{U}_i\left(c_*^0\right) = 1 - \bar{p}(2)$ for every $i$.

For each $t = 0,1$, denote by $\underline{c}_*^t$ the consumption process that each member can receive from period $t$ onwards by quitting on period $t$. Then $\underline{c}_*^0 = (1,1,0,0,\dots)$ and

$$\underline{c}_*^1 = \left(1 + \frac{\bar{p}(0)\bar{p}(2)}{\bar{p}(1)}, 0, 0, \dots\right).$$

Thus $\hat{U}_i\left(\underline{c}_*^0\right) = (1 - \exp(-2\rho_i))^{1/(1-\gamma)}$ and

$$\hat{U}_i\left(\underline{c}_*^1\right) = (1 - \exp(-\rho_i))^{1/(1-\gamma)} \left(1 + \frac{\bar{p}(0)\bar{p}(2)}{\bar{p}(1)}\right).$$

Since $\rho_1 > \rho_2$, the utility level of member 1 (the more myopic member) at the disagreement point is higher than that of member 2 on both periods, as suggested in the introduction.

To estimate the heterogeneity of discount rates, we use the micro data from the Preference Parameters Study of Osaka University's 21st Century COE Program 'Behavioral Macrodynamics Based on Surveys and Experiments' and its Global COE Project 'Human Behavior and Socioeconomic Dynamics'. Surveys were conducted annually from 2002 to 2010 in Japan.[2] The

[2]Surveys were conducted in the US, China, and India as well, albeit with lower frequency.

number of respondents varies from year to year, 1418 on 2002 to 6181 on 2008. Each questionnaire, intended to elicit such preference parameters as subjective discount rates and coefficients of risk aversion, consists of forty questions, of which we use Question 6 for our estimation. It reads, roughly, as follows: Imagine that you can receive either one million yen in one month later or receive another amount, say $X$ yen, in thirteen months later. For which value of $X$ would you prefer to receive one million yen in one month?[3] The values of $X$ listed in the question and the corresponding (not continuously compounded) annual rates of return are listed in Table 1.

Of the 5,123 respondents who made transitive and monotone choices in Question 6 in the survey on 2010,[4] the percentages of those who chose $X$ yen thirteen months later over one million yen one month later for each of the ten values of $X$ are listed in the third column of Table 1. We can see there that 1.35% of the respondents have negative discounts rates of $-5\%$ or lower, and $3.18\,(= 100 - 96.82)\,\%$ of them have the discount rates of 100% or higher. The fourth column is the percentage of those who chose to receive $X$ yen, but not $X'$ yen with any $X' < X$, thirteen months later over one million yen one month later. For example, since 28.91% of the respondents chose to receive 1,001,000 yen thirteen months later and the 43.86% of the respondents chose to receive 1,005,000 yen thirteen months, the difference between the two, 14.95%, is the percentage of those who chose 1,005,000 yen but none of the smaller amounts in the question. Referring to the second column, which lists the annual rates of return of the amounts $X$ in this question, we can infer that these 14.95% of the respondents have the discount rates between 1% and 5%, which are listed, along with those for the other values of $X$, on the fifth column. Then, we simply take the middle point of the two, 3%, as the discount rate of this subgroup. This is the way we assign the discount rate to the respondents who chose $X$ yen thirteen months later but not $X'$ yen for any $X' < X$, which are listed on the last column, except for the subgroups at the top and bottom rows. Those on the top of the table have discount rates $-5\%$ or lower, and we assume, somewhat arbitrarily, that they have discount rates $-7.5\%$; and those on the bottom of the table have discount rates 100% or higher, and we assume, somewhat arbitrarily, that they have discount rates 200%.

Based on these responses, we determine the discount factors of the two members in Assumption 1 in the following two method. First, we simply take their discount factors to be the first and third quantiles of the sample distribution of the discount factors. The first quantile is nothing but the median of the discount factors of the respondents whose discount factors are less than the median of all respondents, and the third quantile is nothing but the median of the discount factors of the respondents whose discount factors are more than the median of all respondents. Thus, the two members in our model represent those with discount rates higher than the median and those with discount rates lower than the median. In this case, the first

[3]The actual wording of the question is different, so that it is in line with other questions in the questionnaire. Its English version, given to subjects in the US, reads as follows: Now let's assume that you have the option to receive $10,000 in one month or receive a different amount in thirteen months. Compare the amounts and timing in Option A with Option B and indicate which amount you would prefer to receive for all ten choices.

[4]We have eliminated the respondents who made the (non-transitive or non-monotone) choices such that, for some $X$ and $X'$ with $X < X'$, choosing $X$ yen thirteen months later over one million yen one month later, and, yet, choosing one million yen one month later over $X'$ yen thirteen months later.

Table 1: Percentages of the respondents who chose $X$ yen thirteen months over one million yen thirteen months later

| response category $X$ | annual rate of return (in percent) | percentages of respondents who chose $X$ yen thirteen months later | percentages of respondents who chose $X$ yen, but not any $X'$ yen with $X' < X$, thirteen months later | range of discount rates (in percent) consistent with the choice | discount rates (in percent) assigned |
|---|---|---|---|---|---|
| 950,000 | $-5$ | 1.35 | 1.35 | $(-\infty, -5]$ | $-10$ |
| 1,000,000 | 0 | 3.81 | 2.46 | $[-5, 0]$ | $-2.50$ |
| 1,001,000 | 0.1 | 28.91 | 25.10 | $[0, 0.1]$ | 0.05 |
| 1,005,000 | 0.5 | 43.86 | 14.95 | $[0.1, 0.5]$ | 0.3 |
| 1,010,000 | 1 | 59.48 | 15.62 | $[0.5, 1]$ | 0.75 |
| 1,020,000 | 2 | 65.49 | 6.01 | $[1, 2]$ | 1.5 |
| 1,060,000 | 6 | 79.58 | 14.09 | $[2, 6]$ | 4 |
| 1,100,000 | 10 | 89.24 | 9.66 | $[6, 10]$ | 8 |
| 1,300,000 | 30 | 94.11 | 4.87 | $[10, 30]$ | 20 |
| 2,000,000 | 100 | 96.82 | 2.71 | $[30, 100]$ | 65 |
| | | | | $[100, \infty)$ | 200 |

Table 1 shows, in the 5,123 respondents on the 2010 survey, the amounts $X$ to be received thirteen months later; the annual rates of return, in percent, of the amounts $X$; the percentages of the respondents who chose $X$ yen thirteen months later over one million yen one month later; the percentages of respondents who chose $X$ yen, but not any $X'$ yen with $X' < X$, thirteen months later; the ranges of the discount rates, in percent, that are consistent with choosing $X$, but not any $X'$ yen with $X' < X$, thirteen months later; and the discount rates with the respondents in the subgroups are assume to have.

member's discount rate is equal to 4% and the second member's discount rate is equal to 0.05%. In the second method, we eliminate all the respondents who exhibited no discounting (that is, those who chose one million yen thirteen months later over one million yen one month later, and constitute 2.46% of all respondents), assume that the remaining responses are drawn from a gamma distribution, and estimate the two parameters that determine the gamma distribution by the monotone likelihood method. Then, we calculate, with respect to the gamma distribution having the parameters we estimated, the conditional mean of the discount rates given that they are higher than the median, and the conditional mean of the discount rates given that they are lower than the median. Finally, we set the first member's discount rate at the first conditional mean and the second member's discount factor at the second conditional mean. The discount rates chosen for the two members are 16.15% and 0.18%.

Of these two methods, the first one is simpler and can accommodate negative discount rates. The second one involves estimation of gamma distributions and is in line with the analysis of Weitzman (2001), who showed that the distribution of the discount rates that more than 1,500

PhD-level economists think should be used for the cost-benefit analysis of mitigating climate change can be well approximated by a gamma distribution. Because of the definition of gamma distributions, however, negative discount rates must be eliminated.

As for the common coefficient of relative risk aversion, or, more appropriately in the present context, the intertemporal elasticity of substitution, we do not use any particular estimate but simply try various values in our estimation. The reason for not attempting to pin down any estimate of intertemporal elasticity of substitution is that we have no data set on it. Although there is a vast literature on the estimates of coefficients of risk aversion, and the questionnaire that we use does indeed contain some questions to elicit coefficients of risk aversion, we should not take its reciprocal as the estimate of the intertemporal elasticity of substitution, because, as we discussed in Example 1, the members may well have recursive utility functions, but not necessarily additive separable expected utility functions, once the risks are taken into consideration. As we will see, the value of the first measure of inefficiency, introduced in Section 6 depends sensitively on the value of intertemporal elasticity of substitution, which is another reason why we do not use any particular estimate for it.

Table 2 shows the nature and extent of welfare losses resulting from the dynamically inconsistent choices of schemes. The first three columns list the members' discount rates and coefficients $\gamma$ of constant relative risk aversion that we use for estimation. The fourth column lists the contribution that each member makes on each of periods 0 and 1 at the Nash bargaining solution obtained on period 0. This contribution is, however, not realized on period 1. The fifth column lists the contributions on period 1 generated by the Nash bargaining solution obtained on that period. The sixth columns list the (common) utility level at the Nash bargaining solution obtained on period 0, which is not realized. The seventh and eighth column list the utility levels generated by the dynamically inconsistent choices of schemes of the two sequential Nash bargaining solutions.

The last column lists lower bounds on the first measure of inefficiency of the consumption process generated by the dynamically inconsistent choices of pension schemes. They measure roughly how much, in percentage, we can reduce the members' incomes (with the same percentage applied to all periods and all members) while allowing them to enjoy the same utility levels as those obtained at the dynamically inconsistent choices of pension schemes, if the fund can commit itself to the scheme it chooses on period zero. They were calculated as follows. Note first that the utility functions $\hat{U}_i$ are homogeneous of degree one, because the felicity functions $u_i$ exhibit constant relative risk aversion. Thus, if the endowments $e_i$ are scaled down by factor $\theta$ and, at the same time, the pension scheme $h$ at the Nash bargaining solution on period 0 for the original income processes $e_i$ is changed to another pension scheme $h'$ such that $h'(x,t) = \theta h(\theta^{-1}x, t)$ for every $(x,t)$, then the consumption processes generated by $h'$ with the scaled down income processes $\theta e_i$ coincide with the consumption processes generated by $h$ with the income processes $e_i$ and then scaled down by factor $\theta$. In short, the utility possibility set with the scaled down income processes $\theta e_i$ can be obtained by scaling down the utility possibility set with the original income process $e_i$ by factor $\theta$. Now, calculate the ratio $\hat{U}_2(c_*)/\hat{U}_1(c_*)$ of the two members' utility levels at the (common) consumption process $c_*$ that is generated when the fund has no commitment to the scheme it chose on period 0. Let $c'_*$ be a (common) consumption process such that $\left(\hat{U}_1(c'_*), \hat{U}_2(c'_*)\right)$ belongs to the utility possibility set (so that the pair of utility levels can be attained by some pension scheme on period 0 if the fund can commit itself to it) and $\hat{U}_2(c'_*)/\hat{U}_1(c'_*) = \hat{U}_2(c_*)/\hat{U}_1(c_*)$. Let $\theta = \hat{U}_1(c_*)/\hat{U}_1(c'_*)$. We then claim that $1-\theta$ is a lower bound on the first measure of inefficiency. Indeed, suppose that the scheme $h$ attains the the pair $\left(\hat{U}_1(c'_*), \hat{U}_2(c'_*)\right)$ of utility levels with the income processes $e_i$. Define another scheme $h'$ by letting $h'(x,t) = \theta h(\theta^{-1}x, t)$ for every $(x,t)$. Then $h'$ attains the (common) consumption process $\theta c'_*$. Since $\hat{U}_i(\theta c'_*) = \theta \hat{U}_i(c'_*) = \hat{U}_i(c_*)$, the pair $\left(\hat{U}_1(c_*), \hat{U}_2(c_*)\right)$ of utility levels can be attained with the scaled-down income processes $\theta e_i$. In addition, if $\left(\hat{U}_1(c'_*), \hat{U}_2(c'_*)\right)$ lies on the frontier of the utility possibility set (so that it cannot be Pareto-dominated by the pair of utility levels generated by any scheme even if the fund can commit itself to it), then $1-\theta$ coincides with the first measure of inefficiency. Indeed, the pair cannot be attained with any scaled-down income processes $\theta' e_i$ with $\theta' < \theta$, because if it were generated by some scheme, say $h''$, then define another scheme $h'''$ by letting $h'''(x,t) = (\theta')^{-1} h''(\theta' x, t)$ for every $(x,t)$. Then $h'''$ would generate the (common) consumption process $(\theta')^{-1} c_*$ with the income processes $e_i$. Since $\hat{U}_i\left((\theta')^{-1} c_*\right) = (\theta')^{-1} \hat{U}_i(c_*) > \theta^{-1} \hat{U}_i(c_*) = \hat{U}_i(c'_*)$, this is a contradiction to the hypothesis that $\left(\hat{U}_1(c'_*), \hat{U}_2(c'_*)\right)$ lies on the frontier of the utility possibility set. Thus $1-\theta$ coincides with the first measure of inefficiency.

Although there are many candidates for a consumption process $c'_*$ such that $\left(\hat{U}_1(c'_*), \hat{U}_2(c'_*)\right)$ belongs to the utility possibility set and $\hat{U}_2(c'_*)/\hat{U}_1(c'_*) = \hat{U}_2(c_*)/\hat{U}_1(c_*)$, the one we use is such that $c'_*(0) = c'_*(1)$. In other words, each member makes, in real term, the same contribution on both periods before retirement. This choice of $c'_*$ is mainly for computational simplicity, but probably not unreasonable, because the Nash bargaining solution obtained on period 0 has this property.

We calculated lower bounds on the first measure of inefficiency introduced for ten cases. The first four cases are for the discount rates derived from the first and third quantiles of the distribution of the 5,123 responses. The second four cases are for the discount rates derived from estimating the gamma distribution generating the responses to which positive discount rates are assigned. In each of these two groups, we have five different coefficients $\gamma$ of relative risk aversion, 0.1, 0.2, 0.5, 0.8, and 0.9. Note first that the measure of inefficiency is an increasing function of the coefficient of constant relative risk aversion, or, in other words, a decreasing function of the intertemporal elasticity of substitution. This is consistent with our intuition. The very purpose of pension schemes is to smooth consumption for members before and after retirement. The less willing they are to accept bumpy consumption, the higher the welfare loss caused by the dynamically inconsistent choices of schemes, relative to the consumption process generated by the dynamically consistent one, which is, by Assumption 1, constant over time. Second, the welfare loss is non-negligible in size. It may be well more than 5% of the total wealth of the members even when $\gamma$ is as small as 0.2, and it may exceed 14% if $\gamma$ is 0.8 or higher. Third, while the contributions that each member makes on periods 0 and 1 are equal (because each has the constant income and consumption over these periods) at the dynamically consistent choice of pension scheme (the Nash bargaining solution obtained on period 0), the contribution that is actually made on period 1 at the dynamically inconsistent choice (the Nash bargaining solution obtained on period 1) is lower than the contribution at the dynamically consistent choice. This can probably be attributed to the fact that the disagreement point shifts to the direction that favors the more myopic member (member 1), although it is not a general phenomenon. According to our separate calculation (not listed here), in some extreme cases, such as member 1's discount rate is 0.1, member 2's discount rate is 0.55, and $\gamma = 0.5$, the contribution that is actually made on period 1 is higher than the contribution planned on period 0.

The consequence of the lower-than-planned contribution on period 1 is that the utility level of the more patient member (member 2) is always lower at the dynamically inconsistent choice than at the dynamically consistent choice, and the utility level of the more myopic member (member 1) has the same property except for the cases in which $\gamma \geq 0.5$ in the first specification of discount rates, and $\gamma = 0.9$ in the second specification of discount rates. In short, unless the members are very averse to consumption fluctuation, the dynamic inconsistency favors the more myopic member, because the fund needs to offer, on their second earning period, a pension scheme which the myopic member has no incentive to quit despite the fact that he has made a (relatively) large contribution on the first earning period (at the Nash bargaining solution obtained on period 0).

Table 2: Welfare losses arising from dynamically inconsistent choices

| discount rate (in percent) of member 1 | discount rate (in percent) of member 2 | CRRA $\gamma$ | contribution (in real term), planned on period 0, to be made on each period before retirement | contribution (in real term) on period 1 | utility level planned on period 0 for each member | utility level of member 1 | utility level of member 2 | lower bound (in percent) on the measure of inefficiency |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.05 | 0.1 | 0.9339 | 0.6670 | 0.0661 | 0.0664 | 0.0570 | 0.50 |
| 4 | 0.05 | 0.2 | 0.9424 | 0.6274 | 0.0576 | 0.0579 | 0.0483 | 8.99 |
| 4 | 0.05 | 0.5 | 0.9590 | 0.5232 | 0.0410 | 0.0393 | 0.0320 | 15.47 |
| 4 | 0.05 | 0.8 | 0.9618 | 0.4887 | 0.0382 | 0.0338 | 0.0291 | 19.38 |
| 4 | 0.05 | 0.9 | 0.9618 | 0.4865 | 0.0382 | 0.0331 | 0.2290 | 20.09 |
| 16.15 | 0.18 | 0.1 | 0.7592 | 0.5976 | 0.2408 | 0.2420 | 0.2174 | 4.26 |
| 16.15 | 0.18 | 0.2 | 0.7781 | 0.5831 | 0.2219 | 0.2244 | 0.1966 | 5.19 |
| 16.15 | 0.18 | 0.5 | 0.8361 | 0.5161 | 0.1639 | 0.1696 | 0.1354 | 9.40 |
| 16.15 | 0.18 | 0.8 | 0.8683 | 0.4488 | 0.1317 | 0.1338 | 0.1027 | 14.29 |
| 16.15 | 0.18 | 0.9 | 0.8688 | 0.4449 | 0.1312 | 0.1310 | 0.1019 | 14.96 |

Table 2 shows, for each profile of the common coefficient $\gamma$ of relative risk aversion, the discount rate of member 1, for the discount rate of member 2: the contribution to be made on each of periods 0 and 1 at the Nash bargaining solution obtained on period 0; the contribution actually made on period 1 at the Nash bargaining solution obtained on that period; the (not realized) common utility level at the Nash bargaining solution obtained on period 0; the (realized) utility level for member 1 generated by the dynamically inconsistent pension schemes; the (realized) utility level for member 2 generated by the dynamically inconsistent pension schemes; and the lower bound, calculated in the way described in this section, on the first coefficient of inefficiency introduced in Section 6.

# 10 Conclusion

We have provided a model of pension fund management to assess its efficacy. Our main results are that depending on the price processes under which the fund can borrow and lend, all members may well be equally treated or any particular member may well be best treated. Based on a set of micro data, we have shown the welfare loss of dynamic inconsistency can be as high as 14% of the members' total wealth, and the dynamically inconsistent choices of pension schemes tend to favor myopic members.

The model of this paper, being restrictive, admits many directions of future research. First and foremost, we should accommodate uncertainty in the model and allow the pension fund to trade not only the riskless asset but also risky ones. The Nash bargaining problem is, then, more difficult to solve, because the utility possibility set is determined the volatility of the risky assets and the market price of risk. Second, recall that the interest rate process $r$ is defined by letting $r(t) = \ln p(t) - \ln p(t+1)$ for every $t$. Then suppose that there are the borrowing and lending rate processes, $\overline{r}$ and $\underline{r}$, such that $\underline{r}(t) \leq r(t) \leq \overline{r}(t)$ for every $t$, and that the members can borrow at the rate $\overline{r}(t)$ and lend at the rate $\underline{r}(t)$. This means that unlike the pension fund, which can borrow and lend at the rate $r(t)$, the members have to incur transactions costs $\overline{r}(t) - r(t)$ when borrowing and $r(t) - \underline{r}(t)$ when lending. This assumption is milder and more realistic than the assumption, maintained throughout the paper, that the members have no access to financial markets. The utility level that each member can attain from quitting the fund, which determines the disagreement point, would be higher, and the more myopic members would benefit at the Nash bargaining solution, under this milder assumption. Third, we should allow the members of the fund to receive not only salaries but also dividends (from stocks he owns) or rents (from apartments he owns), from which he does not contribute to the fund. Then, they can receive nonzero consumption levels after retirement even when quitting the fund and, again, this would affect the disagreement point. Those who receive higher dividends will benefit at the Nash bargaining solution in this more general setting. Fourth, although we assume that the members have additively separable utility functions, it would be better to assume that they have recursive utility functions, because this weaker restriction on utility functions allow us to disentangle intertemporal elasticity of substitution and risk aversion, if the latter can be identified. As discussed in part 3 of Example 1, we can then use the recursive utility function to define the utility possibility set and the less risk-averse members would then benefit at the Nash bargaining solution.

Finally, we should give a fuller empirical analysis of the nature and extent of dynamic inconsistency. Our present analysis is limited to the case of two members, two earning periods, and identical income processes. We should extend it to the case of arbitrary numbers of members, earning periods, and types of income processes.

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