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monotone choice and strategic complementarity"

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A revealed preference theory of monotone choice and strategic complementarity

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Abstract: We carry out a revealed preference analysis of monotone comparative statics. We ask what restrictions on an agent's observed choice behavior are necessary and sufficient to rationalize the data with a preference guaranteeing that choices are always monotone with respect to a parameter. We extend our analysis to a game-theoretic setting where players' chosen actions, the strategy sets from which actions are chosen, and the parameters which may affect payoffs are observed. Variation in the data arises from changes to parameters and/or changes to the strategy sets. We show that an intuitive and easy-to-check property on the data set is necessary and sufficient for it to be consistent with the hypothesis that each observation is a pure strategy Nash equilibrium in a game with strategic complementarity. When a data set obeys this property, we show how to exploit this data to identify the set of possible Nash equilibria in a game outside the set of observations.

Keywords: monotone comparative statics, single crossing differences, interval dominance, supermodular games, lattices

JEL classification numbers: C6, C7, D7

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1 Introduction

In recent years, the techniques of monotone comparative statics have been extensively applied to economic modeling. The most basic result in this theory concerns the monotonicity of an agent's optimal choice as a parameter changes. To be specific, let $\succsim_i$ be a binary relation over $(x_i, \xi_i) \in \mathbb{R} \times \mathbb{R}$, where x_i is interpreted as agent i's action and ξ_i some parameter that may affect the agent's choice. Assume that, for any fixed $\bar{\xi}_i$, the restriction of $\succsim_i$ to the set $\{(x_i, \bar{\xi}_i) : a_i \in \mathbb{R}\}$ is a preference, i.e., a complete, reflexive and symmetric relation. Given ξ_i and given a feasible action set $A_i \subset \mathbb{R}$, agent i's optimal choice (or best response) is

$$\mathrm{BR}(\xi_i, A_i, \succsim_i) = \{x_i' \in A_i : (x_i', \xi_i) \succsim_i (x_i, \xi_i) \text{ for all } x_i \in A_i\}. \tag{1}$$

What conditions guarantee that $\mathrm{BR}(\xi_i, A_i)$ is increasing in ξ_i, in the sense that every element in $\mathrm{BR}(\xi_i'', A_i)$ is greater than every element in $\mathrm{BR}(\xi_i', A_i)$, when $\xi_i'' > \xi_i'$? It is known that, for this to hold on every A_i, where A_i is an arbitrary subset of $\mathbb{R}$, it is necessary and sufficient that $\succsim_i$ obeys *strict single crossing differences* (see Milgrom and Shannon, 1994). This property says that for every $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$,

$$(x_i'', \xi_i') \succsim_i (x_i', \xi_i') \Longrightarrow (x_i'', \xi_i'') \succ_i (x_i', \xi_i''),$$

where $\succ_i$ is the strict preference induced by $\succsim_i$. In the case where we restrict the feasible action sets A_i to intervals of $\mathbb{R}$, then strict single crossing differences can be weakened and replaced by the *strict interval dominance* property (see Quah and Strulovici, 2008), which says that

$$(x_i'', \xi_i') \succsim_i (x_i, \xi_i') \text{ for all } x_i \in [x_i', x_i''] \Longrightarrow (x_i'', \xi_i'') \succ_i (x_i', \xi_i'').$$

Given the central role played by these properties in ensuring monotonicity, it is useful to characterize those observations of an agent's choices that are consistent with them. In other words, our objective is to develop revealed preference tests for these properties, along the lines of Afriat's Theorem, which characterizes consumer data sets that are consistent with the maximization of a locally non-satiated utility function. A test of this type is a natural starting point for a nonparametric investigation of monotone comparative statics. In our setting, we assume that the observer has access to a data set with T observations, $\mathcal{O}_i = \{(a_i^t, \xi_i^t, A_i^t)\}_{t=1}^T$, where a_i^t is the observed choice from the feasible action set A_i^t when the parameter is ξ_i^t. We assume that A_i^t is a compact interval of $\mathbb{R}$. The data set is said to be *monotone-rationalizable* if there is $\succsim_i$ such that (i) $a_i^t \in \text{BR}(\xi_i^t, A_i^t, \succsim_i)$ and (ii) $\succsim_i$ obeys strict interval dominance and is *regular* in the sense that $\text{BR}(\xi_i, A_i, \succsim_i)$ is nonempty and compact for all y_i, whenever A_i is compact. This definition captures the notion that the observations are consistent with the belief that the agent is optimizing according to a preference with strict interval dominance. The requirement (i) guarantees that $\succsim_i$ can actually account for the observed data, while the strict interval dominance requirement in (ii) guarantees that agent i's choice is monotone (with respect to the parameter) on all feasible sets that are compact intervals, even at feasible sets not amongst those observed.

It is clear that monotone-rationalizability is a refutable hypothesis. For example, consider two observations $(a_i^1, \xi_i^1, A_i^1) = (4, 1, [3, 6])$ and $(a_i^2, \xi_i^2, A_i^2) = (3, 2, [3, 6])$. In both observations, the feasible set is $[3, 6]$. The fact that 4 is chosen in the first observation means that $(4, 1) \succsim_i (a_i, 1)$ for all $a_i \in [3, 4]$. Strict interval dominance of $\succsim_i$ then requires $(4, 2) \succ_i (3, 2)$, which means that the choice in the second observation cannot be optimal. A more elaborate example involving three observations is the following: $(a_i^1, \xi_i^1, A_i^1) = (4, 1, [3, 6])$ (as before), $(a_i^2, \xi_i^2, A_i^2) = (3, 2, [0, 3]$ and $(a_i^3, \xi_i^3, A_i^3) = (0, 2, [0, 5])$. Again, the first observation tells us that $(4, 2) \succ_i (3, 2)$. The second observation tells us that $(3, 2) \succsim_i (0, 2)$ (since action 0 was available when 3 was chosen) while the third observation tell us that $(0, 2) \succsim_i (4, 2)$ (since 4 was available when 0 was chosen). Since $\succsim$ is transitive, observations 2 and 3 together tell us that $(3, 2) \succ_i (4, 2)$ and so we obtain a contradiction.

We show that an intuitive and easy-to-check property of the data set we call the *axiom of revealed complementarity* (ARC) is both necessary and sufficient for monotone-rationalizabilty. In essence, the axiom requires the *exclusion* of the phenomena depicted in the examples. In fact, our result is somewhat stronger: whenever ARC holds, we could choose $\succsim_i$ to obey strict single crossing differences (and not just strict interval dominance).[1] When a data set obeys ARC, there will typically be more than one preference that rationalizes the data, so it would be natural to ask

[1]Note that the result hinges on assumption that observed feasible sets are compact intervals. When we allow for A_i^t to be arbitrary compact sets, ARC is *not* sufficient to guarantee rationalization with $\succsim_i$ obeying strict single crossing differences (see Example 1 in Section 3).

what we can robustly infer about agent i's preference from his observed behavior. We provide a way of precisely identifying those pairs of actions that we can safely order, in the sense that the ordering holds for *all* preferences consistent with the data and obeying strict interval dominance.

An important application of monotone comparative statics is to the study of games with strategic complementarity (see Milgrom and Robert (1990) and Vives (1990)). These are games where players' strategies are complements in the sense that an agent's best response increases with the action of other players in the game. These games are known to be very well-behaved: they always have pure strategy Nash equilibria, in fact, there is always a largest and a smallest pure strategy Nash equilibrium and a parameter change that leads to one agent having a greater best response will raise both the largest and smallest equilibrium. Our revealed preference tests can be applied to this context. Specifically, consider a data set where observations are drawn from an n-player game. For each player i ($i = 1, 2, ..., n$), we observe the feasible action set A_i^t (assumed to be a compact interval), the action chosen by the player, $a_i^t \in A_i^t$, and an exogenous parameter y_i^t (drawn from a poset) that affects player i's action. An observation t may be succinctly written as (a^t, y^t, A^t), where $a^t = (a_i^t)_{i=1}^n$, etc, and the data set is $\mathcal{O} = \{(a^t, y^t, A^t)\}_{t=1}^T$. In other words, we observe the outcome in T different games, played by the same players, with games differing according to the feasible action sets available to each player and/or the exogenous parameters affecting each players' behavior.

Our first and most basic task is to develop a test for the hypothesis that the observations constitute Nash equilibria in games with strategic complementarity. Notice that this hypothesis is at least internally consistent since we know that these games always have pure strategy Nash equilibria. The resolution to this problem is straightforward given the single-agent results: all we need to do is to check that each player's choices are monotone-rationalizable. Formally, this involves determining whether the data sets $\mathcal{O}_i = \{(a_i^t, \xi_i^t, A_i^t)\}_{t=1}^T$, where $\xi_i = (a_{-i}^t, y_i^t)$ obey ARC. The data set $\mathcal{O}$ is consistent with strategic complementarity if and only if ARC holds for $\mathcal{O}_i$, for all i.

When the data set $\mathcal{O}$ obeys ARC (in the sense that every player obeys ARC), a natural followup is to ask how this data can be exploited to make predictions of equilibrium play in a new game, with different feasible action sets $A^0 = (A_i^0)_{i=1}^n$ and different parameters $y^0 = (y_i^0)_{i=1}^n$, assuming that the players' preferences remain unchanged. For each player, we provide a procedure for working out the *possible response correspondence.* This specifies, given an exogenous parameter y_i and a profile of other players' strategies, a_{-i}, the set of all actions of player i that *could* be a best response, in the sense that it is a best response according to some preference for player i that obeys strict interval dominance and is consistent with his behavior as observed in $\mathcal{O}_i$. With these correspondences we may identify $\mathcal{E}(y^0, A^0)$, which is the set of all possible Nash equilibria in the new game. Remarkably, we show that $\mathcal{E}(y^0, A^0)$ has properties that echo those of a set of Nash equilibrium of a game with strategic complementarity even if they are not exactly the same. While $\mathcal{E}(y^0, A^0)$ may not have a largest or smallest element, we show that the closure of this set does have a largest and smallest element. Furthermore, these extremal elements increase with y^0.

Our study is not the first to obtain a revealed preference-type result in a monotone choice environment. In particular, Topkis (1998, Theorem 2.8.9) reports an early result of this type. Topkis considers a correspondence $\varphi : T \to \mathbb{R}^\ell$ that maps elements of a totally ordered set T to compact sublattices of the Euclidean space $\mathbb{R}^l$. He shows that this correspondence is increasing in the strong set order if and only if there is a function $f : \mathbb{R}^\ell \times T \to \mathbb{R}$ such that $\varphi(t) = \arg\max_{x \in \mathbb{R}^\ell} f(x, t)$ where f is supermodular in x and has increasing differences in (x, t). In the case where φ is a choice *function*, such a rationalization is possible even when T is a partially (rather than totally) ordered set; this has been noted by Carvajal (2004) and Lazzati (2014), who also exploit this result in a revealed preference analysis of games with strategic complementarity. In our paper, we confine ourselves to the case where actions are totally ordered (rather than elements of a Euclidean space), but the observational possibilities for the observer are allowed to be richer because we assume that he may observe the agent choosing from *different subsets* of the set of all possible actions. Consequently, at a given parameter value, the observer's information may go beyond the set of globally optimal actions; he may also receive partial information on the agent's preference over different actions. This in turn means that the problem we pose is different and (in one respect) more complicated than the one posed by Topkis, because the rationalizing preference we construct has to agree with this wider range of preference information, in addition to obeying single crossing differences or interval dominance.

The paper is organized as follows. After a quick review of monotone comparative statics and strategic complementarity in the next section, Section 3 gives a revealed preference characterization of monotone rationalizability. This section also discusses the preference information that may be robustly inferred from a monotone rationalizable data set. Section 4 extends the analysis to games with strategic complementarity and discusses out-of-sample predictions of Nash equilibria.

2 Basic concepts and theory

Our objective in this section is to give a quick review of some basic concepts and results in monotone comparative statics and of their application to games with strategic complementarities. This will then motivate the revealed preference theory developed later in the paper.

2.1 Monotone choice on intervals

Let $X_i \subset \mathbb{R}$ be the set of all conceivable actions of an agent i. A *feasible action set* of agent i is a subset A_i of X_i. We assume that A_i is compact in $\mathbb{R}$ and that it is an *interval of* X_i. We say that a set $A_i \subseteq X_i$ is an *interval of* X_i if, whenever x'', $x' \in A_i$, with $x'' > x'$, then, for any element $\tilde{x} \in X_i$ such that $x'' > \tilde{x} > x'$, $\tilde{x} \in A_i$. Given that A_i is both compact and an interval, we can refer to it as a *compact interval.* It is clear that there must be $\underline{a}_i$ and $\bar{a}_i$ in A_i such that $A_i = \{a_i \in X_i : \underline{a}_i \leqslant a_i \leqslant \bar{a}_i\}$. Given this it is sometimes convenient to denote A_i by $[\underline{a}_i, \bar{a}_i]$. We denote by $\mathcal{A}_i$ the collection of all compact intervals of X_i. Given a feasible action set A_i, agent i's choice over different actions in A_i is affected by some parameter ξ_i. We assume that ξ_i is drawn from a partially ordered set (or poset, for short) $(\Xi_i, \geqslant)$. For the sake of notational simplicity, we are using the same notation for the orders on X_i and Ξ_i and for any other ordered sets; we do not anticipate any danger of confusion.

A binary relation $\succsim_i$ on $X_i \times \Xi_i$ is said to be a *preference* of agent i if, for every fixed $\xi_i \in \Xi_i$, $\succsim_i$ is a complete, reflexive and transitive relation on X_i. We call a preference $\succsim_i$ *regular* if, for all $A_i \in \mathcal{A}_i$ and ξ_i, the set $\text{BR}_i(\xi_i, A_i, \succsim_i)$ (which we may shorten to $\text{BR}_i(\xi_i, A_i)$ when there is no danger of confusion), as defined by (1), is nonempty and compact in $\mathbb{R}$. We refer to $\text{BR}_i(\xi_i, A_i)$ as agent i's *best response* or *optimal choice* at (ξ_i, A_i). The best response of agent i is said to be *monotone* or *increasing in* ξ_i if, for every $\xi_i'' > \xi_i'$,

$$a_i'' \in \text{BR}_i(\xi_i'', A_i) \text{ and } a_i' \in \text{BR}_i(\xi_i', A_i) \Longrightarrow a_i'' \geqslant a_i'. \tag{2}$$

The preference $\succsim_i$ is said to obey *strict interval dominance* (SID) if, for every $x''_i > x'_i$ and $\xi''_i > \xi'_i$,

$$(x''_i, \xi'_i) \succsim_i (x_i, \xi'_i) \text{ for all } x \in [x'_i, x''_i] \Longrightarrow (x''_i, \xi''_i) \succ_i (x'_i, \xi''_i), \tag{3}$$

where $\succ_i$ is the asymmetric part of $\succsim_i$, i.e., $(x_i, \xi_i) \succ_i (y_i, \xi_i)$ if $(x_i, \xi_i) \succsim_i (y_i, \xi_i)$ and $(x_i, \xi_i) \not\precsim_i (y_i, \xi_i)$. The following result is straightforward adaptation of Theorem 1 in Quah and Strulovici (2009). We shall re-prove it here because of its central role in this paper.

Theorem A. *Suppose $\succsim_i$ is a regular preference on $X_i \times \Xi_i$. Then agent i has a monotone best response correspondence if and only if $\succsim_i$ obeys strict interval dominance.*

Proof. To show that $\succsim_i$ obeys SID, suppose that, for some $x''_i > x'_i$ and $\xi''_i > \xi'_i$, the left side of (3) holds. Letting $A_i = [x'_i, x''_i]$, we obtain $x''_i \in \mathrm{BR}_i(\xi'_i, A_i)$. Hence, by (2), it also holds that $x''_i \in \mathrm{BR}_i(\xi''_i, A_i)$. If $(x''_i, \xi''_i) \sim_i (x'_i, \xi''_i)$ were to hold, then $x'_i \in \mathrm{BR}_i(\xi''_i, A_i)$. However, then we have that $x''_i \in \mathrm{BR}_i(\xi'_i, A_i)$, $x'_i \in \mathrm{BR}_i(\xi''_i, A_i)$, and $x'_i < x''_i$, which contradicts (2). Therefore, $(x''_i, \xi''_i) \succ_i (x'_i, \xi''_i)$. Conversely, suppose $\xi''_i > \xi'_i$, $a''_i \in \mathrm{BR}_i(\xi''_i, A_i)$ and $a'_i \in \mathrm{BR}_i(\xi'_i, A_i)$. If $a''_i < a'_i$, then $(a'_i, \xi'_i) \succsim_i (a_i, \xi'_i)$ for every $x_i \in [a''_i, a'_i] \subset A_i$. SID guarantees that $(a'_i, \xi''_i) \succ_i (a''_i, \xi''_i)$, which contradicts the assumption that $a''_i \in \mathrm{BR}_i(\xi_i, A_i)$. □

Readers familiar with the standard theory of monotone comparative statics will notice that our definition of monotonicity (2) is stronger than the standard notion, which merely requires that $\mathrm{BR}_i(\xi''_i, A_i)$ dominates $\mathrm{BR}_i(\xi'_i, A_i)$ in the *strong set order*. This means that, for any $a''_i \in \mathrm{BR}_i(\xi''_i, A_i)$ and $a'_i \mathrm{BR}_i(\xi'_i, A_i)$, $\max\{a''_i, a'_i\} \in \mathrm{BR}_i(\xi''_i, A_i)$ and $\min\{a''_i, a'_i\} \in \mathrm{BR}_i(\xi'_i, A_i)$. In turn, this weaker notion of monotonicity can be characterized by preferences obeying *interval dominance* (rather than strict interval dominance), which is defined as follows: for every $a''_i > a'_i$ and $\xi''_i > \xi'_i$,

$$(a''_i, \xi'_i) \succsim_i (\succ_i)\, (a_i, \xi'_i) \text{ for every } a_i \in [a'_i, a''_i] \Longrightarrow (a''_i, \xi''_i) \succsim_i (\succ_i)\, (a'_i, \xi''_i). \tag{4}$$

(The reader can verify this claim by a straightforward modification of the proof of Theorem A or by consulting Theorem 1 in Quah and Strulovici (2009).) Throughout this paper we have chosen to work with this stronger notion of monotonicity; the weaker notion does not permit meaningful revealed preference analysis because it does not exclude the possibility that an agent is simply indifferent to all actions at every parameter value.[2]

2.2 Strategic complementarity

An important application of monotone comparative statics is to the study of games with strategic complementarity. Let $N = \{1, 2, ..., n\}$ be the set of agents in a game, and let $X_i \subset \mathbb{R}$ be the set of all conceivable actions of agent i. We assume that i has a feasible action set A_i that is a compact interval of X_i; as before, the family of compact intervals of X_i is denoted by $\mathcal{A}_i$. Agent i's choice over different feasible actions is affected by the actions of other players and also by an exogenous parameter y_i, which we assume is drawn from a poset $(Y_i, \geqslant)$. Let $\Xi_i = X_{-i} \times Y_i$, where $X_{-i} := \times_{j \neq i} X_j$. A typical element of Ξ_i is denoted by $\xi_i = (a_{-i}, y_i)$ and Ξ_i is a poset if we endow it with the product order. We assume that agent i has a preference $\succsim_i$ on $X_i \times \Xi_i$.

Given a profile of regular preferences $\{\succsim_i\}_{i \in N}$, a *joint feasible action set* $A \in \mathcal{A} = \times_{i \in N} \mathcal{A}_i$, and a profile of exogenous parameters $y \in Y = \times_{i \in N} Y_i$, we can define a game

$$\mathcal{G}(y, A) = \left[(y_i)_{i \in N}, (A_i)_{i \in N}, (\succsim_i)_{i \in N}\right].$$

We say that the family of games $\mathbb{G} = \{\mathcal{G}(y, A)\}_{(y,A) \in Y \times \mathcal{A}}$ exhibits *strategic complementarity* if, for every $A \in \mathcal{A}$, the best response of each agent i (as given by (1)) is monotone in $\xi_i = (a_{-i}, y_i)$. It is clear from Theorem A that *the family of games $\mathbb{G} = \{\mathcal{G}(y, A)\}_{(y,A) \in Y \times \mathcal{A}}$ exhibits strategic complementarity if and only if $\succsim_i$ is an SID preference for every agent i.*

As an example of such a family, consider the case of a Bertrand oligopoly with n firms, with each firm producing a single differentiated product. Assume that firm i has constant marginal cost $c_i > 0$, faces the demand function $D_i(p_i, p_{_i}) : \mathbb{R}_+ \times \mathbb{R}^{n-1}_{++} \to \mathbb{R}_+$, and chooses its price $p_i > 0$ to maximize profit $\Pi_i(p_i, p_{-i}, c_i) = (p_i - c_i) D_i(p_i, p_{-i})$. Suppose that the firms' products are substitutes in the sense that the own-price elasticity of demand,

$$-\frac{p_i}{D_i(p_i, p_{-i})} \frac{\partial D_i}{\partial p_i}(p_i, p_{-i})$$

[2] Our stronger assumption here is analogous to the assumption of local non-satiation made in Afriat's Theorem. It is clear that without such an assumption, any type of consumption data is rationalizable since one could simple suppose that the consumer is indifferent across all consumption bundles.

is strictly falling with respect to p_{-i} (the prices charged by other firms). These assumptions guarantee that, on any compact interval of prices, firm i's set of profit-maximizing prices is monotone in (p_{-i}, c_i).[3] If this property holds for every firm in the industry, the collection of Bertrand games generated by different feasible price sets to each firm and different exogenous parameters, $c = (c_i)_{i \in N}$, will constitute a collection of games exhibiting strategic complementarity.

It is known that the set of Nash equilibria of a game with strategic complementarity (even in the weaker sense of best responses increasing in the strong set order) is particularly well-behaved. The following result summarizes some of its properties.[4]

THEOREM B. *Suppose* $\mathbb{G} = \{\mathcal{G}(y, A)\}_{(y,A)\in Y\times\mathcal{A}}$ *exhibits strategic complementarity.*

1. [EXISTENCE] *Then, for every game* $\mathcal{G}(y, A) \in \mathbb{G}$, *the set of pure strategy Nash equilibria* $E(y, A)$ *is nonempty.*

2. [STRUCTURE] *For every game* $\mathcal{G}(y, A)$, $E(y, A)$ *has a largest and smallest element (which we denote by* $\max E(y, A)$ *and* $\min E(y, A)$ *respectively). Furthermore,* $E(y, A)$ *forms a* complete lattice, *i.e., every set* $K \subseteq E(y, A)$ *has a supremum and infimum in* $E(y, A)$, *i.e., the sets*

$$\begin{aligned}\mathcal{U}(K) &= \{z \in E(y, A) | z \geqslant x \text{ for all } x \in K\} \text{ and}\\ \mathcal{L}(K) &= \{z \in E(y, A) | z \leqslant x \text{ for all } x \in K\}\end{aligned}$$

   are nonempty and $\min \mathcal{U}(K)$ *and* $\max \mathcal{L}(K)$ *(respectively the supremum and infimum of* K *in* $E(y, A)$*) both exist.*

3. [COMPARATIVE STATICS] *The extremal equilibria of* $\mathcal{G}(y, A)$ *are both increasing in* y, *i.e., if* $y'' > y'$ *then* $\max E(y'', A) \geqslant \max(y', A)$ *and* $\min E(y'', A) \geqslant \min(y', A)$.

The set of Nash equilibria of $\mathcal{G}(y, A)$ coincides with the fixed points of the joint best response

[3] Specifically, they guarantee that for any $p_i'' > p_i'$, $\ln \Pi(p_i'', p_{-i}, c_i) - \ln \Pi(p_i', p_{-i}, c_i)$ is strictly increasing in (p_{-i}, c_i), which implies SID (see Milgrom and Shannon, 1994).

[4] For the proof of this result see Milgrom and Roberts (1990) and Vives (1990). The complete lattice structure of $E(A, y)$ was first pointed out in Zhou (1994).

correspondence $\text{BR}(\cdot, y, A) : A \rightrightarrows A$, where

$$\text{BR}(a, y, A) = (\text{BR}_1(\xi_1, A_1), \text{BR}_2(\xi_2, A_2), ..., \text{BR}_n(\xi_n, A_n)) .$$

The strong structural properties of $E(y, A)$ follow from the fact that this is a very well-behaved correspondence. Indeed, under strategic complementarity, for each $(y, A) \in Y \times \mathcal{A}$, $\text{BR}_i(\xi_i, A_i)$ is a compact subset of $\mathbb{R}$ and increasing in the sense of (2). Consequently, $\text{BR}(a, y, A)$ is a compact sublattice (hence subcomplete sublattice) of $\mathbb{R}^n$; furthermore the correspondence $\text{BR}(\cdot, y, A)$ is increasing in the sense that if $a'' > a'$, then $\bar{a}'' \geqslant \bar{a}'$ for any $\bar{a}'' \in \text{BR}(a'', y, A)$ and $\bar{a}' \in \text{BR}(a', y, A)$. With these observations, parts 1 and 2 of Theorem B follow immediately from Zhou's (1994) extension (to increasing correspondences) of Tarski's fixed point theorem.

Tarski-Zhou fixed point theorem. *Let $(L, \geqslant)$ be a complete lattice, and let a correspondence $F : L \rightrightarrows L$ be subcomplete sublattice-valued and increasing with respect to the strong set order. Then $(\mathcal{L}, \geqslant)$ is a nonempty complete lattice, where $\mathcal{L}$ is the set of fixed points of F.*

To obtain Part 3 of Theorem B, notice that $\text{BR}(a, y, A)$ is also increasing in y. Part 3 then follows from the following result, with $F(\cdot) = \text{BR}(\cdot, y'', A)$, $G(\cdot) = \text{BR}(\cdot, y', A)$, and $y'' > y'$.

Monotone fixed points theorem. *Suppose that both $F : L \rightrightarrows L$ and $G : L \rightrightarrows L$ obey the assumptions in Tarski-Zhou fixed point theorem and, for each $z \in L$, $F(z)$ dominates $G(z)$ in the strong set order. Then the largest and the smallest fixed points of F are respectively larger than the largest and the smallest fixed points of G.*[5]

In essence, our objective in this paper is to establish the choice-based counterparts of Theorems A and B. Our starting point is a data set drawn from a family of games. We characterize those data sets that are compatible with strategic complementarity and identify, for each player, the set of preferences that are compatible with his observed behavior. With this information, we ask how the same players will play in a new game, with difference strategy sets or parameters; we provide a computable characterization of the set of *possible* pure strategy Nash equilibria (i.e., the set of equilibria compatible with prior observations) and show that this set has a structure similar to that

[5]This result is originally shown in Milgrom and Roberts (1994) for single-valued functions, and extended to correspondences in Topkis (1998).

of the *actual* equilibria in a game with strategic complementarity (as outlined in Theorem B).

3 Revealed monotone choice

Consider an observer who collects a finite data set from agent i, where each observation consists of the action chosen by the agent, the set of feasible actions and the value of the parameter. Formally, the data set is $\mathcal{O}_i = \{(a_i^t, \xi_i^t, A_i^t)\}_{t \in \mathcal{T}}$, where $\mathcal{T} = \{1, 2, ..., T\}$, $a_i^t \in A_i^t$, and $A_i^t \in \mathcal{A}_i$. We say that $\mathcal{O}_i$ (or simply, agent i) is *consistent with monotonicity* or *monotone-rationalizable* if there is a regular and SID preference $\succsim_i$ on $X_i \times \Xi_i$ such that for every $t \in \mathcal{T}$, $(a_i^t, \xi_i^t) \succsim_i (x_i, \xi_i^t)$ for every $x_i \in A_i^t$. The motivation for this definition is clear: if $\mathcal{O}_i$ is monotone-rationalizable then we have found a preference that can (i) account for the observed behavior of the agent and (ii) guarantee that the agent's optimal choice based on this preference is increasing in the parameter, on any feasible action set that is a compact interval. Our principal objective in this section is to characterize monotone-rationalizability.

3.1 The axiom of revealed complementarity

It is useful to introduce the revealed preference relations induced by $\mathcal{O}_i$. The *direct revealed preference* relation $\succsim_i^R$ is defined in the following way: $(x_i'', \xi_i) \succsim_i^R (x_i', \xi_i)$ if $(x_i'', \xi_i) = (a_i^t, \xi_i^t)$ and $x_i' \in A_i^t$ for some $t \in \mathcal{T}$. The *indirect revealed preference* relation $\succsim_i^{RT}$ is the transitive closure of $\succsim_i^R$, i.e., $(x_i'', \xi_i) \succsim_i^{RT} (x_i', \xi_i)$ if there exists a finite sequence $z_i^1, z_i^2, ..., z_i^k$ in X_i such that

$$(x_i'', \xi_i) \succsim_i^R (z_i^1, \xi_i) \succsim_i^R (z_i^2, \xi_i) \succsim_i^R ... \succsim_i^R (z_i^k, \xi_i) \succsim_i^R (x_i', \xi_i). \tag{5}$$

The motivation for this terminology is clear. If we observe, at t, agent i playing x_i'' when x_i' is also feasible and other agents' actions and the the parameters are given by ξ_i, then it must be case that $(x_i'', \xi_i) \succsim_i (x_i', \xi_i)$ if agent i is optimizing with respect to the preference $\succsim_i$. Furthermore, given that $\succsim_i$ is transitive, if $(x_i'', \xi_i) \succsim_i^{RT} (x_i', \xi_i)$ then $(x_i'', \xi_i) \succsim_i (x_i', \xi_i)$.[6]

A relation $\mathcal{R}$ on $X_i \times \Xi_i$ said to have the *interval property* if, whenever $(x_i, \xi_i)\, \mathcal{R}\, (\tilde{x}_i, \xi_i)$, for x_i,

[6]Note, however, that $\succsim_i^R$ and $\succsim_i^{RT}$ are not generally complete on X_i for every fixed ξ_i; as such these relations are not preferences as we have defined them.

$\tilde{x}_i$ in X_i, then $(x_i, \xi_i)\,\mathcal{R}\,(z_i, \xi_i)$ for any z_i between x_i and $\tilde{x}_i$, i.e., $x_i \leqslant z_i \leqslant \tilde{x}_i$ or $\tilde{x}_i \leqslant z_i \leqslant x_i$. This property plays an important role in our results. The lemma below uses the assumption that feasible action sets are compact intervals to guarantee that $\succsim_i^{RT}$ has the interval property.

Lemma 1. *The relation $\succsim_i^{RT}$ in $X_i \times \Xi_i$ induced by $\mathcal{O}_i = \{a_i^t, \xi_i^t, A_i^t\}_{t=1}^T$ has the interval property.*

Proof. If $(x_i'', \xi_i) \succsim_i^R (x_i', \xi_i)$, then there is A_i^t such that $x_i'' = a_i^t$ and $x_i' \in A_i^t$. The interval property the follows immediately from the assumption that A_i^t is a closed interval. Now suppose $(x_i'', \xi_i) \succsim_i^{RT} (x_i', \xi_i)$, but $(x_i'', \xi_i) \not\succsim_i^R (x_i', \xi_i)$. Then, we have a sequence like (5). Suppose also that $x_i'' > x_i'$ and consider x_i such that $x_i'' > x_i > x_i'$. (The case where $x_i'' < x_i'$ can be handled in a similar way.) Letting $z_i^0 = x_i''$ and $z_i^{k+1} = x_i'$, we know that there exists at least one $0 \leqslant m \leqslant k$ such that $z_i^m \geqslant x_i \geqslant z^{m+1}$. Since $(z_i^m, \xi_i) \succsim_i^R (z_i^{m+1}, \xi_i)$, it must hold that $(z_i^m, \xi_i) \succsim_i^R (x_i, \xi_i)$. This in turn implies that $(x_i'', \xi_i) = (z_i^0, \xi_i) \succsim_i^{RT} (x_i, \xi_i)$, since $(z_i^0, \xi_i) \succsim_i^{RT} (z_i^m, \xi_i)$. □

Definition 1. *The data set $\mathcal{O}_i = \{a_i^t, \xi_i^t, A_i^t\}_{t=1}^T$ obeys the* Axiom of Revealed Complementarity (ARC) *if, for every $s, t \in \mathcal{T}$,*

$$\xi_i^t > \xi_i^s,\ a_i^t < a_i^s,\ \text{and } (a_i^s, \xi_i^s) \succsim_i^{RT} (a_i^t, \xi_i^s) \Longrightarrow (a_i^t, \xi_i^t) \not\succsim_i^{RT} (a_i^s, \xi_i^t). \tag{6}$$

The examples presented in the Introduction show that ARC is a non-vacuous restriction on data. So long as the number of observations $\mathcal{O}_i$ is finite (as it is by assumption), checking whether two elements (a_i^s, ξ_i^s) and (a_i^t, ξ_i^s) are related by $\succsim_i^{RT}$ is a finite procedure and, consequently, so is checking for ARC. It is also clear that there are no computational difficulties, whether theoretical or practical, associated with the implementation of this test. The main result of this section characterizes monotone-rationalizability in terms of ARC.

Theorem 1. *A data set $\mathcal{O}_i = \{a_i^t, \xi_i^t, A_i^t\}_{t=1}^T$ is monotone-rationalizable if and only if it obeys ARC.*

The necessity of ARC for monotone-rationalisability is relatively easy to prove and we shall do that first. Notice that the proof makes crucial use of the interval property and hence the requirement that A_i^t are intervals.

Proof: Suppose there are observation s and t such that $\xi_i^t > \xi_i^s$, $a_i^t < a_i^s$, and $(a_i^s, \xi_i^s) \succsim_i^{RT} (a_i^t, \xi_i^s)$. By Lemma 1, $\succsim_i^{RT}$ has the interval property, and so $(a_i^s, \xi_i^s) \succsim_i^{RT} (x_i, \xi_i^s)$ for all $x_i \in [a_i^t, a_i^s]$. Since

$\mathcal{O}_i$ is SID-rationalizable, there is an SID preference $\succsim_i$ on $X_i \times \Xi_i$ such that $(a_i^s, \xi_i^s) \succsim_i (x_i, \xi_i^s)$ for all $x_i \in [a_i^t, a_i^s]$. The SID property on $\succsim_i$ guarantees that $(a_i^s, \xi_i^t) \succ_i (a_i^t, \xi_i^t)$, which means $(a_i^t, \xi_i^t) \not\succsim_i^{RT} (a_i^s, \xi_i^t)$. □

3.2 Sufficiency of ARC in Theorem 1

Our proof of the sufficiency of ARC involves first working out the (incomplete) revealed preference relations on $X_i \times \Xi_i$ that *must* be satisfied by any SID preference that rationalizes the data and then explicitly constructing a rationalizing preference on $X_i \times \Xi_i$ that completes that incomplete relation. In fact the rationalizing preference we construct for each agent will obey *strict single crossing differences* (SSCD), which is a stronger property than SID. A preference relation $\succsim_i$ is said to obey strict single crossing differences if, for every $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$,

$$(x_i'', \xi_i') \succsim_i (x_i', \xi_i') \Longrightarrow (x_i'', \xi_i'') \succ_i (x_i', \xi_i''). \tag{7}$$

It is clear that every preference that obeys SSCD will also satisfy SID.

Given $\mathcal{O}_i$, the *single crossing extension* of the indirect revealed preference relation $\succsim_i^{RT}$ is another binary relation $\succ_i^{RTS}$ defined in the following way:

(i) for $x_i'' > x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$ if there is $\xi_i' < \xi_i$ such that $(x_i'', \xi_i') \succsim_i^{RT} (x_i', \xi_i')$;

(ii) for $x_i'' < x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$, if there is $\xi_i'' > \xi_i$ such that $(x_i'', \xi_i'') \succsim_i^{RT} (x_i', \xi_i'')$.

Let $\succsim_i^{RTS}$ be the binary relation given by $\succsim_i^{RTS} = \succsim_i^{RT} \cup \succ_i^{RTS}$.[7] It follows immediately from its definition that $\succsim_i^{RTS}$ also has strict single crossing differences, in the following sense: if $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$ *or* $x_i'' < x_i'$ and $\xi_i'' < \xi_i'$, then

$$(x_i'', \xi_i') \succsim_i^{RTS} (x_i', \xi_i') \Longrightarrow (x_i'', \xi_i'') \succ_i^{RTS} (x_i', \xi_i''). \tag{8}$$

In addition, let $\succsim_i^{RTST}$ be the transitive closure of $\succsim_i^{RTS}$, i.e., $(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ if there exists

[7]Note that, as defined, $\succ_i^{RTS}$ is *not* the asymmetric part of $\succsim_i^{RTS}$.

a sequence $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (x_i', \xi_i). \tag{9}$$

If we can find at least one strict relation $\succ_i^{RTS}$ in the sequence (9), then, we let $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$. The relevance of these relations flows from the following result, which shows that any rationalizing preference for agent i must respect the ranking implied by these revealed preference relations.

Proposition 1. *Suppose that the preference $\succsim_i$ obeys SID and rationalizes $\mathcal{O}_i = \{a_i^t, \xi_i^t, A_i^t\}_{t \in \mathcal{T}}$. Then $\succsim_i$* extends $\succsim_i^{RTST}$ and $\succ_i^{RTST}$ *in the following sense:*

$$(x_i'', \xi_i) \succsim_i^{RTST} (\succ_i^{RTST}) (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i (\succ_i) (x_i', \xi_i) \tag{10}$$

Proof. Without loss of generality, we may let $x_i'' > x_i'$. Since $\succsim_i$ is transitive, it is clear that we need only show that $(x_i'', \xi_i) \succsim_i (\succ_i) (x_i', \xi_i)$ whenever $(x_i'', \xi_i) \succsim_i^{RTS} (\succ_i^{RTS}) (x_i', \xi_i)$. If $(x_i'', \xi_i) \succsim_i^{RTS} (\succ_i^{RTS}) (x_i', \xi_i)$ then there exists some $\xi_i' \leqslant (<) \xi_i$ such that $(x_i'', \xi_i') \succsim_i^{RT} (x_i', \xi_i')$. By the interval property of $\succsim_i^{RT}$, we obtain $(x_i'', \xi_i') \succsim_i^{RT} (x_i, \xi_i')$ for all $x_i \in [x_i', x_i'']$. Since $\succsim_i$ rationalizes $\mathcal{O}_i$, we also have $(x_i'', \xi_i') \succsim_i (x_i, \xi_i')$ for all $x_i \in [x_i', x_i'']$. By SID of $\succsim_i$, we obtain $(x_i'', \xi_i) \succsim_i (\succ_i) (x_i', \xi_i)$ for $\xi_i' \leqslant (<) \xi_i$. □

At this point, it is reasonable to ask if we could go beyond the revealed preference relations we have already constructed and consider the single crossing extension of $\succsim_i^{RTST}$, the transitive closure of that extension, and so on. The answer to that is 'no' because, as we shall see in Lemma 4, $\succsim_i^{RTST}$ obeys SSCD, so it does not admit a nontrivial single crossing extension. Thus it is intuitive to believe that all the information on agent i's preference conveyed by the data set $\mathcal{O}_i$ is encoded, no more and no less, in the revealed preference relations $\succsim_i^{RTST}$ and $\succ_i^{RTST}$; a formal statement of this claim is in Theorem 2 in Section 2.4. Note also that Proposition 1 has a converse: if there is a regular and SID preference on $X_i \times \Xi_i$ that obeys (10), then $\mathcal{O}_i$ is monotone-rationalizable. This follows immediately from the fact that $\succsim_i^R \subseteq \succsim_i^{RTST}$.

Given these observations, a reasonable way of constructing a rationalizing preference is to begin with $\succsim_i^{RTST}$ and $\succ_i^{RTST}$ and then complete these incomplete relations in a way that gives a preference with the required properties. This is precisely the approach we take. To that end, we first establish some relevant properties of $\succsim_i^{RTS}$ and $\succsim_i^{RTST}$ in Lemmas 2, 3, and 4. These pave the way for the construction of $\succsim_i^*$ in Lemma 5, which is shown to extend $\succsim_i^{RTST}$ and $\succ_i^{RTST}$ and to obey SSCD. We then complete the proof of the sufficiency part of Theorem 1 with Lemma 6, which establishes the regularity of $\succsim_i^*$.

Lemma 2. *The binary relations* $\succsim_i^{RT}$, $\succsim_i^{RTS}$, $\succ_i^{RTS}$, *and* $\succsim_i^{RTST}$ *on* $X_i \times \Xi_i$ *induced by* $\mathcal{O}_i = \{a_i^t, \xi_i^t, A_i^t\}_{t=1}^T$ *all have the interval property.*

Proof. We have already established in Lemma 1 that $\succsim_i^{RT}$ has the interval property. Let $x_i'' > x_i > x_i'$. (The case where $x_i'' < x_i < x_i'$ can be proved in a similar way.) If $(x_i'', \xi_i) \succsim_i^{RTS}$ $(\succ_i^{RTS})\,(x_i', \xi_i)$ holds, there exists some $\xi_i' \leqslant (<)\,\xi_i$ such that $(x_i'', \xi_i') \succsim_i^{RT} (x_i', \xi_i')$. By the interval property of $\succsim_i^{RT}$, we obtain $(x_i'', \xi_i') \succsim_i^{RT} (x_i, \xi_i')$. Since $x_i'' > x_i$ and $\xi_i' \leqslant (<)\,\xi_i$, we have that $(x_i'', \xi_i) \succsim_i^{RTS} (\succ_i^{RTS})\,(x_i, \xi_i)$. So we have shown that $\succsim_i^{RTS}$ and $\succ_i^{RTS}$ have the interval property.

If $(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$, there exists a sequence $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (x_i', \xi_i).$$

Letting $z_i^0 = x_i''$ and $z_i^{k+1} = x_i'$, since $x_i'' > x_i > x_i'$, we can find some $0 \leqslant m \leqslant k$ such that $z_i^m \geqslant x_i \geqslant z_i^{m+1}$. By the interval property of $\succsim_i^{RTS}$, we obtain $(z_i^m, \xi_i) \succsim_i^{RTS} (x_i, \xi_i)$. Thus $(x_i'', \xi_i) \succsim_i^{RTST} (x_i, \xi_i)$ since $(x_i'', \xi_i) \succsim_i^{RTST} (z_i^m, \xi_i) \succsim_i^{RTS} (x_i, \xi_i)$. □

Lemma 2 is used in the next two lemmas, both of which exploit the combination of the interval property and ARC.

Lemma 3. *Suppose that* $\mathcal{O}_i$ *obeys ARC. Then* $\succsim_i^{RTS}$ *is* cyclically consistent, *i.e.,*

$$(z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \Longrightarrow (z_i^k, \xi_i) \not\succ_i^{RTS} (z_i^1, \xi_i). \tag{11}$$

Remark: Cyclical consistency can be equivalently re-formulated as the following:

$$(z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \tag{12}$$

$$\Longrightarrow (z_i^1, \xi_i) \not\succ_i^{RTS} (z_i^2, \xi_i) \not\succ_i^{RTS} ... \not\succ_i^{RTS} (z_i^k, \xi_i) \not\succ_i^{RTS} (z_i^1, \xi_i)$$

Thus, whenever there is a cycle like (12), it *must* be the case that

$$(z_i^1, \xi_i) \succsim_i^{RT} (z_i^2, \xi_i) \succsim_i^{RT} ... \succsim_i^{RT} (z_i^k, \xi_i) \succsim_i^{RT} (z_i^1, \xi_i)$$

Proof. We prove this by induction on the length of the chain, k, on the left side of (11). Whenever (11) holds for chains of length k or less (equivalently, whenever the cycles in (12) have length k or less), we say that $\succsim_i^{RTS}$ is k-consistent. For 2-consistency, we need to show that

$$(z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \Longrightarrow (z_i^2, \xi_i) \not\succ_i^{RTS} (z_i^1, \xi_i).$$

Suppose that $z_i^1 > z_i^2$; the case of $z_i^1 < z_i^2$ can be dealt with in a similar way. By definition, if $(z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i)$ then there is $\xi_i' \leqslant \xi_i$ such that $(z_i^1, \xi_i') \succsim_i^{RT} (z_i^2, \xi_i')$. On the other hand, if $(z_i^2, \xi_i) \succ_i^{RTS} (z_i^1, \xi_i)$, then there is $\xi_i'' > \xi_i$ such that $(z_i^2, \xi_i'') \succsim_i^{RT} (z_i^1, \xi_i'')$ and so we obtain a violation of ARC.

Suppose that $\succsim_i^{RTS}$ is k-consistent for all $k < \bar{k}$. To show that $\bar{k}$-consistency holds, suppose the left side of (11) holds for $k = \bar{k}$ and $z_i^1 < z_i^{\bar{k}}$. Clearly, there must be $m < \bar{k}$ such that $z_i^m < z_i^{\bar{k}}$ and $z_i^{m+1} \geqslant_i z_i^{\bar{k}}$. We consider two cases separately: (A) $z_i^m \geqslant_i z_i^1$ and (B) $z_i^m < z_i^1$. In case (A), by the interval property of $\succsim_i^{RTS}$, we obtain $(z_i^m, \xi_i) \succsim_i^{RTS} (z_i^{\bar{k}}, \xi_i)$. By way of contradiction, suppose also that $(z_i^{\bar{k}}, \xi_i) \succ_i^{RTS} (z_i^1, \xi_i)$. Then the interval property of $\succ_i^{RTS}$ guarantees that $(z_i^{\bar{k}}, \xi_i) \succ_i^{RTS} (z_i^m, \xi_i)$ and so we obtain a violation of 2-consistency. For case (B), since $(z_i^m, \xi_i) \succsim_i^{RTS} (z_i^{m+1}, \xi_i)$, the interval property guarantees that $(z_i^m, \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i)$. So we obtain the cycle

$$(z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^m, \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \tag{13}$$

which has length strictly lower than $\bar{k}$. By the induction hypothesis, we obtain

$$(z_i^1, \xi_i) \not\succ_i^{RTS} (z_i^2, \xi_i) \not\succ_i^{RTS} ... \not\succ_i^{RTS} (z_i^m, \xi_i) \not\succ_i^{RTS} (z_i^1, \xi_i)$$

and so we can replace each $\succsim_i^{RTS}$ in (13) by $\succsim_i^{RT}$. Furthermore, $(z_i^m, \xi_i) \not\succ_i^{RTS} (z_i^1, \xi_i)$ guarantees that $(z_i^m, \xi_i) \not\succ_i^{RTS} (z_i^{m+1}, \xi_i)$, by the interval property of $\succ_i^{RTS}$. Therefore, $(z_i^1, \xi_i) \succsim_i^{RT} (x_i^{m+1}, \xi_i)$ and, by the interval property of $\succsim_i^{RT}$, we obtain $(z_i^1, \xi_i) \succsim_i^{RT} (x_i^{\bar{k}}, \xi_i)$. 2-consistency then ensures that $(z_i^{\bar{k}}, \xi_i) \not\succ_i^{RTS} (z_i^1, \xi_i)$. $\square$

LEMMA 4. *Suppose that $\mathcal{O}_i$ obeys ARC. Then $\succsim_i^{RTST}$ obeys SSCD.*

Proof. By definition, $\succsim_i^{RTST}$ obeys SSCD if whenever $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$ *or* $x_i'' < x_i'$ and $\xi_i'' < \xi_i'$, then

$$(x_i'', \xi_i') \succsim_i^{RTST} (x_i', \xi_i') \Longrightarrow (x_i'', \xi_i'') \succ_i^{RTST} (x_i', \xi_i'').$$

We shall concentrate on the case where $x_i'' > x_i'$; the other case has a similar proof. If $(x_i'', \xi_i) \succsim_i^{RTS} (x_i', \xi_i)$, then we know that there is z_i^j (for $j = 1, 2, ..., k$) such that

$$(x_i'', \xi_i') \succsim_i^{RTS} (z_i^1, \xi_i') \succsim_i^{RTS} (z_i^2, \xi_i') \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i') \succsim_i^{RTS} (x_i', \xi_i'). \tag{14}$$

We can also choose a chain with the property that (writing $z_i^0 = x_i''$ and $z_i^{k+1} = x_i'$) $(z_i^m, \xi_i') \not\succsim_i^{RTS} (z_i^{m'}, \xi_i')$ for $m' > m + 1$; in other words, no link in the chain can be dropped. We claim that, for such a chain, we must have

$$x_i'' > z_i^1 > z_i^2 > ... > z_i^k > x_i'. \tag{15}$$

Once this is established, the rest is straightforward: since $\succsim_i^{RTS}$ obeys SSCD, (14) and (15) imply

$$(x_i'', \xi_i'') \succ_i^{RTS} (z_i^1, \xi_i'') \succ_i^{RTS} (z_i^2, \xi_i'') \succ_i^{RTS} ... \succ_i^{RTS} (z_i^k, \xi_i'') \succ_i^{RTS} (x_i', \xi_i'')$$

and so $(x_i'', \xi_i'') \succ_i^{RTST} (x_i', \xi_i'')$.

It remains for us to establish (15). If this is false then there is m such that $z_i^{m+1} > z_i^m$. Let z_i^{m+n} be the first time after z_i^{m+1} such that $z_i^{m+n} \leqslant z_i^m$. Then we have $z_i^{m+n} \leqslant z_i^m < z_i^{m+n-1}$. Since $(z_i^{m+n-1}, \xi_i') \succsim_i^{RTS} (z_i^{m+n}, \xi_i')$,the interval property of $\succsim_i^{RTS}$ guarantees that $(z_i^{m+n-1}, \xi_i') \succsim_i^{RTS} (z_i^m, \xi_i')$. Thus we obtain a cycle

$$(z_i^m, \xi_i') \succsim_i^{RTS} (z_i^{m+1}, \xi_i') \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^{m+n-1}, \xi_i') \succsim_i^{RTS} (z_i^m, \xi_i').$$

By Lemma 3, we know that the terms in this chain cannot be related by $\succ_i^{RTS}$ and must be related by $\succsim_i^{RT}$. In particular, $(z_i^{m+n-1}, \xi_i') \not\succ_i^{RTS} (z_i^m, \xi_i')$ and thus $(z_i^{m+n-1}, \xi_i') \not\succ_i^{RTS} (z_i^{m+n}, \xi_i')$ (by the interval property of $\succ_i^{RTS}$). We conclude that $(z_i^m, \xi_i') \succsim_i^{RT} (z_i^{m+n}, \xi_i')$ and thus we can shorten the chain in (14) to

$$(x_i'', \xi_i') \succsim_i^{RTS} (z_i^1, \xi_i') \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^m, \xi_i') \succsim_i^{RTS} (z_i^{m+n}, \xi_i') \succsim_i^{RTS} ... (z_i^k, \xi_i') \succsim_i^{RTS} (x_i', \xi_i')$$

which contradicts our assumption that no link in the chain can be dropped. □

We are now ready to define the preference we use for rationalizing $\mathcal{O}$. Define the binary relation $\succsim_i^*$ on $X_i \times \Xi_i$ in the following manner:

$$\begin{aligned}(x_i'', \xi_i) \succsim_i^* (x_i', \xi_i) \text{ if } & (x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i) \\ & \text{or } (x_i'', \xi_i) \parallel_i^{RTST} (x_i', \xi_i) \text{ and } x_i' \geqslant x_i'',\end{aligned} \tag{16}$$

where $(x_i'', \xi_i) \parallel_i^{RTST} (x_i', \xi_i)$ means neither $(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ nor $(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$.

The following two lemmas will complete the proof of Theorem 1.

Lemma 5. *The binary relation $\succsim_i^*$ is a preference that rationalizes $\mathcal{O}_i$. Furthermore, it extends $\succsim_i^{RTST}$ and $\succ_i^{RTST}$ in the sense that*

$$(x_i'', \xi_i) \succsim_i^{RTST} (\succ_i^{RTST}) (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i^* (\succ_i^*) (x_i', \xi_i) \tag{17}$$

and obeys SSCD.

Proof. Clearly, $\succsim^*$ is complete and reflexive, so to demonstrate that it is a preference we need only show that it is transitive. Indeed, suppose

$$(a_i, \xi_i) \succsim^* (b_i, \xi_i) \succsim^* (c_i, \xi_i) \succsim^* (a_i, \xi_i).$$

There are only four fundamentally distinct cases we need to consider:
Case 1. None of the three elements are related by $\succsim_i^{RTST}$. Given the definition of $\succsim^*$, this means that $a_i < b_i < c_i < a_i$, which is impossible.
Case 2. $a_i < b_i < c_i$, $(a_i, \xi_i) \parallel_i^{RTST} (b_i, \xi_i)$, $(b_i, \xi_i) \parallel_i^{RTST} (c_i, \xi_i)$, and $(c_i, \xi_i) \succsim_i^{RTST} (a_i, \xi_i)$. This is again impossible since the interval property of $\succsim_i^{RTST}$ will imply that $(c_i, \xi_i) \succsim_i^{RTST} (b_i, \xi_i)$.
Case 3. $a_i < b_i$, $(a_i, \xi_i) \parallel_i^{RTST} (b_i, \xi_i)$, $(b_i, \xi_i) \succsim_i^{RTST} (c_i, \xi_i) \succsim_i^{RTST} (a_i, \xi_i)$. This is also impossible

because, by the transitivity of $\succsim^{RTST}$, we obtain $(b_i, \xi_i) \succsim_i^{RTST} (a_i, \xi_i)$.
Case 4. $(a_i, \xi_i) \succsim_i^{RTST} (b_i, \xi_i) \succsim_i^{RTST} (c_i, \xi_i) \succsim_i^{RTST} (a_i, \xi_i)$. By Lemma 3, this is only possible if

$$(a_i, \xi_i) \succsim_i^{RT} (b_i, \xi_i) \succsim_i^{RT} (c_i, \xi_i) \succsim_i^{RT} (a_i, \xi_i),$$

but then we also obtain, by the transitivity of $\succsim_i^{RT}$, $(a_i, \xi_i) \succsim_i^{RT} (c_i, \xi_i)$ and, hence, $(a_i, \xi_i) \succsim^* (c_i, \xi_i)$.

Since $\succsim^{RTST} \subset \succsim^*$ by construction, it is clear that $\succsim^*$ rationalizes $\mathcal{O}_i$. To prove (17), first note that $(x_i'', \xi_i) \succsim_i^* (x_i', \xi_i)$ if $(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ by construction. If $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$, then Lemma 3 says that $(x_i', \xi_i) \not\succsim_i^{RTST} (x_i'', \xi_i)$. Thus $(x_i', \xi_i) \not\succsim_i^* (x_i'', \xi_i)$ as obtain $(x_i'', \xi_i) \succ_i^* (x_i', \xi_i)$

Lastly, to show that $\succsim^*$ obeys SSCD, let $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$; then

$$\begin{aligned}(x_i'', \xi_i') \succsim_i^* (x_i', \xi_i') &\Longrightarrow (x_i'', \xi_i') \succsim_i^{RTST} (x_i', \xi_i') \\ &\Longrightarrow (x_i'', \xi_i'') \succ_i^{RTST} (x_i', \xi_i'') \\ &\Longrightarrow (x_i'', \xi_i'') \succ_i^* (x_i', \xi_i''),\end{aligned}$$

in which the first implication follows from the definition of $\succsim_i^*$, the second implication from Lemma 4, and the third from (17). □

LEMMA 6. *For every compact interval A_i and every $\xi_i \in \Xi_i$, $\mathrm{BR}(\xi_i, A_i, \succsim^*)$ is nonempty and has finitely many elements. In particular, $\succsim_i^*$ is a regular preference.*

Proof. Let $[m, n]$ be a compact interval of Ξ_i. If $[m, n] \not\ni a_i^t$ for every $t \in \mathcal{T}$, then, it follows from the definition of $\succsim_i^*$ that $(m, \xi_i) \succsim_i^* (z_i, \xi_i)$. In this case, m is only one maximiser of $\succsim_i^*$ on $[m, n]$. Suppose that $[m, n] \ni a_i^t$ for some t. Since there are a finite number of observations, we can find some $a_i^s \in [m, n]$ such that $(a_i^s, \xi_i) \succsim_i^* (a_i^t, \xi_i)$ for every $a_i^t \in [m, n]$. We claim that either m or a_i^s is the maximiser of $\succsim_i^*$ on $[m, n]$ for ξ_i. There are two cases to consider.

Suppose $(m, \xi_i) \succsim_i^* (a_i^s, \xi_i)$ and there is $z_i \in [m, n]$ such that $(z_i, \xi_i) \succ_i^* (m, \xi_i)$. Then, since $m < z_i$, it must hold that $(z_i, \xi_i) \succ_i^{RTST} (m, \xi_i)$ and there is $t \in \mathcal{T}$ such that $z_i = a_i^t$. Consequently,

$$(a_i^t, \xi_i) \succ_i^* (m, \xi_i) \succsim_i^* (a_i^s, \xi_i) \succsim_i^* (a_i^t, \xi_i),$$

which is a contradiction. Therefore, $(m, \xi_i) \succsim_i^* (x_i, \xi_i)$ for all $x_i \in [m, n]$. Now suppose $(a_i^s, \xi_i) \succsim_i^*$

(m, ξ_i). For every $x_i \in [m, n]$, either $(a_i^s, \xi_i) \succsim_i^{RTST} (x_i, \xi_i)$, in which case $(a_i^s, \xi_i) \succsim_i^* (x_i, \xi_i)$, or $(a_i^s, \xi_i) \|_i^{RTST} (x_i, \xi_i)$, in which case we have $(a_i^s, \xi_i) \succsim_i^* (m, \xi_i) \succsim_i^* (x_i, \xi_i)$. Thus $(a_i^s, \xi_i) \succsim_i^* (x_i, \xi_i)$ for all $x_i \in [m, n]$. □

3.3 ARC and SSCD

Theorem 1 tells us that when an agent has an SID preference, then any data set collected from this agent must obey ARC. It also says that if a data set obeys ARC, then the agent's actions can be accounted for by an SID preference; indeed, we can explicitly construct a preference consistent with those observations that obey the *stronger* property of SSCD (see Lemma 4).[8] We know that SSCD is necessary and sufficient for an agent's optimal action to be increasing with the parameter on all *arbitrary* constraint sets drawn from X_i (see Milgrom and Shannon, 1994). It follows that when a data set $\mathcal{O}_i$ is monotone-rationalizable, we can find a preference that both explains the data and guarantees that the optimal choices based on this preference is monotone, on any arbitrary feasible action set (and not just intervals).

So far we have always maintained the assumption that the observed feasible action sets A_i^t are intervals. Now consider a data set $\mathcal{O}_i = \{a_i^t, \xi_i^t, B_i^t\}_{t \in \mathcal{T}}$, where a_i^t is the observed choice from B_i^t, and B_i^t is a compact subset of X_i that is not necessarily an interval. It is easy to check that if $\mathcal{O}_i$ is rationalizable by an SSCD preference then it must obey ARC and, given the characterization of SSCD preferences, we may be tempted to think that the converse is true. However, as the following example shows, that is *not* the case and so a revealed preference theory built around arbitrary observed feasible action sets and SSCD must involve a data set property different from ARC; we leave this interesting issue to further research.

Example 1. Let $X_i = \{\alpha_i, \beta_i, \gamma_i\}$ with $\alpha_i < \beta_i < \gamma_i$, and let $A_i^1 = \{\alpha_i, \gamma_i\}$, $A_i^2 = \{\alpha_i, \beta_i\}$, and $A_i^3 = \{\beta_i, \gamma_i\}$. Note that A_i^1 is not an interval of X_i. Suppose that $\xi_i^1 < \xi_i^2 < \xi_i^3$, and that $a_i^1 = \gamma_i$, $a_i^2 = \alpha_i$, and $a_i^3 = \beta_i$. Then $(\gamma_i, \xi_i^1) \succsim_i^R (\alpha_i, \xi_i^1)$, $(\alpha_i, \xi_i^2) \succsim_i^R (\beta_i, \xi_i^2)$, and $(\beta_i, \xi_i^3) \succsim_i^R (\gamma_i, \xi_i^3)$. The

[8]This phenomenon, which may seem surprising, is well known in revealed preference analysis, partly because it is also present in Afriat's Theorem. In that context, the data consist of observations of consumer's consumption bundles at different linear budget sets. If the agent is maximizing a locally non-satiated preference then the data set must obey a property called the generalized axiom of revealed preference (GARP, for short); conversely, if a data set obeys GARP then it can be rationalized by a preference that is not just locally non-satiated but also obeys strong monotonicity, quasi-concavity, and other properties.

indirect revealed preference relation $\succsim_i^{RT}$ is equal to the direct revealed preference relation $\succsim_i^{R}$ in this example and, clearly, this set of three observations obeys ARC. However, it cannot be rationalized by an SSCD preference. Suppose, instead that an SSCD preference $\succsim_i$ rationalizes the data. Then, it must hold that $(\gamma_i, \xi_i^1) \succsim_i (\alpha_i, \xi_i^1)$ and, by SSCD, $(\gamma_i, \xi_i^2) \succ_i (\alpha_i, \xi_i^2)$. In addition, we have $(\alpha_i, \xi_i^2) \succsim_i (\beta_i, \xi_i^2)$ and so $(\gamma_i, \xi_i^2) \succ_i (\beta_i, \xi_i^2)$. Since $\succsim_i$ obeys SSCD, we obtain $(\gamma_i, \xi_i^3) \succ_i (\beta_i, \xi_i^3)$, which contradicts the direct revealed preference $(\beta_i, \xi_i^3) \succ_i (\gamma_i, \xi_i^3)$.

3.4 Robust inference

For a given data set $\mathcal{O}_i$ that obeys ARC, there will typically be more than one preference that rationalizes an agent's observed actions. For example, it is quite clear that the following simple variation on (16) is also a regular preference on $X_i \times \Xi_i$ that obeys SSCD (hence SID) and rationalizes $\mathcal{O}_i$ (when $\mathcal{O}_i$ obeys ARC):

$$(x_i'', \xi_i) \succsim_i^* (x_i', \xi_i) \text{ if } (x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i) \text{ or } (x_i'', \xi_i) \parallel_i^{RTST} (x_i', \xi_i) \text{ and } x_i' \leqslant x_i'', \tag{18}$$

Given this, it would be desirable to characterize that part of agent i's preference that an observer could *robustly infer* from the data; by this we mean those preference relationships that are valid for *all* regular preferences that rationalize $\mathcal{O}_i$ and that obey SID or SSCD. The next result shows that this is captured by the revealed preference relations $\succsim_i^{RTST}$ and $\succ_i^{RTST}$.

THEOREM 2. *Suppose that $\mathcal{O}_i$ obeys ARC and let $\mathcal{P}_i^\star$ be the family of regular preferences on $X_i \times \Xi_i$ that obey SID and rationalize $\mathcal{O}_i$. Then*

(i) $(x_i, \xi_i) \succsim_i (x_i', \xi_i)$ *for all* $\succsim_i \in \mathcal{P}_i^\star$ *if and only if* $(x_i, \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$*;*

(ii) $(x_i, \xi_i) \succ_i (x_i', \xi_i)$ *for all* $\succsim_i \in \mathcal{P}_i^\star$ *if and only if* $(x_i, \xi_i) \succ_i^{RTST} (x_i', \xi_i)$*.*

The statements (i) *and* (ii) *remain valid if* $\mathcal{P}_i^\star$ *is replaced with* $\mathcal{P}_i^{\star\star}$*, the family of regular preferences on* $X_i \times \Xi_i$ *that obey SSCD and rationalize* $\mathcal{O}_i$*.*

Proof. In Proposition 1, we show the "if" part of both statements (i) and (ii) for preferences that obey SID and rationalize $\mathcal{O}_i$. Hence they must remain valid for preferences that obey single crossing differences (which is a stronger property than SID). It remains for us to show the "only if" part of both statements for preferences that obey single crossing differences. This completes the proof since $\mathcal{P}_i^{\star\star} \subset \mathcal{P}_i^{\star}$.

To show the "only if" of statement (i), suppose that $(x_i, \xi_i) \not\succsim_i^{RTST} (x_i', \xi_i)$. Then either $(x_i', \xi_i) \succ_i^{RTST} (x_i, \xi_i)$ or $(x_i, \xi_i) \parallel_i^{RTST} (x_i', \xi_i)$. The first case implies that $(x_i', \xi_i) \succ_i (x_i, \xi_i)$ for all $\succsim_i \in \mathcal{P}_i^{\star\star}$, so clearly we do *not* obtain $(x_i, \xi_i) \succsim_i (x_i', \xi_i)$ for all $\succsim_i \in \mathcal{P}_i^{\star\star}$. For the second case, we know that there exists a preference $\succsim^*$ in $\mathcal{P}_i^{\star\star}$ such that $(x_i', \xi_i) \succ_i^* (x_i, \xi_i)$ and hence, again, we do not have $(x_i, \xi_i) \not\succsim_i (x_i', \xi_i)$ for all $\succsim_i \in \mathcal{P}_i^{\star\star}$; indeed, either the preference in (16) or that in (18) can serve as such a preference, depending (respectively) on whether x_i' is smaller or greater than x_i.

We turn now to the "only if" part of statement (ii), for preferences in $\mathcal{P}_i^{\star\star}$. Suppose that $(x_i, \xi_i) \not\succ_i^{RTST} (x_i', \xi_i)$; then either $(x_i, \xi_i) \parallel_i^{RTST} (x_i', \xi_i)$ or $(x_i', \xi_i) \succsim_i^{RTST} (x_i, \xi_i)$. In the first case, we know that we can find a preference $\succsim_i^*$ (using either (16) or (18)) such that $(x_i', \xi_i) \succ_i^* (x_i, \xi_i)$, so we do *not* obtain $(x_i, \xi_i) \succ_i (x_i', \xi_i)$ for all $\succsim_i \in \mathcal{P}_i^{\star\star}$.

For the second case, we suppose that $x_i' > x_i$. (The argument for the case $x_i' < x_i$ is analogous.) We expand the data set $\mathcal{O}$ by adding a fictitious observation s, such that agent i chooses $a_i^s = x_i'$ in the closed interval $A_i^s = [x_i, x_i']$, when other players' actions and the parameter values are given by $\xi_i^s = \xi_i = (a_{-i}, y_i)$. For player $j \neq i$, we choose $A_j^s = \{a_j\}$. We claim that the new agent i data set $\mathcal{O}_i'$ (with the added observation s) continues to obey ARC. In that case, we know by Theorem 1 that there is there is a regular preference $\succsim_i^*$ obeying single crossing differences that rationalizes $\mathcal{O}_i'$ and hence is also in $\mathcal{P}_i^{\star\star}$; furthermore, consistency with observation s requires that $(x_i', \xi) \succsim_i^* (x_i, \xi)$. So it is *not* the case that $(x_i, \xi) \succ_i^* (x_i, \xi)$. To see that $\mathcal{O}_i'$ obeys ARC, note that a violation must imply that there are elements x_i'' and x_i''' in X_i such that following hold: (A) x_i''' is indirectly revealed preferred to x_i'' through a chain of revealed preference (as in (5)) that includes $(x_i', \xi_i) \succsim^R (y_i, \xi_i)$, for some $y_i \in [x_i, x_i']$ i.e.,

$$(x_i''', \xi_i) \succsim_i^{RT} (x_i', \xi_i) \succsim_i^R (y_i, \xi_i) \succsim_i^{RT} (x_i'', \xi_i) \tag{19}$$

and (B) $(x_i'', \tilde{\xi}_i) \succsim_i^{RT} (x_i''', \tilde{\xi}_i)$, where $\tilde{\xi}_i > (<)\, \xi_i$ if $x_i''' > (<)\ \xi_i''$. Note that all the revealed relations listed in (A) and (B) are valid in the original data set $\mathcal{O}_i$, with the exception of $(x_i', \xi_i) \succsim_i^R (y_i, \xi_i)$ (which arises from observation s). Those relations can be re-combined to obtain

$$(y_i, \xi_i) \succsim_i^{RT} (x_i'', \xi_i) \succ_i^{RTS} (x_i''', \tilde{\xi}_i) \succsim_i^{RT} (x_i', \xi_i).$$

Thus, we obtain, from the original data set $\mathcal{O}_i'$, $(y_i, \xi_i) \succ_i^{RTST} (x_i', \xi_i)$, but this is incompatible with our initial assumption that $(x_i', \xi_i) \succsim_i^{RTST} (x_i, \xi_i)$ which implies that $(x_i', \xi_i) \succsim_i^{RTST} (y_i, \xi_i)$ by the interval property. □

4 Revealed strategic complementarity

Let $\mathbb{G} = \{\mathcal{G}(y, A)\}_{(y,A)\in Y\times\mathcal{A}}$ be a collection of games, as defined in the Section 2.2. We consider an observer who has a set of observations drawn from this collection. Each observation consists of agents' action profiles, feasible closed interval action sets, and exogenous parameters, i.e., each observation is a triple (a^t, y^t, A^t), where $a^t \in A^t$, $A^t \in \mathcal{A}$, and $y^t \in Y$. The set of observations is finite and is denoted by $\mathcal{O} = \{a^t, y^t, A^t\}_{t\in\mathcal{T}}$, where $\mathcal{T} = \{1, 2, ..., T\}$.

Definition 2. *A data set* $\mathcal{O} = \{a^t, y^t, A^t\}_{t\in\mathcal{T}}$ *is* consistent with strategic complementarity *(or* SC-rationalizable*) if there exists a profile of regular and SID preferences* $\{\succsim_i\}_{i\in N}$ *such that each observation constitutes a Nash equilibrium, i.e., for every* $t \in \mathcal{T}$, $(a_i^t, \xi_i^t) \succsim_i (x_i, \xi_i^t)$ *for every* $x_i \in A_i^t$.

The motivation for this definition is clear. If $\mathcal{O}$ is SC-rationalizable then we have found a profile of preference $\{\succsim_i\}_{i\in N}$ such that (i) a^t is a Nash equilibrium of $\mathcal{G}(A^t, y^t)$ and (ii) the family of games $\mathbb{G} = \{\mathcal{G}(y, A)\}_{(y,A)\in Y\times\mathcal{A}}$, where $\mathcal{G}(y, A) = [(y_i)_{i\in N}, (A_i)_{i\in N}, (\succsim_i)_{i\in N},]$ exhibits strategic complementarity (in the sense defined in Section 2.2).

For each agent i, we can define the *agent data set* $\mathcal{O}_i = \{(a_i^t, \xi_i^t, A_i^t)\}_{t=1}^T$ *induced by* $\mathcal{O}$, where $\xi_i^t = (a_{-i}^t, y_i^t)$. We say that $\mathcal{O} = \{a^t, A^t, y^t\}_{t\in\mathcal{T}}$ obeys ARC if $\mathcal{O}_i$ obeys ARC, for every agent i. It is clear that $\mathcal{O}$ is SC-rationalizable if and only if $\mathcal{O}_i$ is monotone-rationalizable for every agent i. This leads to the following result, which is an immediate consequence of Theorem 1 and provides with us with an easy-to-implement test of SC-rationalizability.

Corollary 1. *A data set* $\mathcal{O} = \{a^t, y^t, A^t\}_{t=1}^T$ *is SC-rationalizable if and only if it obeys ARC.*

Suppose an observer collects a data set $\mathcal{O} = \{a^t, A^t, y^t\}_{t\in\mathcal{T}}$ that is SC-rationalizable and then, maintaining that hypothesis, asks himself the following question: what do the observations in $\mathcal{O}$ say about the set of possible pure strategy Nash equilibria of the game $\mathcal{G}(y^0, A^0)$, where $A^0 \in \mathcal{A}$ and $y^0 \in Y$? The issue can be formally posed in the following way. For every $i \in N$, $\mathcal{O}_i$ obeys ARC and so the set of regular and SID preferences that rationalize $\mathcal{O}_i$, i.e. $\mathcal{P}_i^\star$, is nonempty. Every observed strategy profile a^t in the original data set $\mathcal{O}$ is supported as a Nash equilibrium by any preference profile $\{\succsim_i\}_{i\in N}$ in $\mathcal{P}^\star := \times_{i\in N}\mathcal{P}_i^\star$. For each $\{\succsim_i\}_{i\in N} \in \mathcal{P}^\star$, we know from Theorem B that the set of pure strategy Nash equilibria, $E(y^0, A^0, \{\succsim_i\}_{i\in N})$, of the game $\mathcal{G}(y^0, A^0)$ is nonempty and hence

$$\mathcal{E}(y^0, A^0) := \bigcup_{\{\succsim_i\}_{i\in N}\in\mathcal{P}^\star} E\left(y^0, A^0, \{\succsim_i\}_{i\in N}\right)$$

is also nonempty. $\mathcal{E}(y^0, A^0)$ is the set of *predicted Nash equilibria of the game* $\mathcal{G}(y^0, A^0)$. This gives rise to two related questions that we shall answer in this section: how can we compute $\mathcal{E}(y^0, A^0)$ from the data and what can we say about the structure of $\mathcal{E}(y^0, A^0)$?

4.1 Computable characterization of $\mathcal{E}(y^0, A^0)$

Let $\mathrm{BR}_i(\xi_i, A_i^0, \succsim_i)$ be player i's best responses in A_i^0 to $\xi_i = (a_{-i}, y_i^0)$, based on the preference $\succsim_i \in \mathcal{P}_i^\star$. Then the *possible (best) responses of player* i is given by

$$\mathrm{PR}_i(\xi_i, A_i^0) := \bigcup_{\succsim_i\in\mathcal{P}_i^\star} \mathrm{BR}_i(\xi_i, A_i^0, \succsim_i) \tag{20}$$

and the *joint possible response correspondence* $\mathrm{PR}(\cdot, y^0, A^0) : A^0 \rightrightarrows A^0$ is defined by

$$\mathrm{PR}(a, y^0, A^0) = \left(\mathrm{PR}_1(a_{-1}, y_1^0; A_1^0), \mathrm{PR}_2(a_{-2}, y_2^0, A_2^0), ..., \mathrm{PR}_n(a_{-n}, y_n^0, A_n^0)\right). \tag{21}$$

The crucial observation to make is that just as the set of Nash equilibria in a game coincides with the fixed points of its joint best response correspondence, so the set of possible Nash equilibria in $\mathcal{G}(y^0, A^0)$, $\mathcal{E}(y^0, A^0)$, coincides with the fixed points of $\mathrm{PR}(\cdot, y^0, A^0)$. Equivalently, one could think of $\mathcal{E}(y^0, A^0)$ as the intersection of the graphs of each player's possible response correspondence, i.e.,

$\mathcal{E}(y^0, A^0) = \bigcap_{i \in N} \Gamma_i(y^0, A^0)$, where

$$\Gamma_i(y^0, A^0) = \{(a_i, a_{-i}) \in A^0 : a_i \in \mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)\}. \tag{22}$$

It follows from Theorem 1 that

$$\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0) = \left\{\tilde{a}_i \in A_i^0 : \overline{\mathcal{O}_i} = \mathcal{O}_i \cup \{\left((\tilde{a}_i, a_{-i}), y_i^0, A_i^0\right)\} \text{ obeys ARC}\right\}, \tag{23}$$

where $\overline{\mathcal{O}_i}$ is the data set $\mathcal{O}_i$ augmented by the (fictitious) observation $\{((\tilde{a}_i, a_{-i}), y_i^0, A_i^0)\}$. Furthermore, we know that any data set that obeys ARC could in fact be rationalized by a regular and SSCD preference. Thus $\mathrm{PR}_i(\xi_i, A_i^0)$, which is defined by (20), also equals $\bigcup_{\succsim_i \in \mathcal{P}_i^{\star\star}} \mathrm{BR}_i(\xi_i, A_i^0, \succsim_i)$, where $\mathcal{P}_i^{\star\star}$ is the set of regular and SSCD preferences that rationalize $\mathcal{O}_i$. As a result,

$$\mathcal{E}(y^0, A^0) = \bigcup_{\{\succsim_i\}_{i \in N} \in \mathcal{P}^{\star\star}} E\left(y^0, A^0, \{\succsim_i\}_{i \in N}\right),$$

where $\mathcal{P}^{\star\star} = \times_{i \in N} \mathcal{P}_i^{\star\star}$. In other words, the set of possible Nash equilibria of $\mathcal{G}(y^0, A^0)$ (and, obviously, the properties of this set such as those outlined in Theorems 3 and 4) do not depend on whether we are allowing all SID preferences rationalizing the data set $\mathcal{O}$ or all SSCD preferences rationalizing $\mathcal{O}$.

The computation of $\mathcal{E}(A^0; y^0)$ hinges on the computation of $\mathrm{PR}_i(\cdot, y_i^0, A^0) : A_{-i} \rightrightarrows A_i^0$. Two features of this correspondence together make it possible for us to compute it explicitly.

(I) For any a_{-i}, one could show that

$$\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0) = \{a_i \in A_i^0 : \nexists\, \hat{a}_i \in A_i^0 \text{ such that } (\hat{a}_i, a_{-i}, y_i^0) \succ_i^{RTST} (a_i, a_{-i}, y_i^0)\}. \tag{24}$$

In other words, $\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$ coincides exactly with those elements in A_i^0 that are not dominated (with respect to $\succ_i^{RTST}$) by another element in A_i^0. Since the data set is finite, $\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$ can be constructed after a finite number of steps and, in fact, one could also show that it consists of a finite number of intervals.

(II) The correspondence $\mathrm{PR}_i(\cdot, y_i^0, A^0)$ takes only finitely many distinct values. For $j \neq i$, let

$$A_j^{\mathcal{T}} = \{a_j \in X_j : \exists\, a_{-j} \text{ such that } (a_j, a_{-j}) = a^t \text{ for some } t \in \mathcal{T}\}$$

We denote by $\mathcal{I}_j$ the collection consisting of *all* subsets of A_j^0 of the following two types: the singleton sets $\{\tilde{a}_j\}$, where $\tilde{a}_j$ is in the set $\underline{A}_j^0 = \left(A_j^{\mathcal{T}} \cap A_j^0\right) \bigcup \max A_j^0 \bigcup \min A_j^0$ and the interval sets $\{a \in A_j^0 : \tilde{a} < a < \tilde{b}\}$, where $\tilde{a} \in \underline{A}_j^0$ and $\tilde{b}$ is the element in $\underline{A}_j^0$ immediately above $\tilde{a}$. We denote by $\mathcal{H}_i$ the collection of hyper-rectangles

$$I_1 \times I_2 \times ... \times I_{i-1} \times I_{i+1} \times ... \times I_N$$

where $I_j \in \mathcal{I}_j$, for $j \neq i$; note that these hyper-rectangles are subsets of $\times_{j \neq i} A_j^0$. Then one could show that for any hyper-rectangle $H_i \in \mathcal{H}_i$,

$$\textit{if } a'_{-i},\ a''_{-i} \in H_i, \textit{ then } \ \mathrm{PR}_i(a'_{-i}, y_i^0; A_i^0) = \mathrm{PR}_i(a''_{-i}, y_i^0; A_i^0). \tag{25}$$

In other words, the correspondence $\mathrm{PR}_i(a_{-i}, y_i^0; A_i^0)$ is constant within each hyper-rectangle H_i. Therefore, to compute $\mathrm{PR}_i(a_{-i}, y_i^0; A_i^0)$ we need only find its value via (24) for a typical element within each hyper-rectangle H_i in the finite collection $\mathcal{H}_i$.

It follows from observations I and II above that the graph of player i's possible response correspondence (as defined by (22)) is also given by

$$\Gamma_i(y^0, A^0) = \{(a_i, a_{-i}) \in A^0 : \nexists\, \hat{a}_i \in A_i^0 \text{ such that } (\hat{a}_i, a_{-i}, y_i^0) \succ_i^{RTST} (a_i, a_{-i}, y_i^0)\} \tag{26}$$

and can be explicitly constructed. Furthermore, because $\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$ consists of a finite union of intervals of A_i^0, $\Gamma_i(y^0, A^0)$ is a finite union of hyper-rectangles in A^0. The theorem below summarizes these observations.

Theorem 3. *Suppose a data set $\mathcal{O} = \{a^t, y^t, A^t\}_{t=1}^T$ obeys ARC and let $(y^0, A^0) \in Y \times \mathcal{A}$.*

(i) $\mathrm{PR}_i(\cdot, y_i^0, A_i^0)$ *obeys (24) and (25) and, for any $a_{-i} \in \times_{j \neq i} A_j^0$, $\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$ consists of a finite union of intervals of A_i^0.*

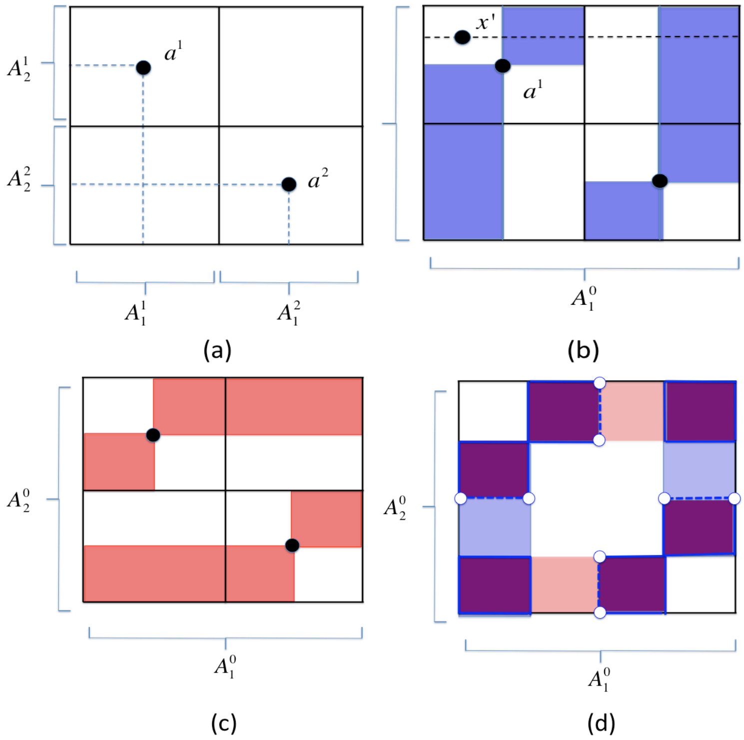

Figure 1: $\mathcal{E}(A^0)$ in Example 2

(ii) *The graph of* $\mathrm{PR}_i(\cdot, y_i^0, A_i^0)$, $\Gamma_i(y^0, A^0)$, *is a finite union of hyper-rectangles in* A^0. *Consequently, the set of possible Nash equilibria,* $\mathcal{E}(y^0, A^0) = \bigcap_{i \in N} \Gamma_i(y^0, A^0)$, *is also a finite union of hyper-rectangles in* A^0.

Example 2. Figure 1(a) depicts two observations, $\{(a^1, A^1)$ and $(a^2, A^2)\}$, drawn from games involving two players. This data set obeys ARC and we would like to compute $\mathcal{E}(A^0)$, where $A_i^0 = A_i^1 \cup A_i^2$ (for $i = 1, 2$). First, we claim that the unshaded area in Figure 1(b) cannot be contained in $\Gamma_1(A^0)$. Indeed, consider the point $x' = (x_1', x_2')$ in the unshaded area, at which $x_1' < a_1^1$, $x_2' > a_2^1$, and $x_1' \in A_1^1$. Therefore, $(a_1^1, a_2^1) \succsim_1^R (x_1', a_2^1)$ and so $(a_1^1, a_2^1) \succsim_1^{RT} (x_1', a_2^1)$. Since $x_2' > a_2^1$, $(a_1^1, a_2^1) \succ_1^{RTS} (x_1', a_2^1)$, which means that $(x_1', x_2') \notin \Gamma_1(A^0)$. Using (26), it is easy to check that $\Gamma_1(A^0)$ corresponds precisely to the shaded area in Figure 1(b). Similarly, $\Gamma_2(A^0)$ consists of the shaded area in Figure 1(c). The common shaded area, as depicted with the darker shade in Figure 1(d), represents $\mathcal{E}(A^0) = \Gamma_1(A^0) \cap \Gamma_2(A^0)$. Note that the dashed lines are *excluded* from $\mathcal{E}(A^0)$, so this set is not closed.

Proof of Theorem 3. Part (ii) follows straightforwardly from part (i), so we shall focus on proving (i), which consists of three claims.

It follows from (23) that (24) holds provided we can show the following: $\overline{\mathcal{O}}_i = \mathcal{O}_i \cup \{(\tilde{a}_i, a_{-i}, y_i^0, A_i^0)\}$ violates ARC if and only if there is $\hat{a}_i \in A_i^0$ such that $(\hat{a}_i, a_{-i}, y_i^0) \succ_i^{RTST} (\tilde{a}_i, a_{-i}, y_i^0)$. Let $\succsim_i^{\overline{R}}$, $\succsim_i^{\overline{RT}}$, $\succsim_i^{\overline{RTS}}$, and $\succsim_i^{\overline{RTST}}$ be the revealed preference relations derived from $\overline{\mathcal{O}}_i = \mathcal{O}_i \cup \{(\tilde{a}_i, a_{-i}, y_i^0, A_i^0)\}$, which must contain the analogous revealed preference relations of $\mathcal{O}_i$. Suppose there is $\hat{a}_i \in A_i^0$ such that $(\hat{a}_i, a_{-i}, y_i^0) \succ_i^{RTST} (\tilde{a}_i, a_{-i}, y^0)$ and so $(\hat{a}_i, a_{-i}, y_i^0) \succ_i^{\overline{RTST}} (\tilde{a}_i, a_{-i}, y_i^0)$. On the other hand, since $\hat{a}_i \in A_i^0$, we have $(\tilde{a}_{-i}, y_i^0) \succsim_i^{\overline{R}} (\hat{a}_i, a_{-i}, y_i^0)$. Thus, the relation $\succsim_i^{\overline{RTS}}$ is not cyclically consistent, which implies (by Lemma 3) that $\overline{\mathcal{O}}_i$ violates ARC. Conversely, suppose that $\overline{\mathcal{O}}_i = \mathcal{O}_i \cup \{(\tilde{a}_i, a_{-i}, y^0, A_i^0)\}$ violates ARC. Since $\mathcal{O}_i$ obeys ARC, this violation can only occur in two ways: there is $\hat{a}_i \in X_i$ such that $(\tilde{a}_i, a_{-i}, y_i^0) \succsim_i^{\overline{RT}} (\hat{a}_i, a_{-i}, y_i^0)$ and $(\hat{a}_i, \bar{a}_{-i}, \bar{y}_i) \succsim_i^{RT} (\tilde{a}_i, \bar{a}_{-i}, \bar{y}_i)$ with either (1) $\hat{a}_i < \tilde{a}_i$ and $(\bar{a}_{-i}, \bar{y}_i) > (a_{-i}, y_i^0)$ or (2) $\hat{a}_i > \tilde{a}_i$ and $(\bar{a}_{-i}, \bar{y}_i) < (a_{-i}, y_i^0)$. We need to show that $\tilde{a}_i$ is dominated (with respect to $\succ_i^{RTST}$) by some element in A_i^0. In either cases (1) or (2), since $(\hat{a}_i, \bar{a}_{-i}, \bar{y}_i) \succsim_i^{RT} (\tilde{a}_i, \bar{a}_{-i}, \bar{y}_i)$, we obtain $(\hat{a}_i, a_{-i}, y_i^0) \succ_i^{RTS} (\tilde{a}_i, a_{-i}, y_i^0)$. If $\hat{a}_i \in A_i^0$, we are done. If $\hat{a}_i \notin A_i^0$ then, given that $(\tilde{a}_i, a_{-i}, y_i^0) \succsim_i^{\overline{RT}} (\hat{a}_i, a_{-i}, y_i^0)$, there exists $\bar{a}_i \in A_i^0$ such that $(\bar{a}_i, a_{-i}, y_i^0) \succsim_i^{RT} (\hat{a}_i, a_{-i}, y_i^0)$. Thus $(\bar{a}_i, a_{-i}, y_i^0) \succ_i^{RTST} (\tilde{a}_i, a_{-i}, y_i^0)$.

To see that (25) holds, first note that $\tilde{a}_i \notin \mathrm{PR}_i(a'_{-i}, y_i^0, A_i^0)$ if and only if $\overline{\mathcal{O}}'_i = \mathcal{O}_i \cup \{(\tilde{a}_i, a'_{-i}, y_i^0, A_i^0)\}$ violates ARC. Since H_i is not a singleton, it must be an interval and so there is *no* a'_i such that $(a'_i, a'_{-i}) = a^t$ for some $t \in \mathcal{T}$. Therefore, $\overline{\mathcal{O}}'_i$ violates ARC if and only if there is $\hat{a}_i \in A_i^0$ and $\bar{a}_{-i}$ such that $(\hat{a}_i, \bar{a}_{-i}, \bar{y}_i) \succsim_i^{RT} (\tilde{a}_i, \bar{a}_{-i}, \bar{y}_i)$ with either (1) $\hat{a}_i < \tilde{a}_i$ and $(\bar{a}_{-i}, \bar{y}_i) > (a'_{-i}, y_i^0)$ or (2) $\hat{a}_i > \tilde{a}_i$ and $(\bar{a}_{-i}, \bar{y}_i) < (a'_{-i}, y_i^0)$. Note that there is $t \in \mathcal{T}$ such that $(\hat{a}_i, \bar{a}_{-i}) = a^t$; in particular, this means that $\bar{a}_{-i} \in \times_{j \neq i} \underline{A}^{\mathcal{T}}$. It follows from our definition of H_i that $(\bar{a}_{-i}, \bar{y}_i) > (a''_{-i}, y_i^0)$ if $(\bar{a}_{-i}, \bar{y}_i) > (a'_{-i}, y_i^0)$ and $(\bar{a}_{-i}, \bar{y}_i) < (a''_{-i}, y_i^0)$ if $(\bar{a}_{-i}, \bar{y}_i) < (a'_{-i}, y_i^0)$. Thus $\overline{\mathcal{O}}''_i = \mathcal{O}_i \cup \{(\tilde{a}_i, a''_{-i}, y_i^0, A_i^0)\}$ also violates ARC. We conclude that $\tilde{a}_i \notin \mathrm{PR}_i(a''_{-i}, y_i^0, A_i^0)$ if $\tilde{a}_i \notin \mathrm{PR}_i(a'_{-i}, y_i^0, A_i^0)$, which establishes (25).

Lastly, we show that $\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$ consists of a finite union of intervals of A_i^0. This is equivalent to showing that $A_i^0 \backslash \mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$ is a finite union of intervals; an element $\tilde{a}_i$ is in this set if and only if there is $t \in \mathcal{T}$ such that $a_i^t \in A_i^0$ and $(a_i^t, \xi_i^0) \succ_i^{RTST} (\tilde{a}_i, \xi_i^0)$, where $\xi_i^0 = (a_{-i}, y^0)$. This is turns holds if and only if is $s \in \mathcal{T}$ such that either (1) $(a_i^t, \xi_i^0) \succsim_i^{RTST} (a_i^s, \xi_i^0)$ and $(a_i^s, \xi_i^0) \succ_i^{RTS} (\tilde{a}_i, \xi_i^0)$ or (2) $(a_i^t, \xi_i^0) \succ_i^{RTST} (a_i^s, \xi_i^0)$ and $(a_i^s, \xi_i^0) \succsim_i^{RT} (\tilde{a}_i, \xi_i^0)$. Notice for a fixed $s \in \mathcal{T}$, the sets $\{a_i \in A_i^0 : (a_i^s, \xi_i^0) \succ_i^{RTS} (a_i, \xi_i^0)\}$ and $\{a_i \in A_i^0 : (a_i^s, \xi_i^0) \succsim_i^{RT} (a_i, \xi_i^0)\}$ both consist of intervals, because of the interval property on $\succ_i^{RTS}$ and $\succ_i^{RTS}$ respectively (see Lemma 2). It follows that $A_i^0 \backslash \text{PR}_i(a_{-i}, y_i^0, A_i^0)$ is a finite union of intervals. □

4.2 The structure of $\mathcal{E}(y^0, A^0)$

As we have pointed out in Section 2, the set of pure strategy Nash equilibria in a game with strategic complementarity forms a nonempty complete lattice, and the largest and smallest equilibria exhibit monotone comparative statics with respect to exogenous parameters. In this subsection, we show that these properties are largely inherited by the set of predicted pure strategy Nash equilibria $\mathcal{E}(y^0, A^0)$. This is illustrated in Example 2, where it is not hard to check from Figure 1(d) that the set of predicted Nash equilibria forms a complete lattice and, in particular, the largest and the smallest possible Nash equilibria exist; while this is not true in general, properties close to this are always true. The next result lists the main structural properties of $\mathcal{E}(y^0, A^0)$; we have consciously presented them in a way that is analogous to Theorem B.

THEOREM 4. *Suppose a data set $\mathcal{O} = \{a^t, y^t, A^t\}_{t \in \mathcal{T}}$ obeys ARC and let $(y^0, A^0) \in Y \times \mathcal{A}$.*

1. [EXISTENCE] *The set of possible pure strategy Nash equilibria, $\mathcal{E}(y^0, A^0)$, is nonempty.*

2. [STRUCTURE] (a) *The set $\overline{\mathcal{E}(y^0, A^0)}$ admits a largest and smallest element (denoted by $\max \overline{\mathcal{E}(y^0, A^0,)}$ and $\min \overline{\mathcal{E}(y^0, A^0)}$ respectively.* (b) *Furthermore, for every set $K \subseteq \mathcal{E}(y^0, A^0)$ that is closed in $\mathbb{R}^n$, the sets*

$$\begin{aligned} \mathcal{U}(K) &= \{z \in \mathcal{E}(y^0, A^0) : z \geqslant x \textit{ for all } x \in K\} \textit{ and} \\ \mathcal{L}(K) &= \{z \in \mathcal{E}(y^0, A^0) : z \leqslant x \textit{ for all } x \in K\} \end{aligned}$$

are nonempty and $\min \overline{\mathcal{U}(K)}$ and $\max \overline{\mathcal{L}(K)}$ both exist.

3. [COMPARATIVE STATICS] *The strategy profiles $\max \overline{\mathcal{E}(y, A^0)}$ and $\min \overline{\mathcal{E}(y, A^0)}$ are both increasing in $y \in Y$.*

Remark: We use $\overline{S}$ to denote the closure of S.

Parts 2(a) and 3 in Theorem 4 tell us that the set of possible Nash equilibria effectively has a largest and smallest element and that these increase as the parameter y increases. Note that because A^0 is a subcomplete sublattice of $(\mathbb{R}^n, \geqslant)$, any set in A^0 will have a supremum and an infimum in A^0. Therefore, the principal content in part 2(a) lies in the claim that the supremum and infimum of $\overline{\mathcal{E}(y^0, A^0)}$ are contained in $\overline{\mathcal{E}(y^0, A^0)}$. Clearly, the analogous statement in Theorem B is stronger since it says that the set of pure strategy Nash equilibria (even when it is not closed) has a largest and smallest element; Example 3 (presented later in this subsection) shows that this conclusion cannot be strengthened. While $\mathcal{E}(y^0, A^0)$ is not generally a complete lattice, part 2(b) of the theorem says that any closed subset K of $\mathcal{E}(y^0, A^0)$ will be bounded above by elements of $\mathcal{E}(y^0, A^0)$ and the closure of this set of upper bounds, $\overline{\mathcal{U}(K)}$, has a smallest element. In this sense, its structure is close to that of a complete lattice. In the special but important case where A^0 is finite, every subset of A^0 is closed and so it follows immediately from Theorem 4 that $\mathcal{E}(y^0, A^0)$ is a bona fide lattice; we record this as a corollary.

COROLLARY 2. *Suppose a data set $\mathcal{O} = \{a^t, y^t, A^t\}_{t \in \mathcal{T}}$ obeys ARC and let $(y^0, A^0) \in Y \times \mathcal{A}$. Then $\mathcal{E}(y^0, A^0)$ is a nonempty complete lattice if A^0 is a finite set.*

The conclusion of Theorem 4 may also be strengthened in the case where the feasible action set of every agent is unchanged throughout the observations, i.e. $A^t = A^0 \in \mathcal{A}$ for all $t \in \mathcal{T}$. In this setting, and allowing for agents to have multi-dimensional actions, Lazzati (2014) shows that $\mathcal{E}(y^0, A^0)$ has a largest and smallest element. Applying Theorem 4 to this case gives the stronger conclusion that $\mathcal{E}(y^0, A^0)$ forms a complete lattice. Indeed, by (23), a necessary and sufficient condition for $\tilde{a}_i \in A^0$ to be contained in $\text{PR}_i(a_{-i}, y_i^0; A_i^0)$ is that $\overline{\mathcal{O}_i} = \mathcal{O}_i \cup \{(\tilde{a}_i, (a_{-i}, y_i^0), A_i^0)\}$ obeys ARC. If $A^0 = A^t$ for all $t \in \mathcal{T}$, then it is straightforward to check that this is equivalent to

$$a_i^s \geqslant \tilde{a}_i \geqslant a_i^t \text{ for all } s, t \in \mathcal{T} \text{ such that } \xi_i^s \geqslant (a_{-i}, y_i^0) \geqslant \xi_i^t, \tag{27}$$

which is precisely the "sandwich" condition obtained by Lazzati (2014). It follows that $\text{PR}_i(a_{-i}, y_i^0, A_i^0)$ must be a closed interval in A_i^0 and (by Theorem 3) its graph $\Gamma_i(y^0, A^0)$ is a finite union of *closed* hyper-rectangles. Therefore, $\mathcal{E}(y^0; A^0) = \bigcap_{i \in N} \Gamma_i(y^0, A^0)$ is also closed and, by Theorem 4, it must contain its largest and smallest element. Furthermore, for any *arbitrary* set $K \subset \mathcal{E}(y^0, A^0)$, its

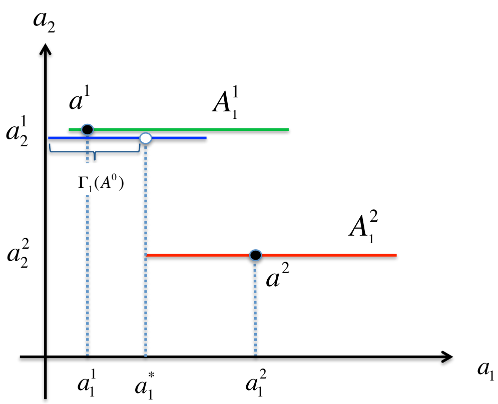

Figure 2: $\mathcal{E}(A^0)$ in Example 3

closure $\bar{K} \subset \mathcal{E}(y^0, A^0)$ because the latter is closed. It follows that $\mathcal{U}(K) = \mathcal{U}(\bar{K})$ and is also a closed set and, by Theorem 4, $\min \mathcal{U}(K)$ exists. By an analogous argument, we may conclude that $\max \mathcal{L}(K)$ exists and thus $\mathcal{E}(y^0, A^0)$ is a complete lattice. The following corollary summarizes our observations.

Corollary 3. *Suppose that a data set $\mathcal{O} = \{a^t, y^t, A^t\}_{t \in \mathcal{T}}$ obeys ARC and let $A^t = A^0$ for all $t \in \mathcal{T}$. Then $\mathcal{E}(y^0, A^0)$ forms a nonempty complete lattice.*

In the case where agents' constraint set changes across observations, the $\mathcal{E}(y^0, A^0)$ is not generally closed and may not contain its largest or smallest element, as the following example illustrates.

Example 3. Suppose that we have two observations as depicted in Figure 2, where A_1^1 and A_1^2 are the strategy sets available to player 1 at observations 1 and 2 respectively, and with player 2 having singleton strategy sets at each observation. Let A_1^0 be the blue segment in the figure, with $A_2^0 = \{a_2^1\}$. It is easy to confirm that observations 1 and 2 obey ARC, and the set of possible equilibria $\mathcal{E}(A^0)$ is equal to $\Gamma_1(A^0)$, which is not a complete lattice because $\max \mathcal{E}(A^0) \notin \mathcal{E}(A^0)$. To see this, it suffices to show that $a_1^* \notin \text{PR}_1(a_2^1; A_1^0)$. Since $(a_1^2, a_2^2) \succsim_1^R (a_1^*, a_2^2)$, we obtain $(a_1^2, a_2^1) \succ_1^{RTS} (a_1^*, a_2^1)$. In addition, $(a_1^1, a_2^1) \succsim_1^R (a_1^2, a_2^1)$ and so $(a_1^1, a_2^1) \succ_1^{RTS} (a_1^*, a_2^1)$. On the other hand, it is clear that $(a_1^*, a_2^1) = \max \overline{\mathcal{E}(A^0)}$.

We end this section with the proof of Theorem 4. The proof uses the following lemma.

LEMMA 7. *Suppose* $\mathcal{O} = \{a^t, y^t, A^t\}_{t=1}^T$ *obeys ARC and let* $A^0 \in \mathcal{A}$. *Then the map* $p_i^{**} : A_{-i}^0 \times Y \to A_i^0$ *given by*

$$p_i^{**}(a_{-i}, y_i) = \sup \mathrm{PR}_i(a_{-i}, y_i, A^0)$$

has the following properties: (i) *it is increasing in* $(a_{-i}, y_i) \in A_{-i}^0 \times Y_i$*;* (ii) *for* a'_{-i} *and* a''_{-i} *in* H_i, $p_i^{**}(a''_{-i}, y_i) = p_i^{**}(a'_{-i}, y_i)$*; and* (iii) *if, for some* $(\bar{a}_{-i}, \bar{y}_i)$, $p_i^{**}(\bar{a}_{-i}, \bar{y}_i) \in \mathrm{PR}_i(\bar{a}_{-i}, \bar{y}_i, A^0)$ *and for some* $(\hat{a}_{-i}, \hat{y}_i) > (\bar{a}_{-i}, \bar{y}_i)$, $p_i^{**}(\bar{a}_{-i}, \bar{y}_i) = p_i^{**}(\hat{a}_{-i}, \hat{y}_i)$*, then* $p_i^{**}(\hat{a}_{-i}, \hat{y}_i) \in \mathrm{PR}_i(\hat{a}_{-i}, \hat{y}_i, A^0)$.

Remark: In a similar way, we define $p_i^* : A_{-i}^0 \times Y_i \to A_i^0$ by $p_i^*(a_{-i}, y_i) = \inf \mathrm{PR}_i(a_{-i}, y_i, A^0)$. This function will obey properties (i) and (ii) and, instead of property (iii), it will have the following property (iii)′: *if, for some* $(\bar{a}_{-i}, \bar{y}_i)$, $p_i^*(\bar{a}_{-i}, \bar{y}_i) \in \mathrm{PR}_i(\bar{a}_{-i}, \bar{y}_i, A^0)$ *and for some* $(\hat{a}_{-i}, \hat{y}_i) < (\bar{a}_{-i}, \bar{y}_i)$, $p_i^*(\bar{a}_{-i}, \bar{y}_i) = p_i^*(\hat{a}_{-i}, \hat{y}_i)$*, then* $p_i^*(\hat{a}_{-i}, \hat{y}_i) \in \mathrm{PR}_i(\hat{a}_{-i}, \hat{y}_i, A^0)$.

Proof. Since $\mathrm{PR}_i(a_{-i}, y_i, A^0)$ is the union of a collection of best response correspondences (see (20)), each of which is increasing in (a_{-i}, y_i), p_i^{**} must be increasing. Claim (ii) is an immediate consequence of (25) (which was proved in Theorem 3). Lastly, if $p_i^{**}(\bar{a}_{-i}, \bar{y}_i) \in \mathrm{PR}_i(\bar{a}_{-i}, \bar{y}_i, A^0)$ then there is $\succsim_i \in \mathcal{P}_i^*$ such that $p_i^{**}(\bar{a}_{-i}, \bar{y}_i) \in \mathrm{BR}_i(\bar{a}_{-i}, \bar{y}_i, A_i^0, \succsim_i)$. Since the best response correspondence is increasing, there is $a'_i \in \mathrm{BR}_i(\hat{a}_{-i}, \hat{y}_i, A_i^0, \succsim_i)$, and thus in $\mathrm{PR}_i(\hat{a}_{-i}, \hat{y}_i, A_i^0)$, such that $a'_i \geqslant p_i^{**}(\bar{a}_{-i}, \bar{y}_i)$. This establishes (iii). □

We are now ready to prove Theorem 4. It is worth pointing out an obvious first approach that will *not* work. Given p_i^{**}, we can define, for each $a \in A^0$, $p^{**}(a, y^0) = (p_i^{**}(a_{-i}, y_i^0))_{i \in N}$, and since p_i^{**} is increasing in a_{-i}, so $p^{**}(a, y^0)$ is increasing in a. By Tarski's fixed point theorem, $p^{**}(\cdot, y^0)$ will have a fixed point and indeed a largest fixed point $a^\star$; thus the existence of $\max \overline{\mathcal{E}(y^0, A^0)}$ is ensured if it could be identified with $a^\star$. However, they are not generally the same points: it is straightforward to construct an increasing (but not compact-valued) correspondence such that its largest fixed point does not coincide with the largest fixed point of its supremum function. Our proof of Theorem 4 takes a different route. We have already explained at the beginning of this section why $\mathcal{E}(y^0, A^0)$ is nonempty, so we shall concentrate on proving statements 2 and 3.

Proof of 2(a) and 3 in Theorem 4: We shall confine our attention to $\max \overline{\mathcal{E}(y^0, A^0)}$; the proof for the other case is similar. Firstly, note that the properties of p_i^{**} listed in Lemma 7 guarantee that there exists a sequence of functions $\{p_i^k(\cdot, y_i^0, A_i^0)\}_{k \in \mathbb{N}}$ selected from $\mathrm{PR}_i(\cdot, y_i^0, A_i^0)$ with the following properties: (i) for a'_{-i} and a''_{-i} in H_i, $p_i^k(a''_{-i}, y_i^0) = p_i^k(a'_{-i}, y_i^0)$; (ii) $p_i^k(a_{-i}, y_i^0, A_i^0)$ is increasing in a_{-i} and in k; (iii) $p_i^k(a_{-i}, y_i^0, A_i^0) = p_i^{**}(a_{-i}, y_i^0, A_i^0)$ if $p_i^{**}(a_{-i}, y_i^0, A_i^0) \in \mathrm{PR}_i(a_{-i}, y_i^0, A_i^0)$; and (iv) $\lim_{k\to\infty} p_i^k(a_{-i}, y_i^0, A_i^0) = p_i^{**}(a_{-i}, y_i^0, A_i^0)$. In other words, there is a sequence of increasing selections from $\mathrm{PR}_i(\cdot, y^0, A^0)$ that has $p_i^{**}(a_{-i}, y^0, A^0)$ as it limit, with the sequence being exactly equal to $p_i^{**}(a_{-i}, y_i^0, A_i^0)$ if the latter is a possible response of player i.

The function $p^k(a, y^0, A^0) = (p_i^k(a_{-i}, y_i^0, A_i^0))_{i\in N}$ is increasing in a, since p_i^k is increasing in a_{-i}. By Tarski's fixed point theorem, p^k has a largest fixed point, which we denote by $x^k(y^0, A^0)$. Since $p_i^k(\cdot, y_i^0, A_i^0)$ is a selection from $\mathrm{PR}_i(\cdot, y_i^0, A_i^0)$, $x^k(y^0, A^0) \in \mathcal{E}(y^0, A^0)$. By the monotone fixed points theorem (see Section 2), the sequence $x^k(y^0, A^0)$ is increasing with k. Since A^0 is compact, this sequence must have a limit. This limit, which we denote by $a^{**}(y^0, A^0)$, lies in $\overline{\mathcal{E}(y^0, A^0)}$.

We claim that $a^{**}(y^0, A^0)) \geqslant \tilde{x}$, for any $\tilde{x} \in \mathcal{E}(y^0, A^0)$. Indeed, since $\tilde{x}_i \in \mathrm{PR}_i(\tilde{x}_{-i}, y_i^0, A_i^0)$ for all $i \in N$, for k sufficiently large, $p_i^k(\tilde{x}_{-i}, y_i^0, A_i^0) \geqslant \tilde{x}_i$. Now consider the map p^k confined to the domain $S = \times_{i\in N}\{a_i \in A_i^0 : a_i \geqslant \tilde{x}_i\}$. Since p^k is increasing, the image of p^k also falls on S; in other words, p^k can be considered as a map from S to itself. It is also an increasing map and, by Tarski's fixed point theorem will have a largest fixed point. The largest fixed point of p^k restricted to S must again be $x^k(y^0, A^0)$ and it follows from our construction that $x^k(y^0, A^0)) \geqslant \tilde{x}$. In turn this implies that $a^{**}(y^0, A^0)) \geqslant \tilde{x}$. So $a^{**}(y^0, A^0))$ is an upper bound of $\mathcal{E}(y^0, A^0)$ and thus also an upper bound of $\overline{\mathcal{E}(y^0, A^0)}$. Given that $a^{**}(y^0, A^0)) \in \overline{\mathcal{E}(y^0, A^0)}$, we conclude that $a^{**}(y^0, A^0)) = \max \overline{\mathcal{E}(y^0, A^0)}$.

To see that $a^{**}(y, A^0)$ is increasing with respect to the parameter, consider $y'' > y'$. Given the properties of p_i^{**} listed in Lemma 7, we can choose functions $\{p_i^k(\cdot, y_i, A_i^0)\}_{k\in\mathbb{N}}$ selected from $\mathrm{PR}_i(\cdot, y_i, A_i^0)$ (for $y_i = y_i'$ and y_i'') satisfying properties (i) – (iv) and, in addition, $p_i^k(a_{-i}, y_i'', A_i^0) \geqslant p_i^k(a_{-i}, y_i', A_i^0)$ for all a_{-i}. The map $p^k(\cdot, y_i'', A_i^0)$ is increasing and will have a largest fixed point $x^k(y'', A^0)$ which, by the monotone fixed points theorem satisfies $x^k(y'', A^0) \geqslant x^k(y', A^0)$. Taking limits as $k \to \infty$, we obtain $a^{**}(y'', A^0) \geqslant a^{**}(y', A^0)$. □

Proof of 2(b) in Theorem 4: For each i, let $z^i = \arg\max_{a\in K} a_i$. Then $\sup K = (z_1^1, z_2^2, ..., z_n^n) \in A^0$ and we shall denote this profile of strategies by $\bar{z}$. Since $z^i \in K$, there is $\succsim_i^* \in \mathcal{P}_i^*$ such that $z_i^i \in \mathrm{BR}_i(z_{-i}^i, y_i^0, A_i^0, \succsim_i^*)$. Since the best response correspondence is increasing, $a_i \geqslant z_i^i$ for all $a_i \in \mathrm{BR}_i(a_{-i}, y_i^0, A_i^0, \succsim_i^*)$, where $a_{-i} \geqslant z_{-i}^i$; this holds, in particular, for all $a_{-i} \geqslant \bar{z}_{-i}$. Thus $\mathrm{PR}_i(a_{-i}, y_i^0, A_i^0) \cap [\bar{z}_i, \max A_i^0]$ is nonempty for all $a_{-i} \geqslant \bar{z}_{-i}$ since it contains $\mathrm{BR}_i(x_{-i}, y_i^0, A_i^0, \succsim_i$

$) \cap [\bar{z}_i, \max A_i^0]$ and the latter is nonempty. Therefore, the correspondence $F_i : \times_{j \neq i}[\bar{z}_j, \max A_j^0] \to [\bar{z}_j, \max A_j^0]$, given by

$$F_i(a_{-i}) = \mathrm{PR}_i(a_{-i}, y_i^0, A_i^0) \cap [\bar{z}_i, \max A_i^0]$$

is well-defined. With this, we may define $F : [\bar{z}, \max A^0] \to [\bar{z}, \max A^0]$, where $F(a) = (F_i(a_{-i}))_{i \in N}$. The fixed points of F coincide with the set $\mathcal{U}(K)$. Firstly, note that F does have a fixed point and thus $\mathcal{U}(K)$ is nonempty. Indeed, F contains the correspondence $G : [\bar{z}, \max A^0] \to [\bar{z}, \max A^0]$ given by $G_i(a) = \mathrm{BR}_i(a_{-i}, y^0, A^0, \succsim_i^*)$; since G is increasing and compact-valued, the Tarski-Zhou fixed point theorem guarantees that G, and thus F, has a fixed point.

To show that $\min \overline{\mathcal{U}(K)}$ exists, we can adopt essentially the same proof strategy as the one used to show the existence of $\min \overline{\mathcal{E}(y^0, A^0)}$. We shall sketch the argument, leaving the details to the reader. By adapting the proof of Lemma 7, we can show that the function $q_i^* : \times_{j \neq i}[\bar{z}_j, \max A_j^0] \to [\bar{z}_j, \max A_j^0]$ defined by $q_i^*(a_{-i}) = \inf F_i(a_{-i})$ has the following properties: (i) it is increasing in a_{-i}; (ii) for a'_{-i} and a''_{-i} in $H_i \cap \times_{j \neq i}[\bar{z}_j, \max A_j^0]$, $q_i^*(a''_{-i}) = q_i^*(a'_{-i})$; and (iii) if, for some $\bar{a}_{-i}$, $q_i^*(\bar{a}_{-i}) \in F_i(\bar{a}_{-i})$ and for some $\hat{a}_{-i} < \bar{a}_{-i}$, $q_i^*(\bar{a}_{-i}) = q_i^*(\hat{a}_{-i})$, then $q_i^*(\hat{a}_{-i}) \in F_i(\bar{a}_{-i})$. Given this, we may then construct a sequence of increasing functions q_i^k selected from F_i that converges monotonically to q_i^* and with the property that $q_i^k = q_i^*$ if $q_i^*(a_{-i}) \in F_i(a_{-i})$. We define the function $q^k : [\bar{z}, \max A^0] \to [\bar{z}, \max A^0]$ by $q^k(a) = (q_i^k(a_{-i}))_{i \in N}$ and denote the smallest fixed point of q^k by x^k. The point x^k is also a fixed point of F and thus $x^k \in \mathcal{U}(K)$. Furthermore, it is decreasing in k, with a limit x^*, which is a lower bound $\overline{\mathcal{U}(K)}$. Consequently, $x^* = \min \overline{\mathcal{U}(K)}$. □

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