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"Apparentement incentives under the d'Hondt method"

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Apparentement incentives under the d'Hondt method[1]

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This paper provides an example of sole apparentement (coalition) that leads to unlimited seat losses. The necessary and sufficient condition of the lack of successful apparentements is discovered. A precise description of an apparentement-proof set is recognized. The probability of the lack of successful apparentement is evaluated. A game-theoretical approach for apparentement formation is applied.

JEL Classification: C71, D71, D72.

Keywords: proportional representation, coalition formation, d'Hondt method.

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1. Introduction

List apparentement is the form of coalition formation before elections day. Votes obtained by parties of apparentement are united and counted as a single list. List apparentements are widely used in the d'Hondt apportionment systems in particular in German States (Bochsler, 2010, Leutgäb and Pukelsheim, 2009).

The d'Hondt method (also known as the Jefferson method, the Hagenbach-Bischoff system, divisor method with rounding down, and the greatest divisors method) is a seats-allocation method for proportional representation electoral systems. Today, it is the predominant method for parliamentary elections in proportional representation systems around the world and is used often for mixed systems for a proportional component (Bormann, Golder, 2013), for local elections (Parigi, Bearman, 2008), and for labor elections (Rosenthal, 1974).

The d'Hondt method arises in different voting systems. Cox (1991) showed that if the district magnitude and distribution of voter support among parties are held constant and some empirically attainable conditions are met, then the single non-transferable vote system (plurality rule) and the d'Hondt method are equivalent. Cumulative voting in corporate board elections leads to the d'Hondt apportionment (Glazer et al, 1984; Cooper 2007). Karpov (2011) showed that seat distribution obtained by the d'Hondt method is the unique seat distribution obtained as a Nash equilibrium in a board election game. Pérez and De la Cruz (2014) achieved full Nash implementation of the Jefferson-d'Hondt rule in a committee formation game.

Empirical studies show that the d'Hondt method is one of the least proportional among the proportional representation methods (Benoit, 2000; Lijphart, 1990). Theoretical seat bias formulas were discovered by Schuster et al. (2003) and Janson (2014). Probabilities of majority and minority violation were obtained by Schwingenschlögl (2007).

The axiomatic properties of the d'Hondt method were studied by Balinski and Young (1978, 1979). They proved that the d'Hondt method is the unique consistent, monotone, stable, and balanced method that encourages coalitions. A method is consistent if it treats tied parties equally. By monotonicity, the number of seats provided to any state or party will not decrease if the house size increases. A method is stable if the merged party has no more than one additional seat and no less than one lost seat. Because the d'Hondt method is stable and encourages coalitions, coalesced parties cannot lose the seat.

Due to the unique properties of the d'Hondt method, coalition formation (vote pooling, apparentement) incentives set up a distinct field of research. The 1951 and 1956 elections for the French National Assembly utilized the d’Hondt method and permitted the formation of distinct formalized coalitions in each of the 95 multimember districts. Examples and game-theoretical analysis focusing on these elections were provided in Rosenthal (1975); Lee and Rosenthal (1976); and Lee, McKelvey, and Rosenthal (1979). They utilized the von Neumann-Morgenstern solution to predict apparentement structure and found some statistical support from the data of the 1951 elections. Strategic coalescing in a corporate board election game with equilibrium resulting in the d’Hondt distribution of seats was demonstrated in (Glazer et al, 1984).

Bochsler (2010) estimated apparentements gains assuming that the remaining fractions that are rounded off in every seat allocation are randomly distributed between 0 and 1. He argued that the sole apparentement wins approximately half a seat, and this gain is allocated within the apparentement proportionally to the parties’ vote shares. Janson (2014) shows that two small parties that form an apparentement gain at most half a seat.

The result changes for two or more list apparentements. Some apparentements can lose seats. Due to the difficulty of computing precise results, Pukelsheim and Leutgäb (2009) called this the lottery effect. They also found that even the total bias of the whole system could increase.

This paper provides example of huge seat deviations in the case of sole apparentement. The distinction between multiple and sole apparentements is blurred out, proving the equivalence of apparentement-proof conditions for these cases. The apparentement-proof set has a precise geometric configuration presented in the paper.

The organization of the paper is as follows. The next section, which begins with an example, provides the model and main results. Section 3 presents the inverse problem. Section 4 utilizes the game-theoretical approach. Section 5 concludes briefly. An Appendix includes all proofs.

2. Apparentement model

The mathematical model of the proportional representation elections conducted by the d'Hondt method is described by a set of parties $N = \{1,...,n\}$, the number of votes $v = (v_1,...,v_n)$, $V = \sum_{i \in N} v_i$, and the number of seats assigned by d'Hondt method $s(S) = (s_1,...,s_n)$, where $S = \sum_{i=1}^{n} s_i$ is the elected body size.

The d'Hondt method solution is obtained recursively (Balinski, Young, 1978):

(i) If $S = 0$ then $s = (0,...,0)$;

(ii) If $s(T) = (s_1,...,s_n)$ is the apportionment for $T < S$ and k is some party for which $\frac{v_k}{s_k + 1} = \max_{i \in N} \frac{v_i}{s_i + 1}$, then party k achieves subsequent seat $s(T+1) = (s_1,..,s_{k-1}, s_k + 1, s_{k+1},...,s_n)$. Step (ii) iterates until $\sum_{i \in N} s_i = S$.

For any party pair and any S, the following condition holds

$$\frac{v_j}{s_j} \geq \frac{v_i}{s_i + 1}. \tag{1}$$

Any additional seat decreases the number of votes per seat. If party *j* receives the last seat according to recursive procedure, then the two inequalities hold for any party pair

$$\frac{v_i}{s_i} \geq \frac{v_j}{s_j} \geq \frac{v_i}{s_i + 1}. \tag{2}$$

All parties but party *j* have the greater or equal number of votes per seat. It is a reason to coalesce. Apparentement $C \subseteq N$ is considered as one party with $\sum_{i \in C} v_i$ votes. The vote per seat ratio of an apparentement can be bigger than party *j*'s ratio. By the d'Hondt method, an apparentement receives the same number of seats or achieves additional seats. An apparentement is successful if it wins at least one additional seat. We do not define a particular tie-breaking rule and we do not consider ties. An apparentement is deemed successful if it has a greater votes per seats ratio in the presence of an additional seat than party *j*.

$$\frac{\sum_{i \in C} v_i}{\sum_{i \in C} s_i + 1} > \frac{v_j}{s_j} \tag{3}$$

Figure 1 represents the d'Hondt method apportionment with $n = 3$ and $S = 3$. Each point of the simplex corresponds to the vote distribution between parties. Point A shows the opportunity of successful apparentement (coalescing). At this point, parties 2 and 3 receive zero seats. Their apparentement has all votes but the votes of party 1. Geometrically, there is point B or C (each triangle side represents a two-party case). The apparentement receives one additional seat, and party 1 loses one seat.

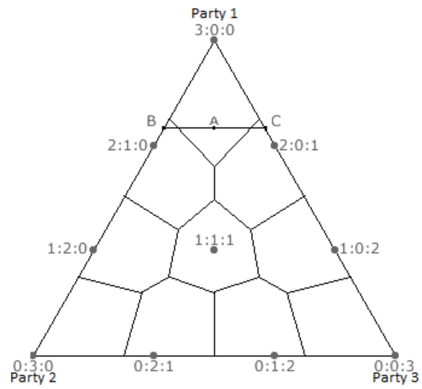

Figure 1

Successful apparentement can gain more than one seat. Consider an example with six parties and vote distribution $v = (179,\ 99,\ 99,\ 99,\ 99,\ 99)$. In the case of $S = 29$, we find the seat distribution $s = (9,\ 4,\ 4,\ 4,\ 4,\ 4)$. If the last five parties create an apparentement, then the new vote distribution is $\hat{v} = (179,\ 495)$, and the new seat distribution is $\hat{s} = (7,\ 22)$. The biggest party loses two seats.

In the general case, there is no limit to apparentement seat gains, the magnitude of loss, and the number of parties that lose seats can be arbitrarily large.

Proposition 1: *The number of additional seats of a successful apparentement is not limited from above.*

Proposition 2: *The number of parties that lose seats is not bounded from above.*

The proof for propositions 1 and 2 and subsequent propositions are given in the appendix. The proofs for propositions 1 and 2 are based on a heuristic example. Approximately every two parties in apparentement can gain one seat. Because the number of parties in apparentement is not limited, the number of additional seats of a successful apparentement is not limited from above.

There are arbitrary consequences of a sole apparentement. This result reflects a lottery effect investigated by Leutgäb and Pukelsheim (2009). There are $2^n - n - 2$ possible apparentements with two or more parties. It is hard to check the success of each apparentement. By propositions 3 and 4, the success of these apparentements is interconnected. It is necessary and sufficient to examine only one apparentement to explore the presence of successful apparentements.

Proposition 3. *If there is a successful apparentement* $C \subseteq N$*, then apparentement* $C' = N \setminus \{j\}$ *is also successful, where party j is a party that receives the last seat in an apportionment without apparentement.*

Apparentement $N \setminus \{j\}$ is not successful if and only if

$$\frac{\sum_{i \neq j} v_i}{\sum_{i \neq j} s_i + 1} < \frac{v_j}{s_j}. \tag{4}$$

Having $\sum_{i \in N} v_i = V$ and $\sum_{i \in N} s_i = S$, we obtain

$$\frac{V}{(S+1)} < \frac{v_j}{s_j}. \tag{5}$$

There are no successful apparentements if and only if apparentement $N \setminus \{j\}$ is not successful. Inequality (5) is called **the apparentement-proof condition**. The left side of this inequality is the Droop quota. The link between the d'Hondt method and the Droop quota method is not unique. De Córdoba and Penadés (2009) proved that the higher threshold functions (the maximum share of the votes that cannot be apportioned more than s seats) for the d'Hondt and Droop methods are identical. Equivalently, the apparentement-proof condition is represented in the biased form

$$\frac{1}{S(S+1)} > \frac{s_j}{S} - \frac{v_j}{V}. \tag{6}$$

The right side is the deviation from the exact proportional case. The apparentement-proof condition constrains disproportionality of apportionment. Because party *j* receives the last seat, it is possible, that party *j* does not have the highest bias among all parties. Other parties can be more malapportioned.

Proposition 3 shows the possibility of successful apparentement enlargement. To find minimal successful apparentement, proposition 4 provides the condition of successful apparentement reduction.

Proposition 4. *If 4 apparentement* $C = N \setminus \{j\}$ *is successful then apparentement* $C' = C \setminus \left\{ i \mid i \in C, \dfrac{V}{(S+1)} > \dfrac{v_i}{s_i} \right\}$ *is also successful.*

The apparentement-proof condition can be extended to the multiple apparentement case. Proposition 5 connects the sole apparentement case with the multiple apparentements case.

Proposition 5. *If the apparentement-proof condition holds, then multiple apparentements are not successful.*

Apparentement-proof set is a subset of a vote simplex in which the apparentement-proof condition holds. The construction of the apparentement-proof set is shown in figures 2-4. The apparentement-proof condition relies on information about the party that receives the last seat. The grey zone in figure 2 indicates that party 1 receives the last seat. The satisfaction of the apparentement-proof condition for party 1 leads to figure 3.

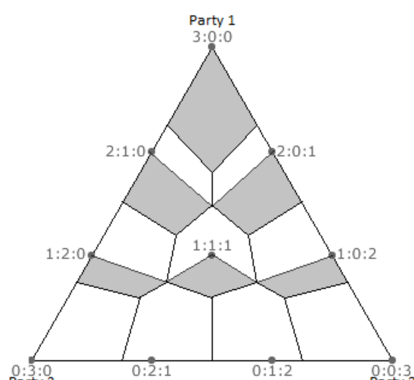

Figure 2

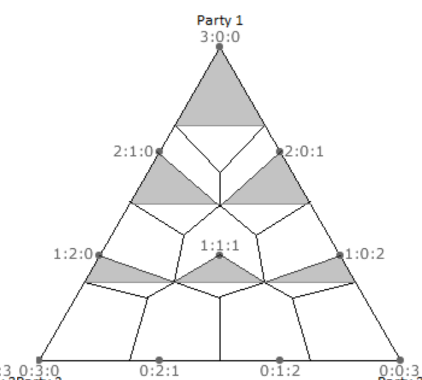

Figure 3

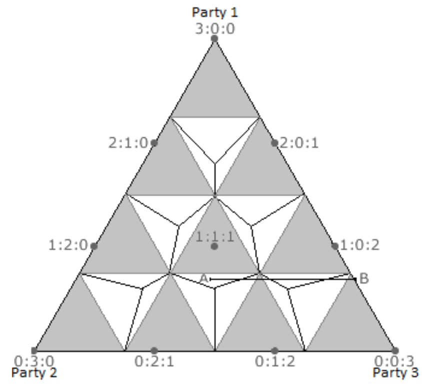

Figure 4

Figure 4 is the union of the grey zones from figure 3 for all parties (the apparentement-proof set). All borders of this set are obtained only with the apparentement-proof condition. The borders separating different seat distributions do not constrain this set. Obviously, exact proportional points with $\frac{v_i}{V} = \frac{s_i}{S}$ marked in figure 4 satisfy the apparentement-proof condition. Every exact proportional point corresponds to a grey triangle. Every point in such a triangle satisfies the apparentement-proof condition. All white triangles consist of three parts with different seat distributions. In each of these triangles, one seat is disputed (it can be assigned to one of three parties). For example, in the triangle with point A at the top, one seat is assigned to party 2, one seat is assigned to party 3, and one seat is disputed. Parties that do not receive this seat form an apparentement and gain an additional seat. In the area including point A, parties 2 and 3 coalesce. Point B from figure 4 is the result of cooperation. The apparentement is successful. It achieves three seats.

The apparentement-proof set consists of multiple fragments. Every exact proportional distribution corresponds to some part of the apparentement-proof set. Proposition 6 provides a general representation of this set.

Proposition 6. *The closure of the apparentement-proof set for each seat distribution forms a shape in the vote space that is similar to the vote simplex with a scale factor equals to* $(S+1)^{-1}$.

Proposition 6 describes the apparentement-proof set for a seat distribution. Having the number of all possible seat distributions, proposition 7 provides a precise share of the apparentement-proof set in the vote simplex.

Proposition 7. *The share of the volume of the shape constructed by the unification of all apparentement-proof sets in the vote simplex is equal to*

$$\alpha(n,S) = \frac{(n+S-1)!}{(n-1)!S!(S+1)^{n-1}}. \tag{7}$$

Corollary. $\lim_{S\to\infty} \alpha(n,S) = \frac{1}{(n-1)!}$, $\lim_{n\to\infty} \alpha(n,S) = 0$.

Assuming a uniform distribution of votes on the vote simplex, proposition 7 provides the probability of satisfaction of the apparentement-proof condition. The satisfaction of the apparentement-proof condition is a rare event. Share $\alpha(n,S)$ decreases with respect to n and S, and it rapidly converges to very small numbers. Table 1 contains example values.

Table 1. The values of $\alpha(n,S)$

	Number of parties					
Number of seats	3	4	5	6	7	8
3	0.625	0.313	0.137	0.055	0.021	0.007
4	0.6	0.28	0.112	0.04	0.013	0.004
5	0.583	0.259	0.097	0.032	0.01	0.003
10	0.545	0.215	0.068	0.019	0.005	0.001
20	0.524	0.191	0.055	0.013	0.003	0
100	0.505	0.172	0.044	0.009	0.002	0

The data from the 2008 Bavarian local elections[3] complies with the theoretical proportions from proposition 7. Elections are held in 2127 districts (communities). Each of them has its own number of seats to allocate and a set of competing parties. The number of parties and number of seats vary significantly. Table 2 provides information about the frequency of satisfaction of the apparentement-proof condition. Frequencies are close to the theoretical limits from corollary 1.

Table 2. The apparentement-proof condition

Number of parties(n)	3	4	5	6	7	8	>=9
Apparentement-proof condition holds	0.513	0.179	0.038	0.007	0.012	0.000	0.000
$1/(n-1)!$	0.500	0.167	0.042	0.008	0.001	0.000	0.000
Number of seats range	8-24	8-60	8-70	12-70	12-70	12-70	16-80

[3] Data is obtained from www.uni-augsburg.de/bazi

3. Inverse problem

Proposition 6 provides a description of the apparentement-proof set. Having a vote distribution, one can find a set of body sizes (different S) whereby the apparentement-proof condition holds. Figure 5 indicates that the minimal number of seats (written inside the figure) leads to satisfaction of the apparentement-proof condition for the three-party case. The size of the electoral body in the case of a small number of seats is frequently odd. Figure 6 presents that the minimal odd number of seats leads to satisfaction of the apparentement-proof condition. Figure 7 is a similar figure for even numbers.

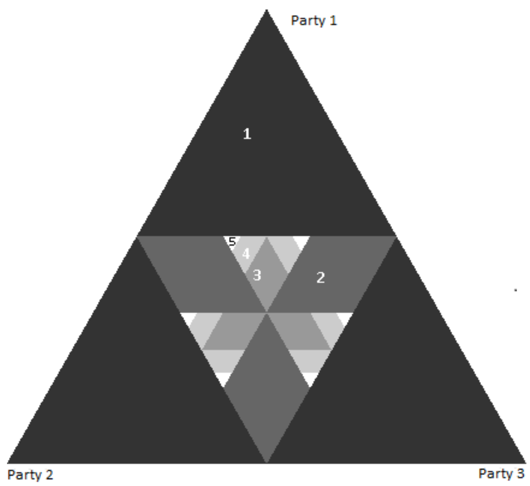

Figure 5

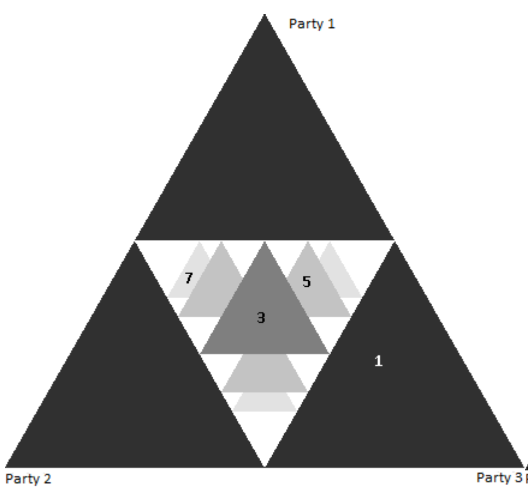

Figure 6

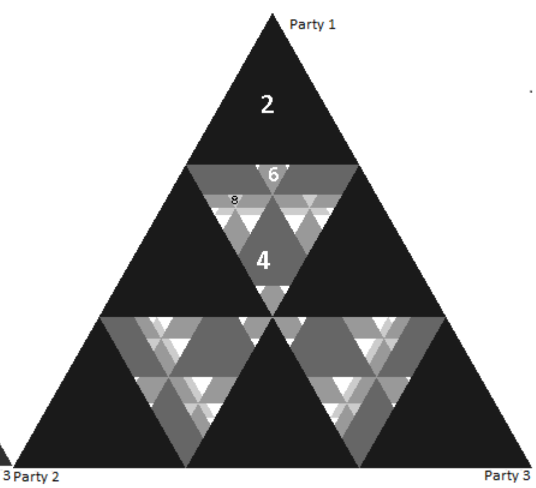

Figure 7

For the odd number of seats, $S+1$ is even and, geometrically, these cases are similar. In the area of almost equal division of votes between two parties, there are vote distributions for which it is impossible to find an odd number of seats that leads to satisfaction of the apparentement-proof condition. The violation of the apparentement-proof condition is correlated for different S, and for odd numbers, it leads to an impossibility result.

Proposition 8. *For any integer $K>1$ there exists a vote distribution that violates the apparentement-proof condition for all odd $S<K$.*

4. Game-theoretical approach

The apparentement problem is a cooperative problem. Let us define an apparentement game $v_A(C)$. An apparentement C corresponds to the number of seats obtained by apparentement C under the d'Hondt method in the new apportionment problem.

Consider an apparentement gain game

$$v_{AG}(C) = v_A(C) - \sum_{i \in C} s_i , \tag{8}$$

in which s_i is the d'Hondt distribution without apparentement. The seat gain $v_{AG}(C)$ is always nonnegative, and, by proposition 1, it can be unrestrictedly high. If the apparentement is not successful then $v_{AG}(C)=0$.

Let us define a full apparentement game $v_{FA}(C)$. All parties are divided on an apparentement C and a counterapparentement $\mathrm{N} \setminus C$. Apparentement C corresponds to the number of seats obtained by apparentement C under the d'Hondt method in the new apportionment problem.

Consider a full apparentement gain game

$$v_{FAG}(C) = v_{FA}(C) - \sum_{i \in C} s_i , \tag{9}$$

in which s_i is the d'Hondt seat distribution without apparentements. The seat gain $v_{FAG}(C)$ can be unrestrictedly high, but it can also be unrestrictedly low. It is a zero-sum game

$$v_{FAG}(C) + v_{FAG}(\mathrm{N} \setminus C) = 0 . \tag{10}$$

In the case of equiprobable apparentements, the Shapley value of games v_{AG} and v_{FAG} is the expected gain in seats. Marginal contribution of party i to apparentement C $v_{AG}(C \cup \{i\}) - v_{AG}(C)$ is limited from above to 1 but is not limited from below. This follows from Proposition 1.

If the apparentement-proof condition holds, then the marginal contribution is equal to zero for all parties and apparentements. If the apparentement-proof condition does not hold, then the Shapley value is positive for parties that form successful apparentements. For example, apparentement $\{2,3\}$ from point A in figure 4 is successful. The Shapley values for this example are equal to

$$\phi(v_{AG}) = (-1/3, \quad 1/6, \quad 1/6) . \tag{11}$$

$$\phi(v_{FAG}) = (-2/3, \quad 1/3, \quad 1/3) . \tag{12}$$

The sum of expected seat gains is always equal to zero. The first party loses on average while other parties gain. If parties 2 and 3 coalesce, then they win a seat. Party 1 also has an incentive to coalesce. With a two party apparentement, it does not lose a seat.

In the general case, losses are not bounded. Proposition 8 reinforces propositions 1 and 2. Even an expected loss of seats can be unrestrictedly large.

Proposition 9. *The Shapley values* $\phi_i(v_{AG})$ *and* $\phi_i(v_{FAG})$ *are not bounded from below.*

Conclusion

This paper shows arbitrary consequences of sole apparentement. One heuristic example lays the foundation of the proofs of the unboundedness of one party seat losses, of the number of losing parties and even of the average seat losses. The necessary and sufficient condition of the lack of sole and multiple successful apparentements discovered in the paper creates a possibility to eliminate the lottery effect of multiple apparentements. The configuration of the apparentement-proof set is simple. The share of the no-apparentement set in the whole vote simplex is very small, even for single-digit-numbers of seats and of parties. Creation of an electoral system that satisfies the apparentement proof condition seems to be problematic.

Appendix:

Proof of proposition 1.

Consider an example with $n = 2x + 1$ parties ($x \geq 3$). $n - 1 = 2x$ parties are equal, and each has $v_i = x + 1$ votes. One party has $v_j = 2x^2 - 4x$ votes. The total number of votes is equal to $V = 4x^2 - 2x$. The total number of seats is equal to $S = 4x^2 - 4x - 1$. Each party obtains one less seat than the number of votes. The vote per seat ratios for both types of parties are higher than one. Additional seats reduce the vote per seat ratio to one. Therefore it is not assigned. The d'Hondt method seat distribution is $s_i = x$ and $s_j = 2x^2 - 4x - 1$.

Consider an apparentement of $n - 1$ equal parties. The sufficient condition that the apparentement wins at least $x - 2$ additional seats is

$$\frac{v_j}{s_j - (x-2) + 1} < \frac{(n-1)v_i}{(n-1)s_i + x - 2}, \tag{13}$$

$$\frac{2x^2 - 4x}{2x^2 - 5x + 2} < \frac{2x^2 + 2x}{2x^2 + x - 2}. \tag{13a}$$

It holds if $x \geq 3$. Taking an unrestrictedly large x, we obtain an unrestrictedly large number of additional seats gained by apparentement.■

Proof of proposition 2.

Consider an example with $n = (2x + 1)y$ parties ($x \geq 3$ $y \geq 1$). $n - y = 2xy$ parties are equal, and each has $v_i = x + 1$ votes. y parties have $v_j = 2x^2 - 4x$ votes. The total number of votes is equal to $V = 4x^2 y - 2xy$. The total number of seats is equal to $S = 4x^2 y - 4xy - y$. Each party obtains one less seat than the number of votes. The vote per seat ratios of both parties are higher than one. Additional seats reduce the vote per seat ratio to one, and therefore, it is not assigned. The d'Hondt method seat distribution is $s_i = x$ and $s_j = 2x^2 - 4x - 1$.

Consider apparentement of $n-y$ equal parties. The sufficient condition that apparentement wins at least y seats is

$$\frac{v_j}{s_j} < \frac{(n-y)v_i}{(n-y)s_i + y}, \tag{14}$$

$$\frac{2x^2 - 4x}{2x^2 - 4x - 1} < \frac{2xy(x+1)}{2xyx + y}. \tag{14a}$$

It is true for any $x \geq 3$ and $y \geq 1$. There are at least y parties that lose seats. Taking an unrestrictedly large y, we obtain an unrestrictedly large number of parties that lose seats. ■

Proof of proposition 3.

Let $s(S) = (s_1,...,s_n)$ be the d'Hondt's seats apportionment before coalition formation. Apparentement C is successful, and therefore, it receives at least one extra seat

$$\frac{\sum_{i\in C} v_i}{\sum_{i\in C} s_i + 1} > \frac{v_j}{s_j}. \tag{15}$$

If $j \notin C$ then

$$\sum_{i\in C} v_i > \frac{v_j}{s_j}\left(\sum_{i\in C} s_i + 1\right). \tag{16}$$

If $j \in C$ then

$$\sum_{i\in C/\{j\}} v_i > \frac{v_j}{s_j}\left(\sum_{i\in C/\{j\}} s_i + 1\right). \tag{16a}$$

From the d'Hondt method property for all $i \in N/(C \cup \{j\})$ we have

$$v_i \geq \frac{v_j}{s_j} s_i, \tag{17}$$

$$\sum_{i\in N/(C\cup\{j\})} v_i \geq \frac{v_j}{s_j}\left(\sum_{i\in N/(C\cup\{j\})} s_i\right). \tag{18}$$

Summing (15) or (15a) and (17) we have

$$\sum_{i\in N\setminus\{j\}} v_i > \frac{v_j}{s_j}\left(\sum_{i\in N\setminus\{j\}} s_i + 1\right), \tag{19}$$

$$\frac{\sum_{i\in N\setminus\{j\}} v_i}{\sum_{i\in N\setminus\{j\}} s_i + 1} > \frac{v_j}{s_j}. \tag{20}$$

Apparentement $N\setminus\{j\}$ is successful.■

Proof of proposition 4.

Apparentement C is successful therefore it receives at least one extra seat

$$\frac{\sum_{i\in N\setminus\{j\}} v_i}{\sum_{i\in N\setminus\{j\}} s_i + 1} > \frac{V}{S+1} > \frac{v_j}{s_j}, \tag{21}$$

Having $s_i > v_i \frac{(S+1)}{V}$ we obtain

$$\sum_{i\in C'} v_i > \frac{V}{S+1}\left(\sum_{i\in C'} s_i + 1\right). > \frac{v_j}{s_j}\left(\sum_{i\in C'} s_i + 1\right), \tag{22}$$

$$\frac{\sum_{i\in C'} v_i}{\sum_{i\in C'} s_i + 1} > \frac{V}{S+1} > \frac{v_j}{s_j}. \tag{23}$$

Apparentement C is successful. ■

Proof of proposition 5.

If party j obtains the last seat according to the d'Hondt method then

$$v_i \geq \frac{v_j}{s_j} s_i, \forall i \in N. \tag{24}$$

For arbitrary apparentement $C \subseteq \mathrm{N}$, $j \notin C$ we have

$$\frac{\sum_{i\in C} v_i}{\sum_{i\in C} s_i} \ge \frac{v_j}{s_j}. \quad (25)$$

Taking into account the apparentement-proof condition we have

$$\frac{\sum_{i\in C} v_i}{\sum_{i\in C} s_i} \ge \frac{v_j}{s_j} > \frac{\sum_{i\ne j} v_i}{\sum_{i\ne j} s_i + 1}. \quad (26)$$

Apparentement C does not change a party that receives the last seat in apportionment. Because the party j receives the last seat, the apparentement-proof condition holds.

The first apparentement does not change the distribution of seats and the apparentement-proof condition remains unchanged. Every subsequent apparentement has the same result. The existence of multiple apparentements does not change the distribution of seats and satisfaction of the apparentement-proof condition remains unchanged.■

Proof of proposition 6.

Because we consider the closure of the apparentement-proof set, the inclusion of the inequality borders does not matter.

On a vote simplex $\sum_{i\in N} v_i = V$, the d'Hondt's seats apportionment $s = (s_1,...,s_n)$, $S = \sum_{i\in N} s_i$ corresponds to a set constrained by inequalities

$$v_j > s_j \frac{v_i}{s_i + 1}. \quad (27)$$

Any point from the set leads to the seat distribution s. These sets form the partition of the vote simplex. A set in which party j in the d'Hondt's seats apportionment s is a party that obtains the last seat is a set described by a system of $2(n-1)$ inequalities

$$\frac{v_j}{s_j} > \frac{v_i}{s_i + 1}, \quad (28)$$

$$v_i > s_i \frac{v_j}{s_j}. \quad (29)$$

The apparentement-proof condition generates an additional inequality

$$\frac{v_j}{s_j} > \frac{V}{(S+1)}. \qquad (30)$$

A set obtained by inequalities (28), (29), (30) is a set $A_j(s)$. The union of these sets forms a set $A(s) = \bigcup_{j \in N} A_j(s)$. Set $A(s)$ is an apparentement-proof set for vote distribution s. Inequality (29) defines a party that obtains the last seat. Set $A(s)$ is constrained by inequalities (28) and (30). Inequality (29) is not constrained.

Having $\frac{v_j}{s_j} = \frac{v_j}{s_j}$ and inequality (30) we have

$$\frac{v_j}{s_j} > \frac{\sum_{i \neq j} v_i}{\sum_{i \neq j} s_i + 1}. \qquad (31)$$

If for some $k \in N, k \neq j$ we have $v_k \geq \frac{v_j}{s_j}(s_k + 1)$, then $\frac{v_j}{s_j} < \frac{\sum_{i \neq j} v_i}{\sum_{i \neq j} s_i + 1}$. This contradicts (31), therefore

$$\frac{v_k}{s_k + 1} < \frac{v_j}{s_j}. \qquad (32)$$

Inequality (32) coincides with inequality (28). Thus, inequalities (29) and (30) imply inequality (28). Set $A(s)$ is bordered only by inequality (30).

The apparentement-proof set $A(s)$ for vote distribution s is a set defined by n linear inequalities (30). Each inequality forms a border that is parallel with a border of the vote simplex. Set $A(s)$ forms a shape that is similar to that of the vote simplex. From inequality (30), the scale factor is equal to $(S+1)^{-1}$. ∎

Proof of proposition 7.

From Proposition 5, the volume of a shape corresponding to the apparentement-proof set for arbitrary seat distribution is equal to $(S+1)^{-(n-1)}$ of the volume of vote simplex. The number of different seat distributions is equal to multiset coefficient $\frac{(n+S-1)!}{(n-1)!S!}$. The share of the volume of the shape constructed by the unification of all apparentement-proof sets in the vote simplex is equal to

$$\alpha(n,S)=\frac{(n+S-1)!}{(n-1)!S!(S+1)^{n-1}}\ .\blacksquare \tag{33}$$

Proof of corollary 1.

$$\lim_{s\to\infty}\alpha(n,S)=\lim_{s\to\infty}\frac{(n+S-1)!}{(n-1)!S!(S+1)^{n-1}}=\lim_{s\to\infty}\frac{\sqrt{2\pi(n+S-1)}\left(\frac{n+S-1}{e}\right)^{n+S-1}}{(n-1)!\sqrt{2\pi S}\left(\frac{S}{e}\right)^{S}(S+1)^{n-1}}=$$

$$=\lim_{s\to\infty}\frac{\sqrt{\frac{n+S-1}{S}}\left(\frac{n+S-1}{e(S+1)}\right)^{n-1}\left(\frac{n+S-1}{S}\right)^{S}}{(n-1)!}=\lim_{s\to\infty}\frac{e^{-(n-1)}e^{n-1}}{(n-1)!}=\frac{1}{(n-1)!}. \tag{34}$$

$$\lim_{n\to\infty}\alpha(n,S)=\lim_{n\to\infty}\frac{(n+S-1)!}{(n-1)!S!(S+1)^{n-1}}=\lim_{n\to\infty}\frac{\sqrt{2\pi(n+S-1)}\left(\frac{n+S-1}{e}\right)^{n+S-1}}{\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}S!(S+1)^{n-1}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{\frac{n+S-1}{n-1}}\left(\frac{n+S-1}{n-1}\right)^{n-1}\left(\frac{n+S-1}{e}\right)^{S}}{S!(S+1)^{n-1}}=\lim_{n\to\infty}\frac{(n+S-1)^{S}}{S!(S+1)^{n-1}}=0.\blacksquare \tag{35}$$

Proof of proposition 8.

Consider an example with $n\geq 3$ parties. The first party has $\frac{V}{2}-\frac{V}{2(K+2)}+\varepsilon$ and the second party has $\frac{V}{2}-\frac{V}{2(K+2)}-\varepsilon$ votes, where $\varepsilon\geq 0$ is sufficiently small. Each of other $n-2$ parties has $\frac{V}{(K+2)(n-2)}$ votes. The necessary condition to have at least one seat in the legislature of the size $S<K$ is $v_i\geq\frac{V}{S+1}$. We have $\frac{V}{S+1}\geq\frac{V}{K+1}\geq\frac{V}{(K+2)(n-2)}$. Each of the

other $n-2$ parties gains no seats. The first party obtains $\frac{S+1}{2}$ seats and the second party obtains $\frac{S-1}{2}$ seats. The second party has an incentive to form an apparentement with all parties but the first party. This apparentement wins more than $\frac{S-1}{2}$. This vote distribution violates the apparentement-proof condition. ■

Proof of proposition 9.

Consider an example with $n = 2x+1$ parties ($x \geq 3$). $n-1 = 2x$ parties are equal, and each has $v_i = x+1$ votes. One party has $v_j = 2x^2 - 4x$ votes. The total number of votes is equal to $V = 4x^2 - 2x$. The total number of seats is equal to $S = 4x^2 - 4x - 1$. Each party obtains one less seat than the number of votes. The vote per seat ratios of both parties are higher than one. Additional seats reduce the vote per seat ratio to one, and therefore, it is not assigned. The d'Hondt method seat distribution is $s_i = x$ and $s_j = 2x^2 - 4x - 1$.

Consider an apparentement of k equal parties ($2 \leq k \leq 2x$). The sufficient condition that the apparentement wins at least $\lfloor k/3 \rfloor$ additional seats is

$$\frac{v_j}{s_j - k/3 + 1} < \frac{kv_i}{ks_i + k/3}, \tag{36}$$

$$\frac{2x^2 - 4x}{2x^2 - 4x - 1 - k/3 + 1} < \frac{kx + k}{kx + k/3}. \tag{36a}$$

If $x \geq 6$ then it holds for any $2 \leq k \leq 2x$.

Consider an apparentement of $k \geq 0$ equal parties and party j. The sufficient condition that the apparentement wins no additional seats is

$$\frac{v_j + kv_i}{s_j + ks_i + 1} < \frac{v_i}{s_i}, \tag{37}$$

$$\frac{2x^2 - 4x + k(x+1)}{2x^2 - 4x - 1 + kx + 1} < \frac{x+1}{x}. \tag{37a}$$

It holds if $x \geq 3$.

The marginal contribution of party j to apparentement C with k equal parties ($2 \le k \le 2x$) in the apparentement gain game is negative, and it is at most to $-\lfloor k/3 \rfloor$. The marginal contribution of party j to apparentement C with k equal parties ($0 \le k \le 1$) in the apparentement gain game is equal to zero. The Shapley value of party j is at most to

$$\phi_j(v_{AG}) \le \frac{1}{2x+1}\sum_{k=2}^{2x}\left(-\left\lfloor \frac{k}{3} \right\rfloor\right) \le \frac{1}{2x+1}\sum_{k=2}^{2x}\left(-\frac{k}{3}+1\right) = \left(\frac{2x-1}{2x+1}\right)\left(\frac{2}{3}-\frac{1}{3}x\right). \quad (38)$$

The Shapley value of party j is not bounded from below.

Consider a full apparentement gain game. Consider an apparentement of k equal parties ($2 \le k \le 2x$). The sufficient condition that the apparentement wins at least $\lfloor k/3 \rfloor$ additional seats is

$$\frac{v_j + (2x-k)v_i}{s_j + (2x-k)s_i - k/3 + 1} < \frac{kv_i}{ks_i + k/3}, \quad (39)$$

$$\frac{2x^2 - 4x + (2x-k)x + (2x-k)}{2x^2 - 4x - 1 + (2x-k)x - k/3 + 1} < \frac{kx+k}{kx + k/3}. \quad (39a)$$

If $x \ge 6$, then it holds for any $2 \le k \le 2x$.

Consider an apparentement of $k \ge 0$ equal parties and party j. The sufficient condition that the apparentement wins no additional seats is

$$\frac{v_j + kv_i}{s_j + ks_i + 1} < \frac{(2x-k)v_i}{(2x-k)s_i}, \quad (40)$$

$$\frac{2x^2 - 4x + k(x+1)}{2x^2 - 4x - 1 + kx + 1} < \frac{x+1}{x}. \quad (40a)$$

It holds if $x \ge 3$. The marginal contribution of party j to apparentement C with k equal parties ($2 \le k \le 2x$) in the full apparentement gain game is negative, and it is at most to $-\lfloor k/3 \rfloor$. The marginal contribution of party j to apparentement C with k equal parties ($0 \le k \le 1$) in the full apparentement gain game is equal to zero. The Shapley value of party j is at most to

$$\phi_j(v_{FAG}) \le \frac{1}{2x+1}\sum_{k=2}^{2x}\left(-\left\lfloor \frac{k}{3} \right\rfloor\right) \le \frac{1}{2x+1}\sum_{k=2}^{2x}\left(-\frac{k}{3}+1\right) = \left(\frac{2x-1}{2x+1}\right)\left(\frac{2}{3}-\frac{1}{3}x\right). \quad (41)$$

The Shapley value of party j is not bounded from below.■

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