KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE OF ECONOMIC RESEARCH

Discussion Paper No.859

“Pricing of Discount Bonds with a Markov Switching Regime”

Robert J. Elliott and Katsumasa Nishide

April 2013

KYOTO UNIVERSITY

KYOTO, JAPAN

Pricing of Discount Bonds with a Markov Switching Regime*

Robert J. Elliott† and Katsumasa Nishide‡

(April 16, 2013)

Abstract. We consider a Markov switching regime and price a discount bond using two popular models for the short rate, the Vasicek- and CIR-dynamics. In both cases, an explicit formula is obtained for the bond price which includes the solution of a matrix ODE. Our model is easy to calculate and captures the effect of regime uncertainty on the price and the term structure.

Keywords: Bond pricing, term structure, Markov switching regime, Vasicek model, CIR model, stochastic flows.

JEL classification: G12, E32,

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*The first author acknowledges the supports of Australian Research Council and the SSHRC. The second author acknowledges the financial support of the Japanese Ministry of Education, Culture, Sports, Science and Technology (MEXT) Grand in Aid for Scientific Research (B) #23310098 and (C) #23530362. This paper was presented at a workshop organized by the 2012 Project Research at KIER of Kyoto University as the Joint Usage and Research Center.

†School of Mathematics, University of Adelaide, Center for Applied Financial Studies, University of South Australia, Haskayne School of Business, University of Calgary. E-mail: `relliott@ucalgary.ca`.

‡Department of Economics, Yokohama National University. 79-4 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan. E-mail: `knishide@ynu.ac.jp`.

1 Introduction

When pricing a discount bond there are two popular models for the short rate, the Vasicek- and CIR-dynamics. However, it is often said that a one-factor Vasicek or CIR model is not flexible enough to describe the term structure of bond prices. To address this issue, Hull and White (1990) extended the two models to the case where the coefficients in the short rate dynamics are deterministic functions of time to match the term structure observed in actual financial markets. Other papers that extended the basic models include Hull and White (1994), Duffie and Kan (1996), Duffie et al. (2000).

Another approach is to introduce a Markov switching regime into the short rate dynamics to capture economic cycles like booms and recessions. There are some papers which consider the case where parameters of the short rate have a Markov switching structure. In Hansen and Poulsen (2000), the mean-reverting level takes one of the two values with a symmetric transition rate. Landén (2000) considers the case where the drift and diffusion parameters are modulated by a Markov process and derives simultaneous PDEs (which are ODEs in some special cases).

In this paper we derive an explicit expression for bond prices where the short rate follows Markov switching dynamics under the risk-neutral probability. The key idea is that we firstly derive a conditional expectation given knowledge of the full history of the Markov chain and then, using the tower property, take a second expectation. Given the information of the chain up to maturity, the conditional expectation of the bond is easily derived as we can use a short rate process in which the exogenous parameters are time-dependent functions. Then the exact bond price is given by the second expected value of this conditional expectation.

As we shall see later, the pricing formula includes a solution of a matrix ODE, or a first-order linear ODEs system with time-dependent coefficients. Unlike previous papers, our formula is numerically tractable because the calculation of a matrix ODE is not difficult compared with simultaneous PDEs or a non-linear ODE system. As a corollary, we obtain a closed-form solution for the bond price in some special cases.

We first obtain the bond price in the case where the short rate follows Vasicek-dynamics. Given the information about the chain up to maturity, the short rate is a conditional normal random variable and so the bond price can be found. When the short rate dynamics have a CIR form with Markov switching, we apply the discussion in Elliott and van der Hoek (2001). A key ingredient is the theory of stochastic flows, and using this theory we obtain a conditional expectation of a bond price given the full information of the chain history.

The remaining part of the paper is organized as follows. Section 2 considers the pricing of a discount bond when the short rate has Vasicek-dynamics. Section 3 gives a bond pricing formula for a CIR-type process. In Section 4 the term structure in our model is calculated and examined. Some concluding remarks are provided in Section 6.

2 Vasicek-Type Short Rate

This section gives a derivation of the bond pricing with Vasicek-type short rate dynamics.

2.1 Model setup

We assume the short rate has dynamics

$$\mathrm{d}r = \kappa(\theta_t - r)\mathrm{d}t + \sigma \mathrm{d}w, \tag{2.1}$$

where w denotes a Brownian motion under the risk-neutral probability measure.

We suppose that the mean-reverting level θ in the dynamics is determined by a Markov chain $\mathbf{z} = \{\mathbf{z}_t, t \geq 0\}$ with n possible states. Without loss of generality the state space of $\mathbf{z}$ can be identifed with the set of standard unit vectors in $\boldsymbol{R}^n$, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_i = (0, \ldots, 0, 1, \ldots, 0)' \in \boldsymbol{R}^n$. If the mean-reverting level θ_t takes one of the n values in $\{\theta^1, \ldots, \theta^n\}$, then $\theta_t = \langle \boldsymbol{\vartheta}, \mathbf{z}_t \rangle$, where $\boldsymbol{\vartheta} = (\theta^1, \ldots, \theta^n)'$. We suppose $\mathbf{z}$ has a transition-rate matrix of the form

$$\Gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{pmatrix}.$$

Here for $i \neq j$ γ_{ij} denotes the intensity of jumping from regime i to j. Recall that $\gamma_{ij} \geq 0$ for $i \neq j$ and $\sum_j \gamma_{ij} = 0$. The two processes w and $\mathbf{z}$ are supposed independent.

It is well-known that the process $\mathbf{z}$ follows the vector stochastic dynamics

$$\mathrm{d}\mathbf{z} = \Gamma' \mathbf{z} \mathrm{d}t + \mathrm{d}\mathbf{m}. \tag{2.2}$$

Here $\mathbf{m} = \{\mathbf{m}_t, t \geq 0\}$ is an $\boldsymbol{R}^n$-valued martingale process.

Suppose $\mathcal{F}_t^w$ and $\mathcal{F}_t^{\mathbf{z}}$, respectively, denote the filtrations generated by the Brownian motion w and the Markov chain process $\mathbf{z}$ up to time t. Then $\mathcal{G}_t = \mathcal{F}_t^w \vee \mathcal{F}_t^{\mathbf{z}}$ describes all the information up to t.

2.2 Derivation

The following proposition gives the pricing formula of a discount bond in the Vasicek case.

Proposition 1. *Let Θ be the diagonal matrix whose i-th component is θ^i and $\mathbf{1} = (1,\ldots,1)'$. When the short rate r has Vasicek dynamics as in (2.1), the price of a zero-coupon bond at time t with maturity τ is given by*

$$p(t,\tau,r,\mathbf{z}) = \left\langle \Phi(t,\tau;\mathrm{e}^{-\kappa(\tau-t)})\mathbf{z}, \mathbf{1}\right\rangle \times \exp\left\{ -\frac{1-\mathrm{e}^{-\kappa(\tau-t)}}{\kappa} r + \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2 \left((\tau - t) - \frac{2(1-\mathrm{e}^{-\kappa(\tau-t)})}{\kappa} + \frac{1-\mathrm{e}^{-2\kappa(\tau-t)}}{2\kappa}\right)\right\}, \tag{2.3}$$

where $\Phi(t,\tau;\eta)$ is the $n\times n$ matrix function defined by the ordinary differential equation

$$\frac{\mathrm{d}\Phi(u,t;\eta)}{\mathrm{d}t} = (\Gamma' - (1-\eta \mathrm{e}^{\kappa t})\Theta)\Phi(u,t;\eta) \tag{2.4}$$

with $\Phi(u,u;\eta) = I$ (identity matrix).

Proof. From (2.1), the short rate satisfies

$$r_t = \mathrm{e}^{-\kappa t} r_0 + \kappa \int_0^t \mathrm{e}^{-\kappa(t-u)}\theta_u \mathrm{d}u + \sigma \int_0^t \mathrm{e}^{-\kappa(t-u)} \mathrm{d}w_u.$$

Thus we have

$$\int_0^\tau r_u \mathrm{d}u = \frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa} r_0 + \int_0^\tau \left(1-\mathrm{e}^{-\kappa(\tau-u)}\right)\theta_u \mathrm{d}u + \frac{\sigma}{\kappa}\int_0^\tau \left(1-\mathrm{e}^{-\kappa(\tau-u)}\right)\mathrm{d}w_u. \tag{2.5}$$

Note that given $\mathcal{F}_\tau^{\mathbf{z}}$, the history of $\mathbf{z}$ to maturity τ, (2.5) is normally distributed with mean

$$\mathbb{E}_0\left[\int_0^\tau r_u \mathrm{d}u \,\middle|\, \mathcal{F}_\tau^{\mathbf{z}}\right] = \frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa} r_0 + \int_0^\tau \left(1-\mathrm{e}^{-\kappa(\tau-u)}\right)\theta_u \mathrm{d}u,$$

and variance

$$\begin{aligned}\mathbb{V}_0\left[\int_0^\tau r_u \mathrm{d}u \,\middle|\, \mathcal{F}_\tau^{\mathbf{z}}\right] &= \left(\frac{\sigma}{\kappa}\right)^2 \int_0^\tau \left(1 - 2\mathrm{e}^{-\kappa(\tau-u)} + \mathrm{e}^{-2\kappa(\tau-u)}\right)^2 \mathrm{d}u \\ &= \left(\frac{\sigma}{\kappa}\right)^2 \left[\tau - \frac{2(1-\mathrm{e}^{-\kappa\tau})}{\kappa} + \frac{1-\mathrm{e}^{-2\kappa\tau}}{2\kappa}\right].\end{aligned}$$

Here $\mathbb{E}_0$ and $\mathbb{V}_0$ are the expectation and variance operators given r_0 and $\mathbf{z}_0$, respectively. Therefore, given $\mathcal{F}_\tau^{\mathbf{z}}$

$$\begin{aligned}\mathbb{E}_0\left[\exp\left\{-\int_0^\tau r_u \mathrm{d}u\right\} \,\middle|\, \mathcal{F}_\tau^{\mathbf{z}}\right] = &\exp\left\{ -\frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa} r_0 - \int_0^\tau \left(1-\mathrm{e}^{-\kappa(\tau-u)}\right)\theta_u \mathrm{d}u \right.\\ &\left. + \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2 \left(\tau - \frac{2(1-\mathrm{e}^{-\kappa\tau})}{\kappa} + \frac{1-\mathrm{e}^{-2\kappa\tau}}{2\kappa}\right)\right\}.\end{aligned}$$

Thus conditioning out $\mathcal{F}_\tau^{\mathbf{z}}$ the bond price is equal to

$$p(0,\tau) = q_\tau \mathbb{E}_0[h_\tau],$$

where

$$q_\tau = \exp\left\{-\frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa} r_0 + \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2 \left(\tau - \frac{2(1-\mathrm{e}^{-\kappa\tau})}{\kappa} + \frac{1-\mathrm{e}^{-2\kappa\tau}}{2\kappa}\right)\right\}$$

and

$$h_\tau = \exp\left\{-\int_0^\tau \left(1-\mathrm{e}^{-\kappa(\tau-u)}\right)\theta_u \mathrm{d}u\right\}. \tag{2.6}$$

To calculate $\mathbb{E}_0[h_\tau]$, consider the processes

$$g_t^\eta = \exp\left\{-\int_0^t (1-\eta\mathrm{e}^{\kappa u})\langle\boldsymbol{\vartheta}, \mathbf{z}_u\rangle \mathrm{d}u\right\}. \tag{2.7}$$

for an arbitrary constant η, and the n-dimensional vector process $g_t^\eta \mathbf{z}_t$. As $\mathbf{z}$ satisfies (2.2) and g_t^η is given by (2.7), the vector process $g_t^\eta \mathbf{z}_t$ has dynamics

$$\begin{aligned} \mathrm{d}(g_t^\eta \mathbf{z}_t) =& g_t^\eta \mathrm{d}\mathbf{z}_t + \mathbf{z}_t \mathrm{d}g_t^\eta \\ =& (\Gamma' - (1-\eta\mathrm{e}^{\kappa t})\langle\boldsymbol{\vartheta}, \mathbf{z}_t\rangle) g_t^\eta \mathbf{z}_t \mathrm{d}t + g_t^\eta \mathrm{d}\mathbf{m}_t. \end{aligned} \tag{2.8}$$

Note that $\langle\boldsymbol{\vartheta}, \mathbf{z}_t\rangle g_t^\eta \mathbf{z}_t = \Theta g_t^\eta \mathbf{z}_t$. Hence (2.8) can be written as

$$g_t^\eta \mathbf{z}_t = \mathbf{z}_0 + \int_0^t (\Gamma' - (1-\eta\mathrm{e}^{\kappa u})\Theta) g_u^\eta \mathbf{z}_u \mathrm{d}u + \int_0^t g_u^\eta \mathrm{d}\mathbf{m}_u. \tag{2.9}$$

Taking the expectation of both sides of (2.9), we obtain

$$\mathbb{E}_0[g_t^\eta \mathbf{z}_t] = \mathbf{z}_0 + \int_0^t (\Gamma' - (1-\eta\mathrm{e}^{\kappa u})\Theta)\mathbb{E}_0\left[g_u^\eta \mathbf{z}_u\right] \mathrm{d}u.$$

Now suppose $\Phi = \Phi(u,t;\eta)$ is the solution of (2.4). Then we have

$$\mathbb{E}_0[g_t^\eta \mathbf{z}_t] = \Phi(0,t;\eta)\mathbf{z}_0. \tag{2.10}$$

Because $\langle \mathbf{z}_t, \mathbf{1}\rangle = 1$, we see that

$$\mathbb{E}_0[g_t^\eta] = \mathbb{E}[\langle g_t^\eta \mathbf{z}_t, \mathbf{1}\rangle] = \langle \Phi(0,t;\eta)\mathbf{z}_0, \mathbf{1}\rangle.$$

Now $\mathrm{e}^{-\kappa\tau}$ is not random, so we can set $\eta = \mathrm{e}^{-\kappa\tau}$. Then we have

$$\mathbb{E}_0[h_\tau] = \left\langle \Phi(0,\tau;\mathrm{e}^{-\kappa\tau})\mathbf{z}_0, \mathbf{1}\right\rangle.$$

The proposition follows because r is a time-homogeneous Markov process. □

Remark 1. The simultaneous PDE system in Proposition 3.2 of Landén (2000) becomes a simultaneous ODE system when the short rate has Vasicek-dynamics, (see Example 3.1 in her paper), with a Markov switching θ. However, her ODE system does not take the form of a matrix ODE but a system of non-linear ODEs and thus requires a condition for the solution of the ODEs to exist. On the other hand, our ODEs are a linear system of a matrix form with a smooth matrix function, and so the solution of the ODEs exists and is unique. The solution of our matrix ODE is easily calculated while solutions of non-linear ODEs or PDEs are not. This difference comes from the treatment of $\eta = \mathrm{e}^{-\kappa(\tau-t)}$. In other words, the ODEs become simple by taking a conditional expectation of the bond price and then setting $\mathrm{e}^{-\kappa\tau}$ in (2.6) as a non ramdom parameter.[1]

Remark 2. If $\Gamma'\Theta = \Theta\Gamma'$, then the solution of (2.4) is given by

$$\begin{aligned}\Phi(u,t;\eta) =& \exp\left\{(t-u)\Gamma' - \left((t-u) - \frac{\eta(\mathrm{e}^{-\kappa t} - \mathrm{e}^{\kappa u})}{\kappa}\right)\Theta\right\}\\ =& \mathrm{e}^{(t-u)\Gamma'} \times \mathrm{e}^{-\left((t-u)-\frac{\eta(\mathrm{e}^{-\kappa t}-\mathrm{e}^{\kappa u})}{\kappa}\right)\Theta}\end{aligned}$$

where the exponential matrix e^A is simply

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

In this case, we have

$$\begin{aligned}\langle \Phi(t,\tau;\mathrm{e}^{-\kappa(\tau-t)})\mathbf{z}, \mathbf{1}\rangle =& \langle \mathrm{e}^{(\tau-t)\Gamma'}\mathbf{z}, \mathrm{e}^{-\left((\tau-t)-\frac{\eta(\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{\kappa t})}{\kappa}\right)\Theta}\mathbf{1}\rangle\\ =& \sum_{i=1}^{n} w_{\tau-t}^i(\mathbf{z})\mathrm{e}^{-\left((\tau-t)-\frac{\eta(\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{\kappa t})}{\kappa}\right)\theta^i},\end{aligned}$$

where

$$\mathbf{w}_\tau(\mathbf{z}) := (w_\tau^1(\mathbf{z}) \dots w_\tau^n(\mathbf{z}))' = \mathrm{e}^{\tau\Gamma'}\mathbf{z}. \tag{2.11}$$

Hence the price formula (2.3) becomes

$$p(t,\tau,r,\mathbf{z}) = \sum_{i=1}^{n} w_{\tau-t}^i(\mathbf{z}) p_{\mathrm{V}}(t,\tau,r;\theta^i),$$

where p_{V} is the pricing formula in the standard Vasicek case with constant mean-reverting level θ^i. That is, the bond price is equal to the weighted average of solutions in the standard Vasicek case when Θ and Γ' commute.

[1] It is worth mentioning that Landén (2000) considers a more general setting and so her PDE system holds for more cases than ours.

As a corollary of Remark 2, we can obtain a closed-form solution for a bond price when for $i \neq j$ γ_{ij} is a constant. That is, when $\gamma_{ij} = \gamma$ for $i \neq j$, the weights are given by

$$w_\tau^i(\mathbf{e}_k) = \begin{cases} \frac{1-\mathrm{e}^{-n\gamma\tau}}{n} & i \neq k, \\ 1 - \sum_{j\neq k} w_\tau^j(\mathbf{e}_k) & i = k. \end{cases}$$

The case $n = 2$ corresponds to the model of Hansen and Poulsen (2000).

Proposition 1 can be generalized to the case where not only θ but also σ are modulated by a Markov chain. We omit the proof because it is a simple variation of Proposition 1.

Proposition 2. *Suppose that the short rate process has Vasicek dynamics with* $\theta_t = \langle \boldsymbol{\vartheta}, \mathbf{z}_t \rangle$ *and* $\sigma_t = \langle \boldsymbol{\sigma}, \mathbf{z}_t \rangle$ *and the matrix* Σ *is defined by*

$$\Sigma = \mathrm{diag}[\boldsymbol{\sigma}].$$

Then the price of a zero-coupon bond at time t *with maturity* τ *is given by*

$$p(t, \tau, r, \mathbf{z}) = \langle \Phi(t, \tau; \mathrm{e}^{-\kappa(\tau-t)})\mathbf{z}, \mathbf{1} \rangle \times \exp\left\{ -\frac{1 - \mathrm{e}^{-\kappa(\tau-t)}}{\kappa} r \right\}.$$

Here $\Phi(t, \tau; \eta)$ *is the* $n \times n$ *matrix function defined by the ordinary differential equation*

$$\frac{\mathrm{d}\Phi(u, t; \eta)}{\mathrm{d}t} = \left(\Gamma' - (1 - \eta\mathrm{e}^{\kappa t})\Theta + \frac{1}{2}\left(\frac{1 - \eta\mathrm{e}^{\kappa t}}{\kappa}\right)^2 \Sigma^2 \right) \Phi(u, t; \eta)$$

with $\Phi(u, u; \eta) = 1$.

3 CIR-Type Short Rate

In this section we shall derive the price of a discount bond when the short rate is given by a CIR-type process with a mean-reverting level which follows a Markov chain. The procedure is similar to the Vasicek case. We first derive a conditional expectation of the bond price given the history of the chain and then take a second expectation. To calculate the conditional expectation, we apply the discussion of Elliott and van der Hoek (2001), who consider CIR dynamics without regime switching.

The price is given in the following proposition:

Proposition 3. *Suppose that the short rate process is given by the CIR dynamics*

$$\mathrm{d}r = \kappa(\langle \boldsymbol{\vartheta}, \mathbf{z}_t \rangle - r)\mathrm{d}t + \sigma\sqrt{r}\mathrm{d}w, \tag{3.1}$$

where $\mathbf{z}$ *follows* (2.2). *Then the price of a discount bond is given by*

$$p(t,\tau,r,\mathbf{z}) = \langle \Phi(t,\tau;\mathrm{e}^{\zeta(\tau-t)})\mathbf{z},\mathbf{1}\rangle \times \exp\left\{-\frac{2(\mathrm{e}^{\zeta(\tau-t)}-1)}{(\kappa+\zeta)(\mathrm{e}^{\zeta(\tau-t)}-1)+2\zeta}r\right\},$$

where $\zeta = \sqrt{\kappa^2+2\sigma^2}$ *and* $\Phi(t,\tau;\eta)$ *is the* $n\times n$ *matrix function defined by the ordinary differential equation*

$$\frac{\mathrm{d}\Phi(u,t;\eta)}{\mathrm{d}t} = \left(\Gamma' - \frac{2\kappa(\eta\mathrm{e}^{-\zeta t}-1)}{(\kappa+\zeta)(\eta\mathrm{e}^{-\zeta t}-1)+2\zeta}\Theta\right)\Phi(u,t;\eta) \tag{3.2}$$

with $\Phi(u,u;\eta) = I$.

Proof. With r given by (3.1), we have

$$\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\tau r_u \mathrm{d}u}\middle|\mathcal{F}_\tau^{\mathbf{z}}\right] = \exp\left\{-\kappa\int_0^\tau \langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle b(u,\tau)\mathrm{d}u - b(0,\tau)r\right\}, \tag{3.3}$$

where

$$b(t,\tau) = \frac{2(\mathrm{e}^{\zeta(\tau-t)}-1)}{(\kappa+\zeta)(\mathrm{e}^{\zeta(\tau-t)}-1)+2\zeta}.$$

See Appendix A for a proof. Therefore, conditioning out $\mathcal{F}_\tau^{\mathbf{z}}$ the price of a discount bond is:

$$p(0,\tau,r,\mathbf{z}) = \mathbb{E}\left[\exp\left\{-\kappa\int_0^\tau \langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle b(u,\tau)\mathrm{d}u\right\}\middle|\,\mathbf{z}_0=\mathbf{z}\right]\mathrm{e}^{-b(0,\tau)r}. \tag{3.4}$$

Define the process g^η by

$$g_t^h = \exp\left\{-\int_0^t \kappa\langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle\frac{2(\eta\mathrm{e}^{-\zeta u}-1)}{(\kappa+\zeta)(\eta\mathrm{e}^{-\zeta u}-1)+2\zeta}\mathrm{d}u\right\}$$

and the n-dimensional vector process $\mathbf{g}^\eta$ by

$$\mathbf{g}_t^\eta = g_t^\eta\mathbf{z}_t.$$

Then $\mathbf{g}^\eta$ satisfies

$$\mathrm{d}\mathbf{g}_t^\eta = -\kappa\langle\boldsymbol{\vartheta},\mathbf{z}_t\rangle\frac{2(\eta\mathrm{e}^{-\zeta t}-1)}{(\kappa+\zeta)(\eta\mathrm{e}^{-\zeta t}-1)+2\zeta}\mathbf{g}_t^\eta\mathrm{d}t + g^\eta(\Gamma'\mathbf{z}_t\mathrm{d}t+\mathrm{d}\mathbf{m}_t).$$

That is,

$$\mathbf{g}_t^\eta = \int_0^t\left(\Gamma' - \frac{2\kappa(\eta\mathrm{e}^{-\zeta u}-1)}{(\kappa+\zeta)(\eta\mathrm{e}^{-\zeta u}-1)+2\zeta}\Theta\right)\mathbf{g}_u^\eta\mathrm{d}u + \int_0^t g_u^\eta\mathrm{d}\mathbf{m}_u. \tag{3.5}$$

Taking the expectation of (3.5), we obtain

$$\hat{\mathbf{g}}_t^\eta = \int_0^t\left(\Gamma' - \frac{2\kappa(\eta\mathrm{e}^{-\zeta u}-1)}{(\kappa+\zeta)(\eta\mathrm{e}^{-\zeta u}-1)+2\zeta}\Theta\right)\hat{\mathbf{g}}_u^\eta\mathrm{d}u,$$

where $\hat{\mathbf{g}}_t^\eta = \mathbb{E}[\mathbf{g}_t^\eta | \mathbf{z}_0]$.

From a similar discussion to the Vasicek case, we have

$$\hat{\mathbf{g}}_\tau^\eta = \mathbb{E}_0[g_\tau^\eta \mathbf{z}_\tau] = \Phi(0,\tau;\eta)\mathbf{z}_0$$

where $\Phi(u,t;\eta)$ is defined by the ODE (3.2). We then obtain

$$\mathbb{E}\left[g_\tau^\eta \middle| \mathbf{z}_0 = \mathbf{z}\right] = \mathbb{E}\left[\langle \mathbf{g}_\tau^\eta, \mathbf{1}\rangle \middle| \mathbf{z}_0 = \mathbf{z}\right] = \langle \Phi(0,\tau;\mathrm{e}^{\zeta\tau})\mathbf{z}, \mathbf{1}\rangle$$

and the proof is completed. □

As in the Vasicek case, the price is equal to the weighted average of those in the constant case when Γ' and Θ commute. That is, if $\Gamma'\Theta = \Theta\Gamma'$, then the price is

$$p(t,\tau,r,\mathbf{z}) = \sum_{i=1}^{n} w_{\tau-t}^i(\mathbf{z}) p_{\mathrm{CIR}}(t,\tau,r;\theta^i).$$

Here p_{CIR} is the price in the standard CIR case and w_τ^i are defined by (2.11).

Note also that, unlike the Vasicek case, we cannot extend our proof of Proposition 3 to the case where other parameters are also modulated by a Markov chain.

4 Term Structure

In this section we give some numerical calculations and examine how the existence of regime shifts affects bond prices and term structure.

The basic parameter setting used in the analysis is presented in Table 1. Here we

Table 1: Basic parameter setting for the numerical calculations.

n	κ	σ	θ_1	θ_2	γ_{12}	γ_{21}
2	0.2	0.02	0.1	0.04	0.1	0.2

set $\theta_1 > \theta_2$, meaning that regime 1 represents an economic boom while regime 2 is a recession. To match actual situations, we set $\gamma_{12} > \gamma_{21}$. In this parameter setting, the long-run mean rate is equal to 8%.

Figure 1 depicts the bond yield up to ten years when the short rate has Vasicek-type dynamics for totally six cases: $r = 0.02$, 0.06, 0.12 and $\mathbf{z} = \mathbf{e}_1$, $\mathbf{e}_2$.

[Figure 1 is inserted here.]

We observe from the figure that even when the initial interest rate is the same level, the ten-year bond yield varies, depending on the current regime. For example, when $r = 0.02$, the ten-year zero rate in the case $\mathbf{z} = \mathbf{e}_1$ is 5.64% while that in the case of $\mathbf{e}_2$ is 4.23%. Note also that in the case $r = 0.06$, the shape of the term structure changes significantly when the regime changes.

Figure 2 depicts the bond yield up to ten years when the short rate has CIR-type dynamics for the six cases.

[Figure 2 is inserted here.]

We also have a similar observation that the effect of the current regime on the price and yield is non-negligible. For example, the ten-year zero rate in the case of regime 1 is 5.82% while that in the case of regime 2 is 4.42%.

In summary, we conclude from the numerical results that the regime structure can have a significant impact on the bond price and the term structure.

5 Option Pricing

This section presents how to calculate the price of derivatives for the case of Vasicek-type dynamics. The key step again is an initial conditional expectation given the information of the chain up to maturity.

Consider a European call option on a discount bond with maturities of the option and the underlying bond t and τ, respectively. The price of a call option at time 0 is given by

$$c(0,t,\tau,r,\mathbf{z}) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t r_u \mathrm{d}u} \{p(t,\tau,r_t,\mathbf{z}_t) - k\}_+ \middle| r_0 = r, \mathbf{z}_0 = \mathbf{z} \right], \tag{5.1}$$

where k is the strike price.

To calculate (5.1), we firstly derive the conditional expectation given the full history of the chain, that is, given $\mathcal{F}_\tau^{\mathbf{z}}$. Let

$$\begin{aligned}
&\tilde{p}(0,\tau,r,\{\mathbf{z}\}) = \mathbb{E}_0\left[\mathrm{e}^{-\int_0^\tau r_u \mathrm{d}u} \middle| \mathcal{F}_\tau^{\mathbf{z}} \right] \\
&= \exp\left\{ -\frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa} r + \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2 \left(\tau - \frac{2(1-\mathrm{e}^{-\kappa\tau})}{\kappa} + \frac{1-\mathrm{e}^{-2\kappa\tau}}{2\kappa} \right) \right. \\
&\quad \left. - \int_0^\tau \left(1 - \mathrm{e}^{-\kappa(\tau-u)}\right) \langle \boldsymbol{\vartheta}, \mathbf{z}_u \rangle \mathrm{d}u \right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}_0\left[\mathrm{e}^{-\int_0^t r_u \mathrm{d}u} \max\{p(t,\tau) - k, 0\} \middle| \mathcal{F}_\tau^{\mathbf{z}} \right] \\
&= \tilde{p}(0,\tau,r,\{\mathbf{z}\}) N(\tilde{d}(0,t,\tau)) - \tilde{p}(0,t,r,\{\mathbf{z}\}) k N(\tilde{d}(0,t,\tau) - \sigma_p),
\end{aligned} \tag{5.2}$$

where N is a cumulative density function of a standard normal,

$$\begin{aligned}\tilde{d}(0,t,\tau) =&\frac{1}{\sigma_p}\log\left(\frac{\tilde{p}(0,\tau,r,\{\mathbf{z}\})}{\tilde{p}(0,t,r,\{\mathbf{z}\})}\right)+\frac{\sigma_p}{2}\\ =&\frac{1}{\sigma_p}\left(\frac{\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{-\kappa t}}{\kappa}r+\frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2\left((\tau-t)+\frac{2(\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{-\kappa t})}{\kappa}-\frac{\mathrm{e}^{-2\kappa\tau}-\mathrm{e}^{-2\kappa t}}{2\kappa}\right)\right.\\ &\left.\int_0^t\left\{1_{\{0\le u<t\}}(\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{-\kappa t})\mathrm{e}^{\kappa u}+1_{\{u\ge t\}}(1-\mathrm{e}^{-\kappa(\tau-u)})\right\}\langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle\mathrm{d}u\right)+\frac{\sigma_p}{2}\end{aligned} \tag{5.3}$$

and

$$\sigma_p^2=\frac{\sigma^2}{\kappa^2}\int_0^t\left(\mathrm{e}^{-\kappa(t-u)}-\mathrm{e}^{-\kappa(\tau-u)}\right)^2\mathrm{d}u.$$

See, for example, Section 9.6 of Elliott and Kopp (2005) for more discussions of the derivation. Therefore the price of the option is expressed as

$$c(0,t,\tau,r,\mathbf{z})=\mathbb{E}_0[\tilde{p}(0,\tau,r,\{\mathbf{z}\})N(\tilde{d}(0,t,\tau))]-\mathbb{E}_0[\tilde{p}(0,t,r,\{\mathbf{z}\})kN(\tilde{d}(0,t,\tau)-\sigma_p)]. \tag{5.4}$$

Consider the first term of (5.4). It can be written as

$$\begin{aligned}\mathbb{E}_0[\tilde{p}(0,\tau,r_0,\{\mathbf{z}\})\Phi(\tilde{d}(0,t,\tau))] =&\mathbb{E}_0[\tilde{p}(0,\tau,r_0,\{\mathbf{z}\})]\mathbb{E}_0\left[\frac{\tilde{p}(0,\tau,r_0,\{\mathbf{z}\})}{\mathbb{E}_0[\tilde{p}(0,\tau,r_0,\{\mathbf{z}\})]}N(\tilde{d}(0,t,\tau))\right]\\ =&p(0,\tau,r_0,\mathbf{z}_0)\mathbb{E}_0^\tau[N(\tilde{d}(0,t,\tau))]\end{aligned}$$

where the probability measure $\mathbb{P}_0^\tau$ is defined by

$$\begin{aligned}\mathbb{P}_0^\tau(A) =&\mathbb{E}_0\left[\frac{\tilde{p}(0,\tau,r_0,\{\mathbf{z}\})}{\mathbb{E}_0[\tilde{p}(0,\tau,r_0,\{\mathbf{z}\})]}1_A\right]=\mathbb{E}_0\left[\frac{\mathrm{e}^{-\int_0^\tau(1-\mathrm{e}^{-\kappa(\tau-u)})\langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle\mathrm{d}u}}{\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\tau(1-\mathrm{e}^{-\kappa(\tau-u)})\langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle\mathrm{d}u}\right]}1_A\right]\\ =&\frac{\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\tau(1-\mathrm{e}^{-\kappa(\tau-u)})\langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle\mathrm{d}u}\times 1_A\right]}{\langle\Phi(0,\tau;\mathrm{e}^{-\kappa\tau})\mathbf{z}_0,\mathbf{1}\rangle}\end{aligned} \tag{5.5}$$

for $A\in\mathcal{G}_\tau$. Equation (5.5) with (5.3) implies that we can calculate the option price if the density function under $\mathbb{P}_0^\tau$ of

$$\int_0^\tau\left(1_{\{0\le u<t\}}(\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{-\kappa t})\mathrm{e}^{\kappa u}+1_{\{u\ge t\}}(1-\mathrm{e}^{-\kappa(\tau-u)})\right)\langle\boldsymbol{\vartheta},\mathbf{z}_u\rangle\mathrm{d}u \tag{5.6}$$

is obtained.

Write

$$\zeta_u=1_{\{0\le u<t\}}(\mathrm{e}^{-\kappa\tau}-\mathrm{e}^{-\kappa t})\mathrm{e}^{\kappa u}+1_{\{u\ge t\}}(1-\mathrm{e}^{-\kappa(\tau-u)}).$$

Since (5.6) is

$$\int_0^\tau \langle \boldsymbol{\vartheta}, \zeta_u \mathbf{z}_u \rangle \mathrm{d}u,$$

its distribution is determined if we can find the characteristic function of the n-dimensional random vector

$$\mathbf{v}_\tau = \int_0^\tau \zeta_u \mathbf{z}_u \mathrm{d}u = \left(\int_0^\tau \zeta_u 1_{\{\theta_u = \theta^1\}} \mathrm{d}u \quad \ldots \quad \int_0^\tau \zeta_u 1_{\{\theta_u = \theta^n\}} \mathrm{d}u \right).$$

The characteristic function of $\mathbf{v}_\tau$ under $\mathbb{P}_0^\tau$ is given by

$$\begin{aligned}
\psi^{\mathbf{v}_\tau}(\boldsymbol{\xi}) =& \mathbb{E}_0^\tau[\mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi}, \mathbf{v}_\tau \rangle}] = \frac{\mathbb{E}_0 \left[\mathrm{e}^{-\int_0^\tau (1-\mathrm{e}^{-\kappa(\tau-u)}) \langle \boldsymbol{\vartheta}, \mathbf{z}_u \rangle \mathrm{d}u + \mathrm{i}\langle \boldsymbol{\xi}, \mathbf{v}_\tau \rangle} \right]}{\langle \Phi(0,\tau)\mathbf{z}_0, \mathbf{1} \rangle} \\
=& \frac{\mathbb{E}_0 \left[\exp\left\{ -\int_0^\tau (1-\mathrm{e}^{-\kappa(\tau-u)}) \langle \boldsymbol{\vartheta}, \mathbf{z}_u \rangle \mathrm{d}u + \mathrm{i} \int_0^\tau \langle \boldsymbol{\xi}, \zeta_u \mathbf{z}_u \rangle \mathrm{d}u \right\} \right]}{\langle \Phi(0,\tau)\mathbf{z}_0, \mathbf{1} \rangle} \\
=& \frac{\mathbb{E}_0 \left[\exp\left\{ -\int_0^\tau \langle (1-\mathrm{e}^{-\kappa(\tau-u)}) \boldsymbol{\vartheta} + \mathrm{i}\zeta_u \boldsymbol{\xi}, \mathbf{z}_u \rangle \mathrm{d}u \right\} \right]}{\langle \Phi(0,\tau)\mathbf{z}_0, \mathbf{1} \rangle}.
\end{aligned} \tag{5.7}$$

To calculate the numerator of (5.7), consider the process

$$g_t^\eta = \exp\left\{ -\int_0^t \langle (1-\eta \mathrm{e}^{\kappa t}) \boldsymbol{\vartheta} + \mathrm{i}\zeta_u \boldsymbol{\xi}, \mathbf{z}_u \rangle \mathrm{d}u \right\}$$

for any arbitrary constant η and the vector process

$$\mathbf{g}_t^\eta = g_t^\eta \mathbf{z}_t.$$

The SDE satisfied by $\mathbf{g}_t^\eta$ is

$$\begin{aligned}
\mathrm{d}\mathbf{g}_t^\eta =& \mathbf{z}_t \mathrm{d}g_t^\eta + g_t^\eta \mathrm{d}\mathbf{z}_t = \mathbf{z}_t g_t^\eta \left\langle -(1-\eta \mathrm{e}^{\kappa t}) \boldsymbol{\vartheta} + \mathrm{i}\zeta_t \boldsymbol{\xi}, \mathbf{z}_t \right\rangle \mathrm{d}t + g_t^\eta (\Gamma' \mathbf{z}_t \mathrm{d}t + \mathrm{d}\mathbf{m}_t) \\
=& \left(\Gamma' - (1-\eta \mathrm{e}^{\kappa t})\Theta + \mathrm{i}\zeta_t \Xi \right) \mathbf{g}_t^\eta \mathrm{d}t + g_t^\eta \mathrm{d}\mathbf{m}_t
\end{aligned}$$

where $\Xi = \mathrm{diag}[\boldsymbol{\xi}]$.

Suppose $\Psi(u,t;\eta)$ is the $n \times n$ complex matrix function defined by

$$\frac{\mathrm{d}\Psi(u,t;\eta)}{\mathrm{d}t} = \left(\Gamma' - (1-\eta \mathrm{e}^{\kappa t})\Theta + \mathrm{i}\zeta_t \Xi \right) \Psi(u,t;\eta)$$

with $\Psi(u,u;\eta) = I$. We have from a similar discussion to the derivation of (2.10) that $\hat{\mathbf{g}}_t^\eta = \mathbb{E}_0[\mathbf{g}_t^\eta]$ is given by

$$\hat{\mathbf{g}}_t^\eta = \Psi(0,t;\eta)\mathbf{z}_0.$$

So

$$\mathbb{E}_0[g_t^\eta] = \mathbb{E}_0[\langle \mathbf{g}_t^\eta, \mathbf{1}\rangle] = \langle \Psi(0,t;\eta)\mathbf{z}_0, \mathbf{1}\rangle.$$

In summary, the characteristic function of $\mathbf{v}_\tau$ under $\mathbb{P}_0^\tau$ is

$$\psi^{\mathbf{v}_\tau}(\xi) = \frac{\langle \Psi(0,\tau;\mathrm{e}^{-\kappa\tau})\mathbf{z}_0, \mathbf{1}\rangle}{\langle \Phi(0,\tau;\mathrm{e}^{-\kappa\tau})\mathbf{z}_0, \mathbf{1}\rangle}.$$

For the second term of (5.4), we can calculate the characteristic function of $\mathbf{v}_\tau$ under the probability measure $\mathbb{P}_0^t$ in a similar way. The ODE system can be solved numerically and so we finally obtain the call option on a discount bond in the case of regime switching Vasicek dynamics

6 Conclusions

In this paper we have obtained the price of discount bonds when the mean-reverting level of the short rate follows a Markov chain. The pricing formula includes a solution of a simple linear matrix ODE, which is easy to handle numerically.

Our model has the advantage that it can capture economic cycles observed in the economy, while the tractability still remains. The methodology is also applicable to pricing of CDS and other derivative securities which depend on the economic conditions. This will be discussed in later work.

A Derivation of (3.3)

Following Elliott and van der Hoek (2001), we use the notation $r_{st}(r)$ to denote that the spot rate at time t depends on $r_s = r$, the spot price at s for $s \le t$. In other words, we have

$$r_{st}(r) = r + \kappa \int_s^t (\theta_t - r_{su}(r))\mathrm{d}u + \sigma \int_s^t \sqrt{r_{su}(r)}\mathrm{d}w_u. \tag{A.1}$$

Suppose that we have the history of θ up to maturity, or $\mathcal{F}_\tau^{\mathbf{z}}$. Using the theory of stochastic flows, we can differentiate (A.1) to obtain

$$\mathcal{D}_{st} \equiv \frac{\partial r_{st}(r)}{\partial r} = 1 - \kappa \int_s^t \mathcal{D}_{su}\mathrm{d}u + \frac{\sigma}{2}\int_s^t \frac{\mathcal{D}_{su}}{\sqrt{r_{su}(r)}}\mathrm{d}w_u. \tag{A.2}$$

Now consider the price of a discount bond

$$p(t,\tau,r) = \mathbb{E}_t\left[\exp\left\{-\int_t^\tau r_{tu}(r)\mathrm{d}u\right\}\right], \tag{A.3}$$

where the expectation operator is conditional on $\mathcal{G}_t$. Differentiating (A.3) with respect to r gives us

$$\frac{\partial p(t,\tau,r)}{\partial r} = \mathbb{E}_t\left[\left(-\int_t^\tau \frac{\partial r_{tu}(r)}{\partial r}\mathrm{d}u\right)\exp\left\{-\int_t^\tau r_{tu}(r)\mathrm{d}u\right\}\right]. \tag{A.4}$$

Recall the forward measure $\mathbb{P}^\tau$ is defined by

$$\frac{\mathrm{d}\mathbb{P}^\tau}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{-\int_0^\tau r_{0u}(r)\mathrm{d}u}}{\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\tau r_{0u}(r)\mathrm{d}u}\right]} = \frac{\mathrm{e}^{-\int_0^\tau r_{0u}(r)\mathrm{d}u}}{p(0,\tau)}.$$

Then (A.4) can be written as

$$\frac{\partial p(t,\tau,r)}{\partial r} = -\mathbb{E}_t^\tau\left[\int_t^\tau \mathcal{D}_{tu}\mathrm{d}u\right]p(t,\tau,r) = -\left(\int_t^\tau \hat{\mathcal{D}}_{tu}\mathrm{d}u\right)p(t,\tau,r),$$

where $\mathbb{E}_t^\tau[\mathcal{D}_{tu}] = \hat{\mathcal{D}}_{tu}$. Integrating in r, we see

$$p(t,\tau,r) = \exp\{a(t,\tau) - b(t,\tau)r\} \tag{A.5}$$

for some functions $a(t,\tau)$ and $b(t,\tau) = \int_t^\tau \hat{\mathcal{D}}_{tu}\mathrm{d}u$.

We now determine the function $\hat{\mathcal{D}}_{tu}$. It follows from Girsanov's theorem that the Brownian motion w^τ under $\mathbb{P}^\tau$ is written with w, the Brownian motion under $\mathbb{P}$, as

$$w_t^\tau = w_t + \int_s^t\left(\int_{u_1}^\tau \hat{D}_{u_1u_2}\mathrm{d}u_t\right)\sigma\sqrt{r_{su_1}(r)}\mathrm{d}u_1. \tag{A.6}$$

Substituting (A.6) into (A.2) yields

$$\mathcal{D}_{st} = 1 - \kappa\int_s^t \mathcal{D}_{su}\mathrm{d}u - \frac{\sigma^2}{2}\int_s^t \mathcal{D}_{su_1}\left(\int_{u_1}^\tau \hat{\mathcal{D}}_{u_1u_2}\mathrm{d}u_2\right)\mathrm{d}u_1 + \frac{\sigma}{2}\int_s^t \frac{\mathcal{D}_{su}}{\sqrt{r_{su}(r)}}\mathrm{d}w_u^\tau.$$

Taking the expectation under $\mathbb{P}_s^\tau$, we obtain

$$\hat{\mathcal{D}}_{st} = 1 - \kappa\int_s^t \hat{\mathcal{D}}_{su}\mathrm{d}u - \frac{\sigma^2}{2}\int_s^t \hat{\mathcal{D}}_{su_1}\left(\int_{u_1}^\tau \mathbb{E}_s^\tau[\hat{\mathcal{D}}_{u_1u_2}]\mathrm{d}u_2\right)\mathrm{d}u_1. \tag{A.7}$$

Recall that the stochastic flow satisfies the property $r_{su_2}(r) = r_{u_1u_2}(r_{su_1}(r))$. Thus we have

$$\mathcal{D}_{su_2} = \frac{\partial r_{u_1u_2}(r_{su_1}(r))}{\partial r} = \frac{\partial r_{u_1u_2}(r_{su_1}(r))}{\partial r_{su_1}}\frac{\partial r_{su_1}(r_{su_1}(r))}{\partial r} = \mathcal{D}_{u_1u_2}\mathcal{D}_{su_1}.$$

and so

$$\mathbb{E}_s^\tau[\mathcal{D}_{su_1}\hat{\mathcal{D}}_{u_1u_2}] = \mathbb{E}_s^\tau[\mathcal{D}_{su_1}\mathbb{E}_{u_1}^\tau[\mathcal{D}_{u_1u_2}]] = \mathbb{E}_s^\tau[\mathcal{D}_{su_1}\mathcal{D}_{u_1u_2}] = \hat{\mathcal{D}}_{su_2}. \tag{A.8}$$

Substituting (A.8) into (A.7), we obtain

$$\hat{\mathcal{D}}_{st} = 1 - \kappa\int_s^t \hat{\mathcal{D}}_{su}\mathrm{d}u - \frac{\sigma^2}{2}\int_s^t\int_{u_1}^\tau \hat{\mathcal{D}}_{su_2}\mathrm{d}u_2\mathrm{d}u_1. \tag{A.9}$$

Differentiating (A.9) twice with respect to t gives us

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{\mathcal{D}}_{st} = -\kappa\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{D}}_{st} + \frac{\sigma^2}{2}\hat{\mathcal{D}}_{st}. \tag{A.10}$$

Noticing that the initial conditions are $\hat{\mathcal{D}}_{ss} = 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{D}}_{st}\bigg|_{t=s} = -1 - \frac{\sigma^2}{2}\int_s^\tau \hat{\mathcal{D}}_{su}\mathrm{d}u,$$

the solution of (A.10) is given by

$$\hat{\mathcal{D}}_{st} = a\mathrm{e}^{\alpha_+(t-s)} + (1-a)\mathrm{e}^{\alpha_-(t-s)},$$

where α_1 and α, $\alpha_1 > \alpha_2$, are the roots of the equation $z^2 + \kappa z - \sigma^2/2 = 0$. Write

$$a = -\frac{\alpha_2 + \kappa + \frac{\sigma^2}{2}\frac{\mathrm{e}^{\alpha_2(\tau-s)}-1}{\alpha_-}}{\alpha_1 - \alpha_2 + \frac{\sigma^2}{2}\left(\frac{\mathrm{e}^{\alpha_1(\tau-s)}-1}{\alpha_1} - \frac{\mathrm{e}^{\alpha_2(\tau-s)}-1}{\alpha_2}\right)}.$$

Then

$$b(t,\tau) = \int_t^\tau \hat{\mathcal{D}}_{tu}\mathrm{d}u = \frac{2(\mathrm{e}^{\zeta(\tau-t)}-1)}{(\kappa+\zeta)\mathrm{e}^{\zeta(\tau-t)} - 2\zeta}.$$

where $\zeta = \sqrt{\kappa^2 + 2\sigma^2}$.

Now recall that the bond price $p(t,\tau,r)$ with short rate process (3.1) satisfies

$$\frac{\partial p}{\partial t} + \kappa(\theta_t - r)\frac{\partial p}{\partial r} + \frac{\sigma^2}{2}r\frac{\partial^2 p}{\partial r^2} - rp = 0 \tag{A.11}$$

with $P(\tau,\tau,r) = 1$. It follows after substituting (A.5) into (A.11) that for any r,

$$\left(\frac{\partial a}{\partial t} - \frac{\partial b}{\partial t}r\right) - r - \kappa(\theta_t - r)b + \frac{\sigma^2}{2}rb^2 = 0.$$

Setting $r = 0$ yields

$$\frac{\partial a}{\partial t} = \kappa\theta_t b(t,\tau),$$

implying from the boundary condition that

$$a(t,T) = -\kappa\int_t^\tau \theta_u b(u,\tau)\mathrm{d}u.$$

Hull and White (1990) and Maghsoodi (1996) also derive a similar formula for the case where the coefficients are deterministic functions of time using a different approach.

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Figures

Figure 1: Term structure in the Vasicek-type.

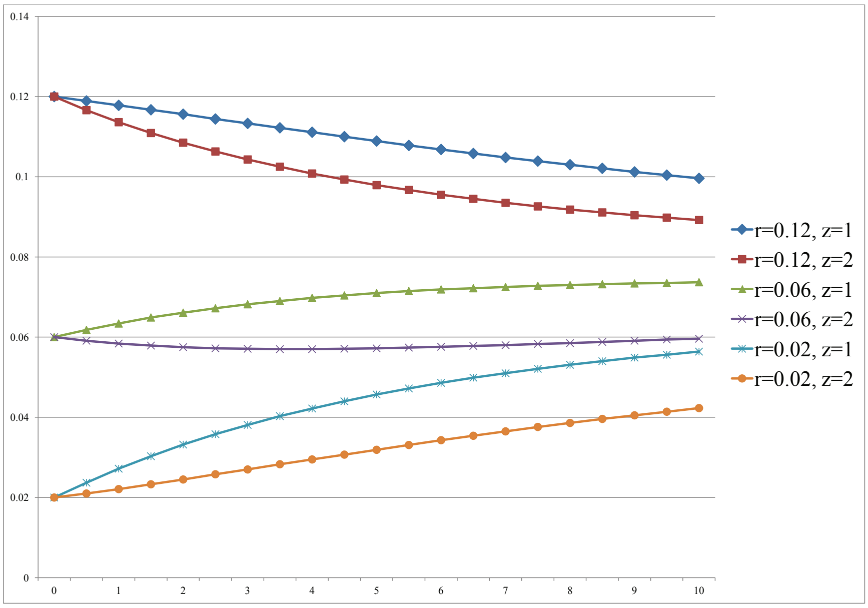

Figure 2: Term structure in the CIR-type.