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"Inequalities and Patience for Tomorrow"

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Inequalities and Patience for Tomorrow

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Abstract

This paper examines how impatience interacts with inequalities in economic development. In a society of intrinsic inequality, we show that (i) poor households tend to benefit more from positive shocks under *decreasing* marginal impatience (DMI) than under *constant* marginal impatience (CMI) and *increasing* marginal impatience (IMI); (ii) an unequal society may be preferable for poor households under DMI; (iii) urbanization can increase the income inequality, while raising overall welfare; (iv) under DMI even if all households are allowed to own assets, with different initial asset holdings, the economy will not converge to the steady state where everybody is a capitalist.

1 Introduction

In fast growing economies such as China, urban and coastal regions experience rapid growth, drastically increasing the income gap between these regions and remote areas. In 2009, the annual per-capita income in China was about $750 for rural residents, while that for urban residents was about $2500. According to China Daily (May 23, 2012), the most affluent 10 percent of the population makes 23 times more than the poorest 10 percent.[1] Such income inequality has caused workers to migrate in large scale from rural to urban areas and from inner to coastal regions. Furthermore, even within city limits, the income inequality has been rising rapidly, especially following the influx of workers from rural areas. Meanwhile, many newly constructed roads, bridges, railways, buildings and even food products are of poor or even hazardous quality, causing fatal accidents; Air and water pollution has reached unbearable

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[1] The inequality in India is slightly better. In the same year, the annual per-capita income in India was about $750 for rural but $1500 for urban residents.

levels. Public morality seems to be in a landslide, culminated by a recent death (October, 2011) of a 2-year-old girl, run over twice by vehicles and subsequently ignored by 18 cyclists and other passers-by (A rag collector eventually came to her aid). These phenomena have generated heated debates in the media, among policy makers and researchers alike. There are soul searching cries in the popular press that China in its rush to modernity should slow down the pace, in order to decrease man-made errors and potential disasters, to reduce ever increasing pollution, and to make more efficient use of its depleting resources. It is also argued that to sustain growth in the longer run, the government must adopt more patient policies and reduce income and other inequalities.[2]

Similar to the popular press, in the academic literature quite a number of empirical studies find strong evidence that households are heterogeneous in terms of impatience (e.g., they discount the future at different rates), and that preference heterogeneity is an important factor in explaining household income inequality, see for instance, Hausman (1979), Becker and Mulligan (1997), Samwick (1997), and Barsky et al. (1997). Lawrance (1991) and Warner and Pleeter (2001) find that more-educated households and individuals tend to have lower discount rates than less-educated ones, thus heterogenous time preference may lead to income and wealth inequality through long-term investment and human capital accumulation. In fact, some studies have found that the marginal propensity to save is considerably higher among wealthier people.[3]

The present paper is motivated by the above heated debates and interesting empirical findings. We consider a society that does not have "equal opportunity" to begin with, as is a fact in many developing countries with strong traditional systems (e.g., some Latin American countries, China and India, etc.).[4] Specifically, there exist two types of households that are symmetric in all aspects except that one type owns asset (e.g., capitalists), while the other type (e.g., workers) is unable to own asset and hence consume all income at each point in time. We examine how such inequality evolves under globalization, in particular faced with technology improvements (such as the telecommunication and Internet revolution in the early 1990s), capital market inefficiency, demand and other shocks. We are especially interested in transition economies where impatience can play important roles during the catch-up process to modernization.

An important phenomenon we examine is the so-called "decreasing marginal impatience (DMI)", under which rich households save and invest more when they become richer. The patience of rich households enables them to invest more and accumulate more capital, generating a trickle-down effect to the poor in the long run. This mechanism thus increases the capital stock, the productivity of workers, and the welfare of all households including the poor when

[2]The above phonomenon is closely related to the so-called "curse to the late comer", who in the rush to become rich picks the easier way out by copying mature technologies and buying assembly lines from rich countries, instead of taking the more difficult route of gradually developing institutional and political systems conducive for innovation and sustainable growth. As such, current growth may be fast, but it is very costly and short-lived, and inevitably it comes with various kinds of social inqualities due to the lack of a fair institution or system.

[3]Frederick et al. (2002) provides a detailed review of this literature.

[4]For instance, although slowly being relaxed, the infamous family registration (*hukou*) system in China determines whether one is a peasant, urban worker, or cadre, etc. at birth (!) following the mother. And India is wellknown for the Caste system, which has largely broken down in cities, but persists in rural areas where 72% of India's population resides. Several Latin American countries have the highest Gini index in the world, and such inqualities have been carried over from generation to generation, some even from the colonial period (Ferreira et al., 2004). As such the top 10% population is estimated to own 50% of the total income while the bottom 10% owns only 1.5%, compared to 30% and 2.5% for corresponding groups in developed countries.

the rich becomes richer. Hence, inequality as a side-effect on a country's catching-up path may not be so bad after all. In fact, we show that when the fraction of rich residents is sufficiently small, subsidizing the rich and taxing the poor raises poor households' welfare in the steady state. These results match the experiences of many developing countries, in whose early years of development, widespread subsidies are provided to business owners, such as those applied in Special Economic Zones (SEZ) where tax holidays, export, import and land subsidies are common.

However, the income gap between the rich and the poor may increase under certain conditions, especially in the short run. Such scenarios may justify government policies such as taxes to redistribute income from the rich to the poor, to keep the income gap within boundaries, otherwise social instability may arise.

A particularly interesting finding arises under urbanization, which causes cities to annex neighboring rural areas and the poor residents there. Such labor migration increases the productivity of capital and the income of the rich, which under DMI induces more capital accumulation and makes all households better off in the long run, although it widens the income inequality. This explains very intuitively and matches the fact that while fast growing economies such as China and India are becoming richer on one hand, their income inequality rises on the other hand.

Next, either a positive productivity shock or an increase in capital market efficiency always raises the income gap, because rich households benefit in more ways or more directly from such shocks while poor households only benefit through changes in the wage rate. This is consistent with Acemoglu (2002), who studies the impacts of skilled labor-biased technology improvement and finds it to be a major cause for income inequality in the twenty's century. Nevertheless, in our model, *poor* households tend to benefit more under DMI than CMI (constant marginal impatience), because the former of which creates a long-run trickle down effect that is absent under CMI.

However, DMI can lead to counter-intuitive and discouraging results: the welfare of an economy where everyone owns capital is lower than if only some people own capital! The logic is, the lower the share of capitalists in the economy, the more they save and the more capital the country accumulates in the steady state. Hence, a country with higher inequality accumulates higher capital stock and enjoys higher levels of welfare, ceteris paribus. On the contrary, if the same amount of capital stock is owned by more capitalists, each capitalist saves less and the steady-state welfare is lowered.

Moreover, even if both types of households own asset to begin with, the economy will not converge to the steady state where all households have some positive level of asset (i.e., everyone becoming a capitalist), as long as their initial asset holdings differ. Thus, asymmetry of households may be a natural consequence of endogenous time preferences with DMI.[5]

Finally on an international scale, patience may impact rich and poor countries differently in terms of income inequality. Developing countries use inferior technology and their capital markets are less efficient and less complete, leading to lower income. An increase in patience may raise the income inequality because rich households tend to gain more asset income (higher interest rates) when capital markets are less complete. Examples are again China and India, where savings and investment are high, but the income inequality is increasing rapidly.[6] In

[5] In contrast, under IMI the economy converges to the steady state where all households have the same level of asset, even when their initial asset holdings differ. See also Epstein (1987).

[6] According to Li, Wei and Jing (2005), the Gini coefficient of wealth inequality in China increased from 0.40 in 1995 to 0.55 in 2002. For India, Bardhan (2006) states that "in the 90's Indian wealth distribution was much

contrast in developed countries, the income inequality may fall when households become more patient, for which case Luxumburg and Switzerland are perhaps good examples.[7]

In the theoretical literature, Krusell and Smith (1998) demonstrate that introducing time preference heterogeneity can significantly improve the Aiyagari (1994) model in explaining income inequality, and Hendricks (2007) incorporates preference heterogeneity into the life-cycle model of Huggett (1996) to account for wealth inequality. Also, Epstein (1987), Das (2003), Hirose and Ikeda (2008) and Chang (2009) investigate equilibrium stability and uniqueness issues under DMI. In Hirose and Ikeda (2012), the Harberger-Laursen-Metzler effect is examined, where at least one country has IMI in order to obtain saddle-point stability.

In contrast, the present paper focuses on the effects of consumer preference and patience and total factor productivity. The interactions of two types of households generate interesting results that are novel in the literature. Our conclusions imply that to attain sustainable economic growth under DMI, developing countries on one hand can allow freer labor migration, improve the efficiency of capital markets, and even educate households to be more patient; on the other hand, they may levy income taxes and adopt lump-sum transfers to reduce inequality across households. However, caution is needed since income transfers may hamper capital accumulation, and eventually lower the steady-state level of welfare for all households.

2 The Basic Model

We consider an economy with two types of households, which are symmetric in all aspects except that one type owns asset, while the other type is unable to own asset for some reason.[8] That is, there exists intrinsic social inequality in this economy to begin with, as in many developing countries with strong traditional systems. Our goal is to examine the relationship between inequality and economic growth and how social inequality evolves in transitional and fast-growing economies such as the so-called BRIC countries (Brazil, Russia, India and China). In the process of catching up, such economies inevitably face technology improvement, capital market inefficiency, labor migration, demand and other shocks. We analyze these issues and government policies including tax reform that might be used to mitigate the existing and possibly widening inequality.

There is only one good that can be consumed and saved as capital, whose output is given by

$$Y = F(K, L),$$

where K and L are respectively capital stock and labor supply. Production exhibits constant returns to scale technology, so that we have

$$k \equiv \frac{K}{L} \quad \text{and} \quad f(k) \equiv F\left(\frac{K}{L}, 1\right).$$

more unequal than that in China."(p16) In addition, there exists more severe educational inequality in India.

[7]A note of caution is in order though. The above results are obtained in the absence of any international credit market. If however rich households can invest in foreign markets or assets, then the result that the welfare of poor households increases when the rich becomes richer should be weakened.

[8]In Appendix 2, this is relaxed and we show that (i) even if all households are allowed to own assets with different initial asset holdings, under DMI the economy will not converge to the steady state where everybody is a capitalist; (ii) the steady state we shall examine in the main text is the same as the one when the borrowing constraint is binding only for one type of households.

Then, the capital rental rate R and the wage rate w are respectively

$$R = f'(k) \quad \text{and} \quad w = f(k) - kf'(k).$$

The household's inelastic labor endowment (and supply) is normalized to one, so that the number of households is denoted by L, and k is the capital stock per household or simply capital stock.

2.1 The Capital Market

Poor households consume all their current income so they do not participate in the capital market. In contrast, rich households save a portion of their income as assets. Historically, a major problem facing any economy during and post takeoff is how to accumulate capital to sustain economic development, because financial markets have not been developed and there are limits for entrepreneurs to borrow from relatives and friends. Thus, we assume there exists inefficiency in the capital market that lowers the return on investments, perhaps due to transaction costs of some sort,

$$r = e[R - \delta],$$

where r is the interest rate, $e \in (0, 1]$ can be interpreted as the efficiency of the capital market, and δ is the depreciation rate on capital. Then the income for rich households (with asset a) is given by $w + ra$, while that for poor households (without asset) is only w.

The government may levy a tax τ on the capital income of rich households (simply named 'the capital tax'), intended to reduce the income gap. The tax revenue is used as lump-sum transfer, T, to poor households. Let $\theta \in (0, 1]$ denote the exogenous share of households with asset.[9] The post-tax income for rich households is then

$$I = w + (1 - \tau)ra,$$

while that for poor households becomes

$$I^* = w + T,$$

for which the government budget constraint $\tau ra\theta = T(1-\theta)$ holds. Also, it is natural to assume the income of rich households to be higher than that of poor households, $I \geq I^*$, for which $\tau \leq 1 - \theta$ is required.

Each household owning asset maximizes the discounted sum of utility

$$\int_0^\infty u(c)X dt, \tag{1a}$$

subject to

$$\dot{a} = w + (1 - \tau)ra - c, \tag{1b}$$

$$\dot{X} = -\rho(c)X, \tag{1c}$$

where for $^\forall c > 0$, $u(c) > 0$; $u'(c) > 0$; $u''(c) < 0$. Also, $X \equiv \exp[-\int_0^t \rho(c)ds]$ is the discount factor at time t which depends on the past and present levels of consumption through the function $\rho(\cdot)$.

[9] In the case of $\theta = 1$, all households own the same level of asset.

Following Das (2003) and Chang (2009), we assume that household preference exhibits DMI (diminishing marginal impatience) as follows,[10]

$$\rho'(c) < 0 < \rho''(c) \text{ for } {}^{\forall}c > 0,\ \rho(0) < \infty,\ \text{and } \lim_{c\to\infty} \rho(c) = 0. \tag{2}$$

Intuitively, it says households with a higher income discount future less, since they can afford to defer consumption of additional income and wealth. As mentioned in the Introduction, this assumption is supported by several empirical studies, such as Lawrance (1991), Samwick (1997), and Barsky et al. (1997), etc. We are interested in how the degree of impatience affects the evolvement of inequalities in the process of catching up and modernization, especially for developing countries.

Further, expression (1c) implies the rate at which X decreases is $\rho(c)$. To be more precise, with an increase in consumption c, expression X decreases at a normal speed under CMI, but at a higher speed under IMI and a lower speed under DMI.

The Hamiltonian associated with our optimization problem is

$$\mathcal{H} \equiv u(c)X + \lambda[w + (1-\tau)ra - c] - \mu\rho(c)X,$$

and the necessary conditions for optimality are

$$\frac{\partial \mathcal{H}}{\partial c} = u'(c)\,X - \lambda - \mu\rho'(c)X = 0, \tag{3a}$$

$$\frac{\partial \mathcal{H}}{\partial a} = \lambda(1-\tau)r = -\dot{\lambda}, \tag{3b}$$

$$\frac{\partial \mathcal{H}}{\partial X} = u(c) - \mu\rho(c) = -\dot{\mu}. \tag{3c}$$

Let $Z \equiv \lambda/X$ to simplify notation. Then (3a) and (3b) can be rewritten as

$$Z = u'(c) - \mu\rho'(c),$$
$$\dot{Z} = Z[\rho(c) - (1-\tau)r].$$

Using the above, our dynamic general equilibrium system can be described as

$$\dot{a} = w(k) + (1-\tau)r(k)a - c, \tag{4a}$$

$$\dot{Z} = Z[\rho(c) - (1-\tau)r(k)], \tag{4b}$$

$$\dot{\mu} = \mu\rho(c) - u(c), \tag{4c}$$

$$0 = u'(c) - \mu\rho'(c) - Z, \tag{4d}$$

where $w(k) \equiv f(k) - kf'(k)$, $r(k) \equiv e[f'(k) - \delta]$, and $k = a\theta$ from the market clearing condition for asset, $a\theta L = K$.[11] Notice that $w'(k) = -kf''(k) > 0$, and $r'(k) = ef''(k) < 0$; that is,

[10] In the case of IMI, we follow Uzawa's assumption, i.e. $\rho(c) \equiv \chi(u(c))$ with $\chi(u) > 0$, $\chi'(u) > 0$, $\chi''(u) > 0$, $\chi(u) - \chi'(u)u > 0$ for all $u > 0$, and $\chi(u(0)) > 0$. See also Ikeda and Hirose (2008) for assumptions on u and ρ that can be applied to both DMI and IMI.

[11] Each household unable to own asset consumes all of his income at each point in time, and hence his consumption is $c^* = w(k) + \theta\tau r(k)a/(1-\theta)$. Then, the goods market clears when:

$$(c + \dot{a})\theta L + c^*(1-\theta)L + \delta K = wL + RK - [(1-e)(R-\delta)K],$$

where the last term on the right-hand side denotes the loss of output due to the inefficiency in the capital market.

when the capital stock rises, the wage rate increases but the interest rate decreases.[12] The interactions of these two opposite effects determine the changes of household income, which will become clear soon.

3 The Steady State

We define the steady state of the model as when all variables for households with asset, i.e., a (or k), Z, μ, and c, are constant, and the consumption of households without asset is also constant at $c^* = i^*(k, \theta, \tau)$, where

$$i^*(k, \theta, \tau) \equiv w(k) + \frac{\tau r(k) k}{1-\theta}.$$

Then, the steady state is a solution to the following system of equations[13]

$$0 = w(k) + \frac{1-\tau}{\theta} r(k) k - c, \tag{5a}$$

$$0 = Z[\rho(c) - (1-\tau) r(k)], \tag{5b}$$

$$0 = \mu \rho(c) - u(c), \tag{5c}$$

$$Z = u'(c) - \mu \rho'(c). \tag{5d}$$

These conditions say that at the steady state, consumption must be equal to the post-tax income (condition (5a)), the post-tax interest rate must be equal to the discount factor of households with asset (5b), utility is constant ($\dot{\mu} = 0$, (5c)), and (5d) equates the current value of the shadow price to the marginal-utility increase. In what follows, we use "$\sim$" to denote the steady state value of each variable.

More specifically, conditions (5a) and (5b) give the steady state solution pair $(\tilde{k}, \tilde{c})$, which can be rewritten as $c = i(k, \theta, \tau)$ and $k = \kappa(c, \tau)$, where

$$i(k, \theta, \tau) \equiv w(k) + \frac{1-\tau}{\theta} r(k) k,$$

$$\kappa(c, \tau) \equiv r^{-1}\left(\frac{\rho(c)}{1-\tau}\right).$$

When θ is small the income of rich households largely consists of asset income, and hence it declines when the interest rate falls due to capital accumulation. However, when e is small but τ is large, it depends also on the wage income and increases with capital accumulation. Indeed, we have

Lemma 1 *If $\theta < e_0 \equiv (1-\tau)e$, capital accumulation can reduce the income of households holding asset, even when the interest rate is positive:*

$$\frac{\partial i(k, \theta, \tau)}{\partial k} = \frac{(\theta - e_0) w'(k)}{\theta} + \frac{(1-\tau) r(k)}{\theta}. \tag{6}$$

[12] In standard models with $\theta = e = 1$ and $\tau = 0$, these two effects are exactly cancelled out, and hence households' income, given by $f(k) - \delta k$, always increases with capital accumulation, for any positive interest rate.

[13] Our model does not have a "satiated" steady state (Z being equal to zero) as discussed in Hirose and Ikeda (2008), since our assumption on u and ρ ensures that the steady state value of Z must be positive.

The first term on the right-hand side of (6) can be negative, because the wage rate increases but the interest rate decreases and the latter dominates if $\theta < e_0$. Then, capital accumulation may reduce the income of rich households, if the fraction of such households is too small. Indeed, with $\theta < e_0$ ($\theta > e_0$) the income of rich households is maximized at some level of capital stock that is lower (higher) than the level at which the capital rental is equal to the depreciation rate. In the case of $\theta = e_0 = 1$, it is the same as the level given by the Golden rule because it maximizes households' income. In contrast, capital accumulation increases the income of poor households, i^*, given $\tau \in [0, 1-\theta]$.

The capital stock that equates the interest rate to the discount factor, κ, is increasing in c due to DMI:

$$\frac{\partial \kappa(c,\tau)}{\partial c} = \frac{\rho'(c)}{(1-\tau)r'(k)} > 0.$$

Once the steady state $\tilde{c}$ is determined, we see from (5c) and (5d) that

$$\tilde{\mu} = \frac{u(\tilde{c})}{\rho(\tilde{c})} \quad \text{and} \quad \tilde{Z} = \frac{u'(\tilde{c})\,\rho(\tilde{c}) - u(\tilde{c})\,\rho'(\tilde{c})}{\rho(\tilde{c})},$$

where $\tilde{\mu}$ (increasing in $\tilde{c}$) can be interpreted as the steady state level of welfare in the sense that the discounted sum of utility, $\int_0^\infty u(c)X dt$, is equal to $\tilde{\mu}$ when $c(t) = \tilde{c}$ for $^\forall t \geq 0$.

Finally we examine the stability and uniqueness of the steady state. Let k_1 and k_2 be the values of the capital stock that equate the post-tax interest rate to $\rho(0)$ and zero respectively:

$$k_1 \equiv r^{-1}\left(\frac{\rho(0)}{1-\tau}\right) \quad \text{and} \quad k_2 \equiv r^{-1}(0), \tag{7}$$

where $k_1 < k_2$ holds.[14] Then, for any $\theta \in (0, 1]$,

$$\begin{aligned} i(k_1,\theta,\tau) &= w(k_1) + \frac{\rho(0)k_1}{\theta} > 0, \\ i(k_2,\theta,\tau) &= w(k_2) < \infty. \end{aligned}$$

It is then apparent from $\partial\kappa/\partial c > 0$, $k_1 = \kappa(0,\tau)$, and $k_2 = \lim_{c\to\infty} \kappa(c,\tau)$ that there necessarily exists an intersection of the two graphs, $c = i(k,\theta,\tau)$ and $k = \kappa(c,\tau)$, as in Figure 1.

In the rest of the paper, we assume[15]

Assumption 1: The intersection between $c = i(k,\theta,\tau)$ and $k = \kappa(c,\tau)$ is unique for any pair (θ,τ) with $\theta \in (0,1]$ and $\tau \leq 1-\theta$.

Remark 1 *This is true when i and κ are strictly concave respectively in k and c. An example is provided in Section 4 (see Assumptions 2 and 3 below).*

Under Assumption 1, at the intersection, the slope of $c = i(k,\theta,\tau)$, i.e., $\partial i/\partial k$, must be smaller than that of $c = \rho^{-1}((1-\tau)r(k))$, i.e., $(\partial\kappa/\partial c)^{-1}$, and hence the following inequality

[14] The Inada conditions: $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$ ensure the existence of k_1 and k_2.

[15] If Assumption 1 is violated, there may be an odd number of steady states. (8) holds at the first steady state, but fails at the second one, ..., and it also holds at the last one. One can verify from the proof of Lemma 2 (Appendix 9.1) that the steady states with (8) are saddle points, while the others are either unstable or indeterminacy arises around them. Assuming the uniqueness of the steady state, we will focus on a saddle point.

holds:[16]

$$\frac{(\theta - e_0)w'(k) + \rho(c)}{\theta}\rho'(c) > e_0 f''(k). \tag{8}$$

Combining this with the stability analysis in Appendix 9.1, we obtain:[17]

Lemma 2 *Under Assumption 1, the steady state of the dynamic system is a saddle point and unique.*

The above completes the basic setup of the model.

4 Technology, Credit Constraints and Demand Shocks

It is not uncommon for transition economies to experience technology and demand shocks as well as credit constraints, especially facing the current wave of globalization. In this section, we examine how technology advancement, capital market inefficiency, and demand shifts impact the steady state, in the presence of DMI. Precisely due to the impact of DMI, any change that makes households with asset richer also has a positive effect on the steady state income of households without asset, as we shall clearly demonstrate below.

Redefine

$$f(k) = A\hat{f}(k) \text{ for } {}^\forall k \text{ and } \rho(c) = B\hat{\rho}(c) \text{ for } {}^\forall c,$$

where $A > 0$ and $B > 0$ can be interpreted as respectively the total productivity and the level of impatience.

We now consider the effects of changes in A, B, and e on the steady state level of welfare for both types of households,[18] where changes in B can represent impatience changes that cause a demand shift of all households. Totally differentiating $c = i(k, \theta, \tau)$ and $k = \kappa(c, \tau)$ with respect to A, B, e_0, and θ to give

$$\begin{pmatrix} -\frac{e_0-\theta}{\theta}\tilde{k}f'' - \frac{\rho}{\theta} & 1 \\ e_0 f'' & -\rho' \end{pmatrix} \begin{pmatrix} dk \\ dc \end{pmatrix}$$
$$= \begin{pmatrix} \hat{f} + \frac{e_0-\theta}{\theta}\tilde{k}\hat{f}' & 0 & \frac{\rho\tilde{k}}{e_0\theta} & -\frac{\rho\tilde{k}}{\theta^2} \\ -e_0\hat{f}' & \hat{\rho} & -\frac{\rho}{e_0} & 0 \end{pmatrix} \begin{pmatrix} dA \\ dB \\ de_0 \\ d\theta \end{pmatrix},$$

where each element of the matrices is evaluated at the steady state. Then, we obtain:

[16] Condition (8) holds at any intersection and hence the steady state is unique, if the following is met:

$$\frac{1}{\rho(0)} \min_{k\in[k_1,k_2]}[-f''(k)] \geq -\frac{\rho'(0)}{e_0\theta} \max_{k\in[k_1,k_2]}\left[1 + (e_0-\theta)\frac{kf''(k)}{\rho(0)}\right].$$

This inequality degenerates to the bounded slope assumption in Chang (2009), when $e_0 = \theta = 1$. Then the steady state of his model (of endogenous time preference with DMI) is unique and is a saddle point.

[17] Notice that with IMI or CMI, the intersection also exists, and under Assumption 1, (8) holds at the unique intersection (Assumption 1 is redundant in the case of CMI). In both cases, one can easily verify that Lemma 2 remains valid.

[18] In the rest of the paper, we define the steady state level of welfare for households without asset as $u(\tilde{c}^*)/\rho(\tilde{c}^*)$, analogous to that for households with asset.

Lemma 3 *Under Assumption 1,*

$$\frac{\partial \tilde{k}}{\partial A} = \frac{-\theta\rho' f + [e_0\theta - (e_0-\theta)\rho'\tilde{k}]f'}{AD\theta} > 0, \tag{9a}$$

$$\frac{\partial \tilde{k}}{\partial B} = -\frac{\rho}{BD} < 0, \tag{9b}$$

$$\frac{\partial \tilde{k}}{\partial e_0} = \frac{\rho(\theta - \rho'\tilde{k})}{De_0\theta} > 0, \tag{9c}$$

$$\frac{\partial \tilde{k}}{\partial \theta} = \frac{\rho\rho'\tilde{k}}{D\theta^2} < 0, \tag{9d}$$

$$\frac{\partial \tilde{c}}{\partial A} = \frac{e_0(\rho f' - \theta f f'')}{AD\theta} > 0, \tag{9e}$$

$$\frac{\partial \tilde{c}}{\partial B} = -\frac{\rho[(e_0-\theta)\tilde{k}f'' + \rho]}{BD\theta}, \tag{9f}$$

$$\frac{\partial \tilde{c}}{\partial e_0} = \frac{\rho(\rho - \theta\tilde{k}f'')}{De_0\theta} > 0, \tag{9g}$$

$$\frac{\partial \tilde{c}}{\partial \theta} = \frac{e_0\rho\tilde{k}f''}{D\theta^2} < 0, \tag{9h}$$

where

$$D \equiv \left(\frac{e_0-\theta}{\theta}\tilde{k}f'' + \frac{\rho}{\theta}\right)\rho' - e_0 f'' > 0 \quad \textit{and} \quad e_0\theta - (e_0-\theta)\rho'\tilde{k} > 0$$

hold from (8).

From Lemma 3, one sees that the steady state capital stock increases when either technology A or the efficiency e of the capital market improves (or the tax rate falls), which must in turn raise the steady state level of welfare for all households, since $\tilde{c}^* = i^*(\tilde{k}, \theta, \tau)$ and $\partial i^*/\partial k > 0$. Observe a slight difference here. Both labor and asset income change at once when a productivity shock arises, while only the latter changes immediately after shocks in the capital market occur, because labor income gradually changes following the adjustment in the capital stock.

In contrast, when households become more patient for *all* consumption levels (i.e., a decrease in B), the steady state capital stock increases, but the steady state consumption for rich households may decrease (see (9f)). To see this more precisely, from (6) and (7), we have

$$\left.\frac{\partial i(k,\theta,\tau)}{\partial k}\right|_{k=k_2} = \frac{(e_0-\theta)k_2 f''(k_2)}{\theta}$$

That is, under $\theta < e_0$ and Assumption 1, the steady state income for rich households decreases in $\tilde{k}$ when $\tilde{k}$ is sufficiently close to k_2, implying $\partial\tilde{c}/\partial B > 0 > \partial\tilde{c}^*/\partial B$ since $\partial\tilde{k}/\partial B < 0$. Hence a decrease in impatience B lowers their consumption, while it raises that of poor households.

Intuitively, when rich households become more patient, they save more, increasing the capital stock in the economy. If $\theta < e_0$, by Lemma 1, the income of rich households falls as the capital stock rises when the interest rate is sufficiently small. Thus an increase in patience tends to lower their consumption. In contrast, poor households' consumption increases due to the rise in the wage rate.

4.1 Income Gaps

We have shown that when positive shocks occur, all households become better off at the steady state. A related issue is whether their income gap widens or not, which we investigate now.

The incomes of households with and without assets at the steady state are respectively

$$\tilde{I} = w(\tilde{k}) + \frac{\rho(\tilde{c})\tilde{k}}{\theta}$$
$$\text{and} \;\; \tilde{I}^* = w(\tilde{k}) + \frac{\tau\rho(\tilde{c})\tilde{k}}{(1-\theta)(1-\tau)}.$$

We can define the income gap in terms of both *level* and *ratio* differences, respectively as:

$$g \equiv \tilde{I} - \tilde{I}^* = \frac{1-\theta-\tau}{\theta(1-\theta)(1-\tau)}\rho(\tilde{c})\tilde{k}$$
$$\text{or} \;\; \hat{g} \equiv \frac{\tilde{I}}{\tilde{I}^*} = \frac{\theta + \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}}{\theta + \frac{\theta\tau}{(1-\theta)(1-\tau)} \cdot \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}}.$$

Then the Gini coefficient can be calculated as,

$$G = 1 - \frac{\tilde{I}^*(1-\theta)^2/2 + \tilde{I}^*\theta(1-\theta) + \tilde{I}\theta^2/2}{[\tilde{I}^*(1-\theta) + \tilde{I}\theta]/2}$$
$$= \frac{\theta\hat{g}}{\theta\hat{g} + 1 - \theta} - \theta,$$

which gives:

Lemma 4 *The Gini coefficient G is increasing in the ratio income-gap $\hat{g}$.*

Further, to obtain clear results, we assume

Assumption 2: The production technology takes the Cobb-Douglas form: $f(k) = Ak^\alpha$, $\alpha \in (0,1)$.

Next, using Lemma 3, we have

Lemma 5 *Under Assumptions 1 and 2,*

$$\frac{\partial}{\partial A}\left[\rho(\tilde{c})\tilde{k}\right] = \frac{\rho(\rho+e_0\delta)}{AD}\left[1 + \frac{(1-\alpha)\delta}{\alpha\rho}\rho'\tilde{k}\right], \tag{10a}$$
$$\frac{\partial}{\partial B}\left[\rho(\tilde{c})\tilde{k}\right] = -\frac{\alpha\rho^2}{BD}\left[1 - \frac{(1-\alpha)e_0\delta}{\alpha\rho}\right], \tag{10b}$$
$$\frac{\partial}{\partial e_0}\left[\rho(\tilde{c})\tilde{k}\right] = \frac{\rho^2}{e_0 D}\left[1 + \frac{(1-\alpha)(\rho+e_0\delta)}{e_0\rho}\rho'\tilde{k}\right], \tag{10c}$$
$$\frac{\partial}{\partial A}\left[\frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}\right] = \frac{e_0\delta\left[e_0\alpha\rho + \theta(1-\alpha)(\rho+e_0\delta)\right]}{AD\theta(1-\alpha)(\rho+e_0\delta)}\rho' < 0, \tag{10d}$$
$$\frac{\partial}{\partial B}\left[\frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}\right] = \frac{e_0\delta\rho}{BDf} > 0, \tag{10e}$$
$$\frac{\partial}{\partial e_0}\left[\frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}\right] = \frac{\alpha\rho^2}{D\tilde{k}(\rho+e_0\delta)}\left[1 + \frac{\delta\rho'\tilde{k}}{\rho} + \frac{\alpha e_0 + (1-\alpha)\theta}{(1-\alpha)e_0\theta}\rho'\tilde{k}\right], \tag{10f}$$

where $\hat{g}$ (and hence G) is increasing in $\rho\tilde{k}/w$ when $\tau < 1-\theta$.

The above conditions give rise to two contrasting results. (i). Under CMI and IMI, we get

$$G_A \geq 0$$

and

$$g_A,\ g_e,\ G_e > 0, \tag{11}$$

where to simplify notation, we have used g_j, $\hat{g}_j$ and G_j to denote $\partial g/\partial j$, $\partial\hat{g}/\partial j$ and $\partial G/\partial j$ respectively, for $j = A$, B, e, τ and θ. These expressions imply that positive shocks in productivity A or efficiency e in the capital market increases the *level* income-gap, and the latter also raises the Gini coefficient since it favorably affects the income of rich households.

(ii). In contrast under DMI, poor households benefit more from positive shocks because such shocks make rich households more patient and hence accumulate more capital. It follows that the Gini coefficient is reduced by a rise in productivity increase: $G_A < 0$.

In addition, (11) is true for weak DMI (in the sense that Assumptions 2 and Assumption 3 below hold).[19]

Assumption 3: $\rho(c) = B(c+1)^{-\beta}$, $\beta \in (0, 1-\alpha)$.[20]

Assumption 3 implies that as capital accumulates, the rental rate falls faster than the speed at which rich households become patient (as their consumption levels rise), i.e.,

$$\left|\frac{d\rho}{\rho}\bigg/\frac{d(c+1)}{c+1}\right| < \left|\frac{dR}{R}\bigg/\frac{dk}{k}\right|.$$

To sum up, the effects of A and e on the Gini coefficient can be restated as:

Proposition 1 *Given $\tau \in [0, 1-\theta)$ and Assumptions 1 and 2, (i). An increase in productivity A reduces the Gini coefficient G under DMI; (ii). An increase in capital-market efficiency e raises G under CMI and IMI, and also under DMI given Assumption 3.*

Proof. See Appendix 9.2. ■

Acemoglu (2002) argues that the widening of income gap is a result of technology improvement over the past century. Our results on the increase of A partly reconfirm his prediction: It is only true for the gap in terms of income level ($g_A > 0$), and with regards to the Gini coefficient G, only true under CMI and IMI, but not under DMI as shown in Proposition 1 ($G_A < 0$). Also, Proposition 1 says that an increase in capital market efficiency widens the income gap. Thus, under some mild DMI such as $\beta < 1-\alpha$, the benefit of poor households from the positive shocks is small, and the level income-gap is magnified by the shocks, as under CMI or IMI. And during a financial crisis such as the current one, the income gap can actually fall, because when e falls the rich takes fewer risks and invests less.

Next, we turn to the impact of rich households' degree of impatience B.

[19] See Appendix 9.2.

[20] One can verify that Assumption 3 is consistent with (2), and that the assumptions on technology and preference yield strictly concave functions i and κ in k and c, respectively; The steady state must be unique and a saddle point.

Proposition 2 *Let $\tau \in [0, 1-\theta)$ and Assumptions 1 and 2 hold. A decrease in impatience B always reduces the Gini coefficient, while it increases the level income-gap iff $\rho(\tilde{c}) > (1-\alpha)e_0\delta/\alpha$.*

Proof. Straightforward from Lemma 5. ■

Proposition 2 gives interesting implications. Developing countries tend to have inferior production technology (low A) and less efficient capital market (low e), thus their steady-state consumption $\tilde{c}$ is low. A decrease in impatience B may widen the level income-gap in these countries. This arises because the interest rate tends to be high when the capital market is incomplete, resulting in a low capital stock (recalling $\rho(\tilde{c}) = (1-\tau)r(\tilde{k})$), which in turn leads to a low wage income and a high interest income, widening the income gap between the haves and the have-nots.[21] For instance, in China and India, both savings and investment are at unbelievably high levels, and the income gap is also rapidly increasing. In contrast, in some developed countries (usually with high A and e, and hence high $\tilde{c}$), a decrease in impatience B may reduce the level income-gap; Examples for this case are probably Luxemburg, Switzerland, etc., with low saving rates and low income gaps too.

5 Capital Tax and Poor Households

Since the income gap between rich and poor households increases when the economy grows (such as caused by a rise in A or e), the government may use taxes to reduce the inequality. We now examine the effects of τ on the level income gap, the Gini coefficient and the steady state welfare.

5.1 Capital Tax

Partial differentiation gives the effect of the capital tax τ on the income gap as:

$$g_\tau = -\frac{g}{1-\tau}\left\{\frac{\theta}{1-\theta-\tau} + \frac{eg_e}{g}\right\},$$
$$\hat{g}_\tau = -\frac{\hat{g}}{1-\tau}\left\{\frac{\tilde{I}^* - w}{\tau\tilde{I}^*} + \frac{e\hat{g}_e}{\hat{g}}\right\},$$

where $\lim_{\tau\to 0}[(\tilde{I}^* - w)/\tau\tilde{I}^*] = \rho\tilde{k}/(1-\theta)w$. From the above, we find the total effects can be divided into the sum of a direct effect and an indirect effect (as contained in the curled braces). Notice that the indirect effect can be negative under DMI, when a tax hike on rich households reduces the capital stock by a large scale (i.e., under strong DMI), and hence the wage rate falls sharply. Then it is possible for the income gap to widen when the elasticity of the gap with respect to the capital efficiency e is negative.

On the contrary, under mild DMI, the government can reduce the income gap by raising the capital tax on rich households as follows.

Proposition 3 *Let $\tau \in [0, 1-\theta)$ and Assumptions 2 and 3 hold. An increase in τ reduces the Gini coefficient under DMI, as well as under CMI or IMI.*

[21] Since the difference is given as $[(1-\theta-\tau)/\theta(1-\theta)]r(\tilde{k})\tilde{k}$, it is increasing in $\tilde{k}$ if $d[r(\tilde{k})\tilde{k}]/d\tilde{k} = r'(\tilde{k})\tilde{k} + r(\tilde{k}) > 0$.

Proof. Straightforward from Proposition 1. ■

Proposition 3 to some extent justifies the capital tax on rich households, in order to reduce social inequality, as one might expect. However, it has a negative effect on the steady state capital stock, and hence it may reduce poor households' income including transfers. Then it is necessary to look into the effect of τ on poor households' steady state welfare.[22]

As a particularly interesting finding, it is possible for the capital tax to lower the steady state welfare of poor households, especially in an economy where almost all households are poor and unable to own asset (e.g., a low θ). In such a case, the government can increase the steady state welfare for all households by setting τ to be negative, in effect subsidizing the rich and taxing the poor!

To be more specific, partially differentiating $\tilde{c}^*$ with respect to τ yields

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \frac{\rho \tilde{k}}{(1-\theta)(1-\tau)^2} + w' \frac{\partial \tilde{k}}{\partial \tau} + \frac{\tau \rho}{(1-\theta)(1-\tau)} \cdot \frac{\partial \tilde{k}}{\partial \tau} + \frac{\tau \tilde{k} \rho'}{(1-\theta)(1-\tau)} \cdot \frac{\partial \tilde{c}}{\partial \tau},$$

where the first term on the RHS denotes a positive direct effect, and the rest is the sum of several indirect effects: the second and third terms are negative, which comes from the fact that an increase in τ reduces the capital stock, and hence both the wage rate and the amount of transfer fall, while the last term is positive due to a rise in the rental rate under DMI.

From Lemma 3 and $\partial e_0/\partial \tau = -e$, we see

$$\begin{aligned}
\frac{\partial \tilde{c}^*}{\partial \tau} &= \hat{D} \Big\{ \tilde{k} \Big[(e_0 - \theta)\rho' \tilde{k} f'' + \rho\rho' - e_0 f'' \theta \Big] + (1-\theta)(1-\tau)\tilde{k} f''(\theta - \rho' \tilde{k}) \\
&\quad - \tau \rho (\theta - \rho' \tilde{k}) - \tau \tilde{k} \rho' (\rho - \theta \tilde{k} f'') \Big\} \\
&= \hat{D} \Big[-(1-\theta-e)(1-\tau)\rho' \tilde{k}^2 f'' - \theta \rho' \tilde{k}^2 f'' + \rho\rho' \tilde{k} + (1-\theta-e)(1-\tau)\tilde{k} f'' \theta \\
&\quad - \tau\theta\rho + \tau\theta\rho' \tilde{k}^2 f'' \Big] \qquad (12) \\
&= \hat{D} \Big[-(1-e)(1-\tau)\rho' \tilde{k}^2 f'' + \rho\rho' \tilde{k} + (1-\theta-e)(1-\tau)\tilde{k} f'' \theta - \tau\rho\theta \Big], \qquad (13)
\end{aligned}$$

where $\hat{D} \equiv \rho / D\theta(1-\theta)(1-\tau)^2$.

We are now in a position to state:

Proposition 4 *Let Assumption 1 hold. (i). If the capital market is imperfect ($e < 1$) and $\theta < 1 - e$, then under DMI or CMI, there exists some $\varepsilon > 0$ such that for $\tau \in (-\varepsilon, 1-\theta)$, reducing the capital tax increases the steady state welfare of all households; (ii). If the capital market is perfect ($e = 1$), a positive capital tax on rich households raises the welfare of the poor under CMI and IMI, but may not do so under DMI: for a sufficiently small θ, subsidizing the rich and taxing the poor may raise all households' welfare in the steady state under DMI.*

Proof. See Appendix 9.3. ■

In fact, under Assumptions 2 and 3 and $e = 1$, we find that for $\tau \in (-\beta, 1-\theta)$, reducing the capital tax or raising the capital subsidy increases the steady state welfare of poor households

[22] Hamada (1967) considers the effect of transfers between capitalists and workers on the latter's income on the equilibrium growth path with constant saving ratios, and he finds the optimal tax rate is zero, $\tau = 0$.

in an economy with some θ, and the welfare is maximized at some $\tau \in (-\beta, 0)$,[23] rather than by an income transfer to them!

The results under DMI contrast sharply with those under CMI and IMI. Poor households' steady state income is more likely to decrease under DMI than under CMI or IMI, when the government raises the tax on rich households (the first and second terms in (13)). This can be divided into two cases. First, if $\theta < 1 - e$, under DMI or CMI, the positive direct effect is dominated by the indirect effect which comes from the decrease in $w(k)$ (see (12)), and from (13), $\partial\tilde{c}^*/\partial\tau < 0$ for $\tau \geq 0$, implying that subsidizing the rich and taxing the poor will raise poor households' welfare in the steady state.

The above results remind us of the Chinese experience in the past 35 years. Until the early 1980s, most Chinese were very poor and owned almost zero assets. The government opened Special Economic Zones to allow business owners (a tiny fraction of the population then) to do business tax free, with land, import and export other subsidies. It is especially worth mentioning that even North Korea opened such Special Economic Zones along the borders with China and South Korea.

Second, if $\theta > 1 - e$, which necessarily holds when $e = 1$, a positive capital tax on rich households is preferable for poor households under CMI and IMI by increasing their present and future incomes. But under DMI, it is possible that $\partial\tilde{c}^*/\partial\tau < 0$ for $\tau \geq 0$ even with $e = 1$.

On the other hand, in a more mature economy with a higher θ (a higher fraction of population owning assets), then under Assumptions 2 and 3,[24]

$$\frac{\partial\tilde{c}^*}{\partial\tau} > 0 \text{ for any feasible } \tau \leq 0;$$

that is, some positive capital tax will maximize poor households' steady state welfare. This arises because (13) yields

$$\frac{\partial\tilde{c}^*}{\partial\tau} = \hat{D}\theta\left\{\left[\left(\frac{e_0 - 1}{\theta}\tilde{k}f'' + \frac{\rho}{\theta}\right)\rho' - e_0 f''\right]\tilde{k} - \tau\left(\rho - \frac{\rho'\tilde{k}^2 f''}{\theta}\right) + (1-\theta)(1-\tau)\tilde{k}f''\right\},$$

where, if θ is sufficiently close to 1, (i). the term in [...] is positive from $D > 0$; (ii) $\rho > \rho'\tilde{k}^2 f''$ holds under Assumptions 2 and 3, and hence the second term is also non-negative given $\tau \leq 0$; (iii) the first term in braces {...} dominates the last term when θ goes to 1 (see footnote 24). Intuitively, the above arises because when θ is sufficiently high, subsidizing the rich causes an even higher tax burden on the poor. Thus, in present-day China, when the fraction of the rich has reached a certain high level, taxing the rich may be a good policy to reduce the ever-increasing income gap.

5.2 Unequal but Preferred

In this subsection we show that an economy with a small θ may be preferable for poor households, and such an economy comes with a sufficiently low Gini coefficient.

As mentioned earlier, the decrease in θ raises the capital stock and the wage rate in the steady state, which implies that some level of inequality may be preferable under DMI. Indeed, we can show that an economy with an *uneven* distribution of income among households is

[23] See the proof of Proposition 4 in Appendix 9.3.

[24] Here, "feasible" means that $i^*(\tilde{k}, \theta, \tau) = w(\tilde{k}) + \tau r(\tilde{k})\tilde{k}/(1-\theta) \geq 0$, i.e., $\tau \geq -(1-\theta)w(\tilde{k})/r(\tilde{k})\tilde{k}$, the right-hand side of which goes to zero when θ goes to 1.

preferable for *all* households to an economy with each household having an identical level of asset.

Suppose that all households own the same level of asset. Then, households' income is given by

$$\begin{aligned} I(k) &\equiv w(k) + r(k)k \\ &= f(k) - (1-e)kf'(k) - e\delta k, \end{aligned}$$

which is increasing in k with $k < k_2$:[25]

$$I'(k) = e[f'(k) - \delta] - (1-e)kf''(k).$$

Let $\tilde{k}(1)$ and $\tilde{c}(1)$ be the steady state levels of capital and consumption in the economy with $\theta = 1$. Then, $\tilde{k}(1) < k_2$ holds, and hence the income (and consumption) in the steady state must be smaller than $I(k_2) = w(k_2)$:

$$\tilde{c}(1) = I(\tilde{k}(1)) < w(k_2).$$

On the other hand, if there are two types of households and only few households own asset (θ is sufficiently small), capital accumulates at almost the same level as k_2, because the very-wealthy households have an extremely low discount rate due to DMI and they will accumulate capital until the interest rate reaches almost zero. It can be verified as follows. For $k < k_2$,

$$\frac{\partial i(k,\theta,\tau)}{\partial \theta} = -\frac{(1-\tau)r(k)k}{\theta^2} < 0 \quad \text{and} \quad \lim_{\theta \to 0} i(k,\theta,\tau) = \infty,$$

and κ does not depend on θ. Therefore,

$$\lim_{\theta \to 0} \tilde{k} = k_2 \quad \text{and} \quad \lim_{\theta \to 0} \tilde{c} = \infty. \tag{14}$$

Also, notice that

$$\lim_{\theta \to 0} \tilde{c}^* = w(k_2),$$

which implies that

$$\lim_{\theta \to 0} \tilde{c}^* > \tilde{c}(1).$$

Thus we have

Proposition 5 *Under Assumption 1, the steady state welfare in the economy without inequality (i.e., everyone becoming a capitalist with an identical level of asset), is lower than the poor households' welfare in the steady state with a sufficiently small θ.*

The Proposition implies that under DMI, some level of inequality is preferable for the economy, basically because rich households are more patient and save more, which lowers the interest rate and in turn raises the wage income of poor households through production linkages. On the contrary if the capital stock is spread over more capital owners (i.e., lowering inequality), then each capitalist saves and invests less, resulting in lower welfare in the long run.[26] The

[25] Notice that k_2 corresponds to the golden-rule level of capital (per household) when $e = 1$.

[26] In the case of $\tau = 1 - \theta$, both types of households have the same level of income and the steady-state consumption is given by the intersection between $c = i(k, \theta, 1-\theta)$ and $k = \kappa(c, 1-\theta)$, which must be smaller than $\tilde{c}(1)$.

generated consequences from this Proposition are similar to those of the "Trickle Down Theory" (Aghion and Bolton, 1997), albeit via a starkly different mechanism.

Note that there is a non-monotonic relationship between θ and the Gini coefficient G, because $G = 0$ when either $\theta = 1$, or $\theta = 0$, i.e.,

$$\lim_{\theta \to 0} \theta \tilde{I} = \lim_{\theta \to 0} \left[\theta w(\tilde{k}) + \rho(\tilde{c})\tilde{k} \right] = 0.$$

Differentiation gives,

$$\begin{aligned} G_\theta &= \frac{\hat{g} + (1-\theta)\theta \hat{g}_\theta}{(\theta \hat{g} + 1 - \theta)^2} - 1 \\ &= \frac{(1-2\theta)(\hat{g}-1) + (1-\theta)\theta \hat{g}_\theta - [\theta(\hat{g}-1)]^2}{(\theta \hat{g} + 1 - \theta)^2}, \end{aligned}$$

where $\hat{g}_\theta < 0$ by Proposition 8 below. Notice that G_θ must be negative for $\theta \geq 1/2$, since $\hat{g} > 1$; and it can be positive for a sufficiently small θ, because $G > 0$ for any $\theta \in (0, 1)$ and $\lim_{\theta \to 0} G = 0$.[27] Therefore we have

Proposition 6 *Let $\tau \in [0, 1-\theta)$, and Assumptions 2 and 3 hold. The Gini coefficient falls for $\theta \geq 1/2$ but rises as the share of rich households increases when θ is sufficiently small.*

The Proposition implies that the Gini coefficient exhibits a sharp inverted-U shape, first increasing then decreasing, following a rise in the percentage owning assets, θ. It again stems from DMI and its trickle down effect, as discussed extensively before. This relationship is similar to the Kuznets' curve in the literature, and it may be good news for developing countries such as China, India and Latin America, who are trying to catch up with developed countries. As the fraction of population owning assets rises to above a certain level, inequality gradually falls, even without any government intervention.

6 Urbanization

Rapid urbanization is a common feature of some newly emerging economies, especially China and India. Inequality attracts large-scale migration of poor households from rural to urban areas. In this section, we look into how such urbanization and domestic migration affects the economy in terms of inequality and welfare.

Consider an urban economy that has both rich and poor households. Urbanization enables some rural poor people to become city residents, though still remaining poor without asset. By Lemma 3, we have

Proposition 7 *Given Assumption 1, under DMI, urbanization (i) raises the steady state welfare of all households in the urban area, if θ or τ is sufficiently small; (ii) lowers (raises) the steady state welfare of poor (rich) households in the urban area, if θ is sufficiently close to 1 and τ is sufficiently close to $1-\theta$. In contrast, under CMI, it has no (a negative) effect on the steady state welfare of poor households when $\tau = 0$ ($\tau > 0$), though it raises that of rich households.*

[27] Indeed, one can verify that under Assumptions 2 and 3, $\lim_{\theta \to 0} \theta \hat{g}_\theta / (\hat{g} - 1) = -(1+\beta)^{-1} > -1$.

Proof. See Appendix 9.4. ■

To give an intuitive explanation for Proposition 7, we suppose that $\tau = 0$,[28] and the urban economy has reached a steady state such that

$$\tilde{k} = \frac{K}{L}, \quad \tilde{a} = \frac{K}{\theta L}, \quad \tilde{c} = w\left(\frac{K}{L}\right) + \rho\frac{K}{\theta L}, \quad \text{and} \quad \tilde{c}^* = w\left(\frac{K}{L}\right),$$

where $\rho = r(K/L)$.

As the city becomes bigger and annexes a neighboring rural region, the urban labor input L rises to L' (the ratio θ falls to θ'), and hence the wage rate falls and the capital rental rises. The whole economy thus accumulates capital and will eventually reach a new steady state, which can be characterized by

$$\tilde{k} = \frac{K'}{L'}, \quad \tilde{a} = \frac{K'}{\theta L}, \quad \tilde{c} = w\left(\frac{K'}{L'}\right) + \rho\frac{K'}{\theta L}, \quad \text{and} \quad \tilde{c}^* = w\left(\frac{K'}{L'}\right),$$

where $\rho = r(K'/L')$, and the rich households own more asset than in the old steady state.

Under CMI, $K'/L' = K/L$ must hold, and hence rich households' steady state consumption will rise, while that of poor households does not change. As a consequence, existing poor households are harmed by urbanization due to the short-term fall in the wage rate.

Under DMI, however, an increase in the capital labor ratio ($K'/L' > K/L$) is accompanied by an increase in the consumption level of rich households since they become more patient, and therefore poor households will be made better off in the long run through the rich's added investment. In other words, DMI generates a further effect above CMI, which in this case is a positive trickle-down, from the haves to the have-nots, similar to those predicted by the "Trickle Down Theory," even though our mechanism is different.

Next, consider the effect of urbanization on the income gap among households. Since a decrease in θ shifts the graph $c = i(k, \theta, \tau)$ upward while leaving unchanged the graph $k = \kappa(c, \tau)$, it increases the steady state levels of k and c along $k = \kappa(c, \tau)$ (see Figure 1). Notice that Assumptions 2 and 3 yield strictly concave functions for i^* in k and κ in c, and that a decrease in θ lowers the amount of income transfer for a given k. Thus we have

Proposition 8 *Let $\tau \in [0, 1-\theta)$, and Assumptions 2 and 3 hold. Urbanization (lowering θ) widens both the level and the ratio income-gap under DMI as well as under CMI and IMI.*

Notice that Propositions 7 (i) and 8 together imply that urbanization, which adds poor people to the city, increases the steady-state welfare of all city households due to capital accumulation stimulated by labor migration, but it also widens the income gap between the rich and the poor. These are consistent with observations in China and India in recent years, where urban (and coastal) regions experience rapid growth in size, population and average income, but the income gap also rises drastically, as documented in the Introduction section.

7 Concluding Remarks

In an economy with intrinsic inequality to begin with, we have examined how endogenous time preference affects the social inequality, with special focus on DMI. Our analysis has shown that

[28] If τ is positive, a fall in θ implies a decrease in the amount of income transfer, $T = \tau r(k)k/(1-\theta)$, for fixed k and τ.

(i) poor households tend to benefit more from positive shocks under DMI than under CMI; (ii) positive shocks widen the income difference between the rich and the poor when the effect of DMI is small; (iii) reducing the inequality may not be preferable for all households; (iv) urbanization widens the income inequality while increasing welfare.

Also, we show in Appendix 2 that under DMI, even if both types of households own different levels of asset to begin with, the economy will not converge to the steady state where all households have some positive levels of asset. That is, the steady state is the same as the one when the borrowing constraint is binding only for one type of households. Thus, asymmetry of households may be a natural consequence of endogenous time preferences under DMI.

Our result that expanding inequality (a fall in θ) makes the poor households better off is derived in the absence of international credit market. If international lending and borrowing are available, this result may be altered, mainly because, a fall in θ implies a decrease in the rich's discount factor but an increase in the their consumption as well as the capital stock in the whole economy, the last of which also raises the income of poor households. However, if international credit markets are available, a fall in θ does not necessarily imply the last result.

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9 Appendix 1

9.1 Proof of Lemma 2

We evaluate the elements of a Jacobian in the system (equations (4a)–(4d)) to study the local dynamics around the steady state. Differentiation gives,

$$\begin{aligned} J(x) &= \det[J - xI] \\ &= \det \begin{bmatrix} (\theta - e_0)w' + \rho - x & 0 & 0 & -1 \\ -\theta Z(1-\tau)r' & -x & 0 & Z\rho' \\ 0 & 0 & \rho - x & -Z \\ 0 & -1 & -\rho' & -M \end{bmatrix} \\ &= \Gamma(x, \theta, \tau), \end{aligned}$$

where $M \equiv -u'' + \mu\rho'' > 0$[29] and

$$\begin{aligned} \Gamma(x,\theta,\tau) \;\equiv\; & Mx^3 - M[(\theta - e_0)w' + 2\rho]x^2 + \{M\rho[(\theta - e_0)w' + \rho] \\ & -Z[\rho\rho' - \theta(1-\tau)r']\}x + \rho Z[(\theta - e_0)w'\rho' + \rho\rho' - \theta(1-\tau)r']. \end{aligned}$$

Notice that (8) implies $\Gamma(0, \theta, \tau) > 0$. This characteristic equation can be used to derive the local dynamics of the system in the neighborhood of the steady state.

[29] In the case of IMI, we assume $M > 0$ to make the Hamiltonian $\mathcal{H}$ strictly concave in c.

It is clear from $M > 0$ and $J(0) = \Gamma(0, \theta, \tau) > 0$ that $J(x) = 0$ has at least one negative root, x_1. If $(\theta - e_0)w' + 2\rho = 0$, then

$$\begin{aligned} J(x) &= Mx^3 + J'(0)x + J(0) \\ &= M(x - x_1)\left[x^2 + x_1 x - \frac{J(0)}{Mx_1}\right]. \end{aligned}$$

Thus the other two roots, x_2 and x_3, satisfy $x_2 + x_3 = -x_1 > 0$ and $x_2 x_3 = -J(0)/Mx_1 > 0$, which implies that they have positive real parts.

Suppose that $(\theta - e_0)w' + 2\rho \neq 0$. Then, applying Routh's (1905) theorem, the number of the roots of $J(x) = 0$ with positive real parts equals the number of changes in signs in the following sequence:

$$M, \quad -M[(\theta - e_0)w' + 2\rho], \quad \frac{\Gamma((\theta - e_0)w' + 2\rho, \theta, \tau)}{(\theta - e_0)w' + 2\rho}, \quad \Gamma(0, \theta, \tau).$$

(i). $(\theta - e_0)w' + 2\rho > 0$. Then, the number of changes in signs is two irrespective of the sign of the third term.

(ii). $(\theta - e_0)w' + 2\rho < 0$. Then, $(\theta - e_0)w' + \rho < 0$ and

$$\begin{aligned} &\frac{\Gamma((\theta - e_0)w' + 2\rho, \theta, \tau)}{(\theta - e_0)w' + 2\rho} \\ &= \frac{M\rho[(\theta - e_0)w' + \rho][(\theta - e_0)w' + 2\rho] + Z\theta(1 - \tau)r'[(\theta - e_0)w' + \rho] - Z\rho^2\rho'}{(\theta - e_0)w' + 2\rho} \\ &< 0, \end{aligned}$$

which implies that the number of changes is two.

9.2 Proof of Proposition 1

When $\rho' \geq 0$, we see from (10a), (10c), and (10f) that g_A, g_e, and $\hat{g}_e$ are all positive. We will show that this is also true under DMI with Assumption 3.

The sign of g_A

We show that under Assumptions 2 and 3,

$$\frac{\rho'(\tilde{c})\tilde{k}}{\rho(\tilde{c})} > -\frac{\alpha\beta}{(1 - \alpha)\delta} \tag{15}$$

holds for any pair of parameters, and hence g_A must be positive due to (10a) and $\beta < 1 - \alpha$. Under Assumption 3, we have

$$\frac{\rho'(\tilde{c})\tilde{k}}{\rho(\tilde{c})} = -\frac{\beta\tilde{k}}{\tilde{c} + 1} \tag{16}$$

and from Lemma 3 and Assumption 2,

$$\frac{\partial}{\partial A}\left(-\frac{\beta\tilde{k}}{\tilde{c} + 1}\right) = -\frac{\beta\tilde{k}^{\alpha - 1}\{[\alpha e_0 + (1 - \alpha)\theta]\beta\rho(\tilde{c})\tilde{k} + \alpha e_0\theta\}}{D\theta(\tilde{c} + 1)^2} < 0. \tag{17}$$

Next we show that

$$\lim_{A \to \infty}\left(-\frac{\beta\tilde{k}}{\tilde{c} + 1}\right) = -\frac{\alpha\beta}{(1 - \alpha)\delta}. \tag{18}$$

One can easily verify that $\lim_{A\to\infty} \tilde{k} = \lim_{A\to\infty} \tilde{c} = \infty$. This implies

$$\lim_{A\to\infty} A\hat{f}'(\tilde{k}) = \delta,$$

because $\lim_{c\to\infty} \rho(c) = 0$ and $\rho(c) = e_0[A\hat{f}'(k) - \delta]$ holds at any steady state. Then (18) holds since

$$\frac{\tilde{k}}{\tilde{c}+1} = \frac{1}{(1-\alpha)A\hat{f}'(\tilde{k})/\alpha + e_0[A\hat{f}'(\tilde{k}) - \delta]/\theta + 1/\tilde{k}}.$$

From (16)–(18), we may conclude that (15) holds for any pair of parameters.

The signs of g_e and G_e

Under Assumptions 2 and 3, (10f) becomes

$$\begin{aligned}
\frac{\partial}{\partial e_0}\left[\frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}\right] &= \frac{\alpha\rho^2}{D\tilde{k}(\rho+e_0\delta)}\left[1 - \frac{\beta\delta\tilde{k}}{\tilde{c}+1} - \frac{\alpha}{(1-\alpha)\theta}\cdot\frac{\beta\rho\tilde{k}}{\tilde{c}+1} - \frac{\beta\rho\tilde{k}}{e_0(\tilde{c}+1)}\right] \\
&= \frac{\alpha\rho^2}{D\tilde{k}(\rho+e_0\delta)(\tilde{c}+1)}\left[(1-\alpha)A\tilde{k}^{\alpha} + \frac{\rho\tilde{k}}{\theta} + 1 - \beta\delta\tilde{k} - \frac{\alpha\beta\rho\tilde{k}}{(1-\alpha)\theta} - \frac{\beta\rho\tilde{k}}{e_0}\right] \\
&= \frac{\alpha\rho^2}{D\tilde{k}(\rho+e_0\delta)(\tilde{c}+1)}\left\{1 + \tilde{k}\left[(1-\alpha)A\tilde{k}^{\alpha-1} - \beta\left(\delta + \frac{\rho}{e_0}\right)\right] + \frac{\rho\tilde{k}}{\theta}\left(1 - \frac{\alpha\beta}{1-\alpha}\right)\right\} \\
&= \frac{\alpha\rho^2}{D\tilde{k}(\rho+e_0\delta)(\tilde{c}+1)}\left(1 + \frac{1-\alpha-\alpha\beta}{1-\alpha}\tilde{c}\right),
\end{aligned}$$

which is positive due to $1-\alpha > \beta$. Therefore $\hat{g}_e$ and G_e must be positive under $\tau < 1-\theta$. Note that $\hat{g}_e > 0$ and $\hat{g} > 1$ together imply $g_e > 0$.

9.3 Proof of Proposition 4

Let $e < 1$ and $\theta < 1 - e$. Then, for $\tau \in [0, 1-\theta)$,

$$\frac{\partial \tilde{c}^*}{\partial \tau} < 0$$

clearly holds under $\rho' \leq 0$. This proves Proposition 4 (i).

Next, suppose that $e = 1$. Then,

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \frac{\rho\theta}{D(1-\theta)(1-\tau)^2}\left[\frac{\rho\rho'\tilde{k}}{\theta^2} - (1-\tau)\tilde{k}f'' - \frac{\tau\rho}{\theta}\right]. \tag{19}$$

Thus for $\tau \leq 0$,

$$\frac{\partial \tilde{c}^*}{\partial \tau} > 0$$

holds under $\rho' \geq 0$.

We now show that if θ is sufficiently small, then the first term in square brackets of (19) dominates the second one, and hence we have

$$\frac{\partial \tilde{c}^*}{\partial \tau} < 0 \text{ for } \tau \in [0, 1-\theta) \tag{20}$$

under $\rho' < 0$.

First, we see from (14) that

$$\lim_{\theta\to 0} \tilde{k} f''(\tilde{k}) = k_2 f''(k_2) \quad \text{and} \quad \lim_{\theta\to 0} \rho(\tilde{c}) = 0.$$

Using L'Hôpital's rule, we have

$$\begin{aligned} \lim_{\theta\to 0} \frac{\rho(\tilde{c})}{\theta} &= \lim_{\theta\to 0} \rho' \frac{\partial \tilde{c}}{\partial \theta} \\ &= \lim_{\theta\to 0} \frac{(1-\tau)\tilde{k} f''}{\frac{\theta}{\rho}(1-\tau-\theta)\tilde{k} f'' + \theta - \frac{\theta^2}{\rho\rho'}(1-\tau) f''}. \end{aligned} \tag{21}$$

However, (14) and the fact that $\tilde{c} = w(\tilde{k}) + \rho(\tilde{c})\tilde{k}/\theta$ together imply

$$\lim_{\theta\to 0} \frac{\rho(\tilde{c})}{\theta} = \infty. \tag{22}$$

From (21) and (22), we have

$$\lim_{\theta\to 0} \frac{\rho(\tilde{c})\rho'(\tilde{c})}{\theta^2} = -\infty. \tag{23}$$

If θ is sufficiently small, then (20) holds, which proves the last part of Proposition 4.

Finally, under Assumptions 2 and 3, we have

$$\begin{aligned} \lim_{\theta\to 0} \left[-\frac{\rho'(\tilde{c})\tilde{k}}{\theta} \right] &= \lim_{\theta\to 0} \frac{\beta \tilde{k}(1-\tau) r(\tilde{k})}{\theta(\tilde{c}+1)} \\ &= \lim_{\theta\to 0} \frac{\beta \tilde{k}(1-\tau) r(\tilde{k})}{\theta w(\tilde{k}) + \tilde{k}(1-\tau) r(\tilde{k}) + \theta} \\ &= \beta. \end{aligned} \tag{24}$$

Since (19) yields

$$\frac{\partial \tilde{c}^*}{\partial \tau} = -\frac{\rho\theta}{D(1-\theta)(1-\tau)^2} \left[(1-\tau)\tilde{k} f'' + \frac{\rho}{\theta}\left(\tau - \frac{\rho' \tilde{k}}{\theta} \right) \right],$$

we see from (22) and (24) that for any $\tau > -\beta$, there exists some θ such that

$$\frac{\partial \tilde{c}^*}{\partial \tau} < 0.$$

Also, if $\tau \leq -\beta$, then $\partial \tilde{c}^*/\partial \tau > 0$: the value of τ that maximizes the steady state levels of welfare for poor households must be greater than $-\beta$. This is because

$$\begin{aligned} \frac{\partial}{\partial \theta} \left[\frac{\rho'(\tilde{c})\tilde{k}}{\theta} \right] &= \frac{(\rho'' \frac{\partial \tilde{c}}{\partial \theta} \tilde{k} + \rho' \frac{\partial \tilde{k}}{\partial \theta})\theta - \rho' \tilde{k}}{\theta^2} \\ &= \frac{(\rho')^2 \tilde{k} f''}{D\theta^3} \left[\frac{\rho\rho''}{(\rho')^2} e_0 \tilde{k} - (e_0 - \theta)\tilde{k} + \frac{\theta e_0}{\rho'} \right] \\ &= \frac{(\rho')^2 \tilde{k} f''}{D\theta^3 \beta\rho} \Big([(\beta+1)e_0 - \beta(e_0-\theta)] e_0 (\alpha A \tilde{k}^\alpha - \delta \tilde{k}) - e_0 \{ A\tilde{k}^\alpha [(1-\alpha)\theta + \alpha e_0] - e_0 \delta \tilde{k} + \theta \} \Big) \\ &= -\frac{e_0 (\rho')^2 \tilde{k} f''}{D\theta^2 \beta\rho} [(1-\alpha-\alpha\beta) A \tilde{k}^\alpha + \beta\delta\tilde{k} + 1] \\ &> 0. \end{aligned}$$

9.4 Proof of Proposition 7

It is apparent from (9h) in Lemma 3 that irrespective of the sign of ρ', labor migration of poor households without asset from rural to urban areas raises the steady state welfare of rich households.

For that of poor households, we have

$$\frac{\partial \tilde{I}^*}{\partial \theta} = \left[w' + \frac{\tau\rho}{(1-\theta)(1-\tau)}\right]\frac{\partial \tilde{k}}{\partial \theta} + \frac{\tau\rho'\tilde{k}}{(1-\theta)(1-\tau)} \cdot \frac{\partial \tilde{c}}{\partial \theta} + \frac{\tau\rho\tilde{k}}{(1-\theta)^2(1-\tau)}$$
$$= \frac{1}{D\theta^2}\left(-\tilde{k}f''\rho\rho'\tilde{k} + \frac{\tau\rho'\tilde{k}}{1-\theta}e\rho\tilde{k}f''\right) + \frac{\tau\rho\tilde{k}}{(1-\theta)^2(1-\tau)}\left[\frac{(1-\theta)\rho\rho'}{D\theta^2} + 1\right]$$
$$= \frac{1}{(1-\theta)^2(1-\tau)D}\left\{-\frac{\tilde{k}^2 f''\rho\rho'[(1-\theta-\tau)^2 + \tau(1-\tau)(1-e)]}{\theta^2} + \tau\rho\tilde{k}\left(\frac{\rho\rho'}{\theta^2} - e_0 f''\right)\right\},$$

which is negative for sufficiently small θ and/or τ due to (23) under DMI.

Let $\tau = 1-\theta$. Then,

$$D = \left[(e-1)\tilde{k}f'' + \frac{\rho}{\theta}\right]\rho' - \theta e f'',$$

and

$$\frac{\partial \tilde{I}^*}{\partial \theta} = \frac{\rho\tilde{k}}{(1-\theta)\theta^2 D}\left\{\left[(e-1)\tilde{k}f'' + \frac{\rho}{\theta}\right]\rho' - \theta^2 e f''\right\},$$

which is positive when θ is sufficiently close to 1 since $D > 0$.

Under CMI, we have $\partial\tilde{k}/\partial\theta = 0$ and $\rho' = 0$, and therefore

$$\frac{\partial \tilde{I}^*}{\partial \theta} = \frac{\tau\rho\tilde{k}}{(1-\theta)^2(1-\tau)},$$

the sign of which corresponds to that of τ.

10 Appendix 2: Endogenous Difference among Households

So far, we have examined the effects of fundamental shocks, government policies and migration of poor households, when 'intrinsic' inequality exists in an economy under DMI. In this section, we further show that the existence of these two types of households may be a natural outcome based on the assumptions on preferences.

Specifically, we relax the model by allowing both types of households to own non-negative asset (though they are not allowed to borrow from future wage incomes): They are symmetric in all aspects except for their initial asset holdings. We shall show that the steady state we have examined in the previous sections is the one where the borrowing constraint is binding only for one type of households, giving rise to an asymmetric steady state endogenously.

There does exist a symmetric steady state where both types of households own the same level of asset. However, there is no steady state where each type of households has a different level of asset. And we will prove that *under endogenous time preference with DMI,* the symmetric one is unstable in the sense that the economy converges to it only if the two types of households initially have an identical level of asset, i.e., there is only one type of households in the economy.[30]

10.1 The Dynamic System

We add the following constraint on each household's asset holdings to the original problem ((1a)–(1c)), since both types can own asset now,

$$a \geq 0, \quad \text{for } {}^{\forall}t > 0.$$

We assume away the income tax here by setting $\tau = 0$. The associated Lagrangian is

$$\mathcal{L} \equiv \mathcal{H} + \nu a.$$

The necessary conditions for optimality are

$$\frac{\partial \mathcal{L}}{\partial c} = u'(c)\, X - \lambda - \mu\rho'(c)X = 0, \tag{25a}$$
$$\frac{\partial \mathcal{L}}{\partial a} = \lambda r + \nu = -\dot{\lambda}, \tag{25b}$$
$$\frac{\partial \mathcal{L}}{\partial X} = u(c) - \mu\rho(c) = -\dot{\mu}, \tag{25c}$$
$$a \geq 0, \ \nu \geq 0, \ \text{and } \nu a = 0. \tag{25d}$$

Let $z \equiv \nu/X$. Then, (25b) can be rewritten as

$$\dot{Z} = Z[\rho(c) - r] - z.$$

Our new dynamic general equilibrium model can be described as

$$\dot{a} = w(k) + r(k)a - c, \tag{26a}$$
$$\dot{a}^* = w(k) + r(k)a^* - c^* \tag{26b}$$
$$\dot{Z} = Z[\rho(c) - r(k)] - z, \tag{26c}$$
$$\dot{Z}^* = Z^*[\rho(c^*) - r(k)] - z^*, \tag{26d}$$
$$\dot{\mu} = \mu\rho(c) - u(c), \tag{26e}$$
$$\dot{\mu}^* = \mu^*\rho(c^*) - u(c^*), \tag{26f}$$
$$0 = u'(c) - \mu\rho'(c) - Z, \tag{26g}$$
$$0 = u'(c^*) - \mu^*\rho'(c^*) - Z^*, \tag{26h}$$
$$0 = az, \tag{26i}$$
$$0 = a^*z^*, \tag{26j}$$

where "$*$" denotes the corresponding behavioral relations for the other type of households and $k = a\theta + a^*(1-\theta)$ from the market clearing condition for asset,

$$a\theta L + a^*(1-\theta)L = K.$$

[30] Epstein (1987) proved that local stability fails if there are two (or more) individuals with DMI, in his central planner problem with $N > 1$ agents.

10.2 The Steady States

The steady state is the solution for the system of equations

$$0 = w(k) + r(k)a - c, \tag{27a}$$
$$0 = w(k) + r(k)a^* - c^*, \tag{27b}$$
$$0 = Z[\rho(c) - r(k)] - z, \tag{27c}$$
$$0 = Z^*[\rho(c^*) - r(k)] - z^*, \tag{27d}$$
$$0 = \mu\rho(c) - u(c), \tag{27e}$$
$$0 = \mu^*\rho(c^*) - u(c^*), \tag{27f}$$
$$Z = u'(c) - \mu\rho'(c), \tag{27g}$$
$$Z^* = u'(c^*) - \mu^*\rho'(c^*), \tag{27h}$$
$$0 = az, \tag{27i}$$
$$0 = a^*z^*. \tag{27j}$$

Suppose that $\tilde{a}$ and $\tilde{a}^*$ are positive. Then, $\tilde{z} = \tilde{z}^* = 0$ and $\tilde{c} = \tilde{c}^*$ from (27c) and (27d), and hence we have $\tilde{a} = \tilde{a}^*$ $(= \tilde{k})$ from (27a) and (27b). Notice that under Assumption 1, $(\tilde{k}, \tilde{c})$ is uniquely determined as the solution to

$$c = i(k, 1, 0) \quad \text{and} \quad k = \kappa(c, 0).$$

Thus, a symmetric steady state exists.

Next, let $\tilde{a} > 0$ and $\tilde{a}^* = 0$. Then, $\tilde{z} = 0$ and $\tilde{a} = \tilde{k}/\theta$. The steady state pair $(\tilde{k}, \tilde{c})$ is given as the solution to

$$c = i(k, \theta, 0) \quad \text{and} \quad k = \kappa(c, 0).$$

This corresponds to the steady state discussed in the previous sections.

Also, there is another asymmetric steady state with $\tilde{a} = 0$ and $\tilde{a}^* > 0$.

Thus, we have

Lemma 6 *Under Assumption 1, there exist three steady states, one of which is symmetric and the others are asymmetric. However, there is no steady state where each type of households has a different level of asset. Furthermore, the symmetric equilibrium is characterized as* $\tilde{a} = \tilde{a}^* = \tilde{k}$, $\tilde{c} = \tilde{c}^* = w(\tilde{k}) + r(\tilde{k})\tilde{k}$, $\tilde{\mu} = \tilde{\mu}^* = u(\tilde{c})/\rho(\tilde{c})$, $\tilde{Z} = \tilde{Z}^* = u'(\tilde{c}) - \tilde{\mu}\rho'(\tilde{c})$, *and* $\tilde{z} = \tilde{z}^* = 0$, *where* $(\tilde{k}, \tilde{c})$ *is determined as the solution to*

$$c = i(k, 1, 0) \quad \textit{and} \quad k = \kappa(c, 0).$$

One of the two asymmetric steady state is $\tilde{a} = \tilde{k}/\theta$, $\tilde{a}^* = 0$, $\tilde{c} = w(\tilde{k}) + r(\tilde{k})\tilde{k}/\theta$, $\tilde{c}^* = w(\tilde{k})$, $\tilde{\mu} = u(\tilde{c})/\rho(\tilde{c})$, $\tilde{\mu}^* = u(\tilde{c}^*)/\rho(\tilde{c}^*)$, $\tilde{Z} = u'(\tilde{c}) - \tilde{\mu}\rho'(\tilde{c})$, $\tilde{Z}^* = u'(\tilde{c}^*) - \tilde{\mu}^*\rho'(\tilde{c}^*)$, $\tilde{z} = 0$, *and* $\tilde{z}^* = \tilde{Z}^*[\rho(\tilde{c}^*) - \rho(\tilde{c})] > 0$, *where* $(\tilde{k}, \tilde{c})$ *is determined as the solution to*

$$c = i(k, \theta, 0) \quad \textit{and} \quad k = \kappa(c, 0).$$

The other asymmetric steady state can be analogously derived from the intersection of

$$c^* = i(k, 1-\theta, 0) \quad \textit{and} \quad k = \kappa(c^*, 0).$$

10.3 Stability of the Symmetric Steady State

Differentiating equations (26a)–(26h) and setting $z = z^* = 0$, we obtain

$$\hat{J}(x) = \det[\hat{J} - xI]$$

$$= \det \begin{bmatrix} (w'+r'a)\theta+r-x & (w'+r'a)(1-\theta) & 0 & 0 & 0 & 0 & -1 & 0 \\ (w'+r'a^*)\theta & (w'+r'a^*)(1-\theta)+r-x & 0 & 0 & 0 & 0 & 0 & -1 \\ -Zr'\theta & -Zr'(1-\theta) & \rho-r-x & 0 & 0 & 0 & Z\rho' & 0 \\ -Z^*r'\theta & -Z^*r'(1-\theta) & 0 & \rho^*-r-x & 0 & 0 & 0 & Z^*\rho^{*\prime} \\ 0 & 0 & 0 & 0 & \rho-x & 0 & -Z & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^*-x & 0 & -Z^* \\ 0 & 0 & -1 & 0 & -\rho' & 0 & -G & 0 \\ 0 & 0 & 0 & -1 & 0 & -\rho^{*\prime} & 0 & -G^* \end{bmatrix}$$

$$= \det \begin{bmatrix} r-x & (w'+r'a)(1-\theta) & 0 & 0 & 0 & 0 & -1 & 0 \\ -\frac{\theta(r-x)}{1-\theta} & (w'+r'a^*)(1-\theta)+r-x & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -Zr'(1-\theta) & \rho-r-x & 0 & 0 & 0 & Z\rho' & 0 \\ 0 & -Z^*r'(1-\theta) & 0 & \rho^*-r-x & 0 & 0 & 0 & Z^*\rho^{*\prime} \\ 0 & 0 & 0 & 0 & \rho-x & 0 & -Z & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^*-x & 0 & -Z^* \\ 0 & 0 & -1 & 0 & -\rho' & 0 & -G & 0 \\ 0 & 0 & 0 & -1 & 0 & -\rho^{*\prime} & 0 & -G^* \end{bmatrix}. \quad (28)$$

Let us evaluate (28) at the symmetric steady state. Then, we obtain the Jacobian $\hat{J}$ for the dynamic system and the characteristic equation,

$$\hat{J}(x) = (x - \rho)\Theta(x)\Gamma(x, 1, 0),$$

where $\Theta(x) \equiv Mx^2 - M\rho x - Z\rho\rho'$ with $M > 0$. Then we have,

Proposition 9 *Under Assumption 1, the symmetric steady state uniquely exists but is unstable in the sense that the economy does not converge to it if the initial asset endowments differ for the two types of households.*

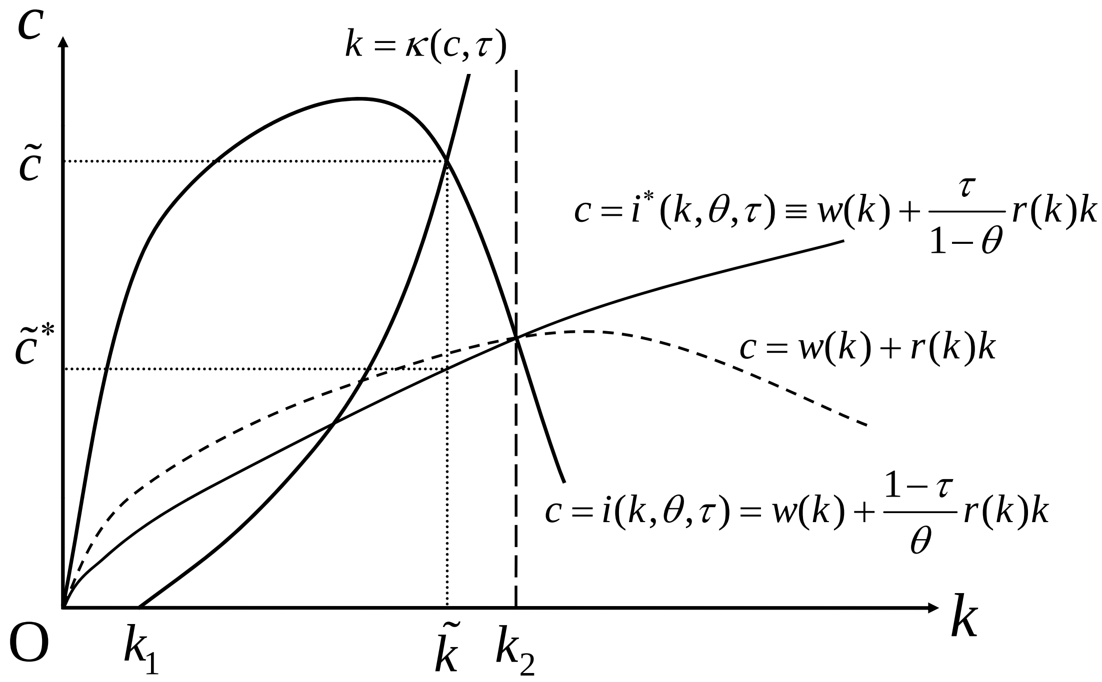
Proof. As shown in the proof of Lemma 2, $\Gamma(x, 1, 0) = 0$ has one negative root and two roots with positive real parts. Clearly, $\Theta(x) = 0$ has two roots with positive real parts since $\Theta(0) > 0$ due to DMI. And $\hat{J}(x) = 0$ has only one root with a negative real part when it is evaluated at the symmetric steady state. ■

Remark 2 *The Proposition implies the economy converges to a symmetric steady state only if households' initial asset holdings are the same.*

Remark 3 *Assuming IMI, $\rho' > 0$, we see that both $\Gamma(x, 1, 0) = 0$ and $\Theta(x) = 0$ have one negative root, and hence the symmetric steady state is a saddle point. Since rich households become less impatient than poor households under IMI, the poor will eventually catch up with the rich.*

From Lemma 6 and Proposition 9, it is clear that in the economy where each household is endowed with a different level of asset, no government policy can make all households own positive asset in the long run, unless the government equalizes the level of households' initial asset endowments. Also, as shown in Proposition 5, an economy without income inequality may not be desirable from a social point of view under DMI.

Figure 1



c
$k = \kappa(c, \tau)$
$\tilde{c}$
$c = i^*(k, \theta, \tau) \equiv w(k) + \frac{\tau}{1-\theta} r(k)k$
$\tilde{c}^*$
$c = w(k) + r(k)k$
$c = i(k, \theta, \tau) = w(k) + \frac{1-\tau}{\theta} r(k)k$
k
O
k_1
$\tilde{k}$
k_2