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"Investment for Patience in an Endogenous Growth Model"

Taketo Kawagishi

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Investment for patience in an endogenous growth model

Taketo Kawagishi*

Abstract

This paper explores a one-sector AK model in which time preference depends on private investment in future-oriented resources along the lines of Becker and Mulligan (1997). Assuming that time preference is also affected by the social level of such investment and that of consumption, we show that multiple balanced growth path (BGP henceforth) equilibria can exist, and provide the conditions for multiple BGP equilibria. Furthermore, we clarify that the equilibrium path is indeterminate in the high-growth BGP equilibrium, while it is determinate in the low-growth BGP equilibrium. We also discuss the effect of a subsidy policy to private investment in future-oriented resources on an endogenous growth rate.

Keywords: One-sector AK model; Endogenous time preference; Multiple balanced growth path equilibria; Subsidy policy

JEL classification: E32; H23; O40

*Graduate School of Economics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan.
E-mail address: kawagishi.taketo.23n@st.kyoto-u.ac.jp

1 Introduction

In economic modeling, most of the literature has used the constant time preference rate, which leads to intertemporal independence. Empirical studies, however, have often claimed that the assumption of time-additive preferences may be inadequate in practice.[1] From this aspect, the constant time preference rate is assumed for mathematical convenience rather than for innate plausibility. Because of this, intertemporally dependent preferences are introduced into economic modeling. Employing endogenous time preference is one way to cause intertemporal dependence.

The pioneering study on endogenous time preference in the neoclassical growth model is Uzawa (1968), which has been reconstructed by Epstein (1983, 1987). Uzawa (1968) and Epstein (1983, 1987) introduce endogenous time preference by assuming that the subjective discount rate is an increasing function of individual consumption (increasing marginal impatience). On the other hand, Becker and Mulligan (1997) assume that the subjective discount rate is a decreasing function of individual investment in future-oriented resources (decreasing marginal impatience).[2] Since the role of future-oriented resources is to distract the individual's attention from current pleasures to future ones, the individual who invests in future-oriented resources becomes more patient.

In contrast, this paper studies endogenous time preference in the one-sector AK model. Among existing studies, Meng (2006) is most closely related to the present study. He assumes that time preference depends on both average consumption and average income in the economy, and clarifies the case in which equilibrium indeterminacy arises.[3] This specification of time preference embodies in a simple and tractable way the spirit of Rae (1834), which saw culture as a critical determinant of differences in time preferences across various economies. The analysis of Meng (2006), however, is limited to the case where the individual time preference is socially determined. In fact, as a future research, he states that "in addition to the external factors, allowing also for dependence of the discount rate on variables taken as

[1]Since tests of the stochastic intertemporal Euler equations have produced strong statistical rejections under the assumption of time-additive preferences, a number of researches posit some form of intertemporal dependence in tastes (for example, Hayashi (1985), and Heaton (1993)).

[2]Typical examples of future-oriented resources are education, time to imagine the future, magazines, newspapers, the internet and so forth.

[3]Meng (2006) analyzes both the neoclassical growth model and the AK model with the socially determined individual time preference.

internal by the agent would be of more interest."

Taking into account this statement of Meng (2006), we assume that time preference (more precisely, the subjective discount rate) of this paper depends on a variable taken as internal as well as variables taken as external by the individual.[4] Specifically, as a variable taken as internal, we consider private investment in future-oriented resources.[5] This is along the lines of Becker and Mulligan (1997). As variables taken as external, on the other hand, we consider average investment in future-oriented resources in the economy at large (investment externalities) and average consumption in the economy at large (consumption externalities). The former is the extension of Becker-Mulligan type and captures the "spillover" effect of investment externalities. The latter, which is the same as in Meng (2006), reflects the modified Uzawa type and captures the "jealousy (admiration)" effect of consumption externalities.[6]

Under this setting, we examine whether the introduction of private investment for patience (the introduction of the variable taken as internal by the individual) causes a qualitative change compared to Meng (2006), which shows that there exists at most one balanced growth path (BGP henceforth) equilibrium in the AK model in which the subjective discount rate is a function of the ratio of average consumption to average income.[7] In the present study, we regard the qualitative change as the emergence of multiple BGP equilibria, and focus on whether the introduction of private investment for patience generates multiple BGP equilibria.[8]

As a result of the analysis, we show that multiple BGP equilibria can exist, and

[4]Regarding the instantaneous utility function, we assume the CIES (constant intertemporal elasticity of substitution) type.

[5]In this paper, "private investment in future-oriented resources" is also called "private investment for patience."

[6]Concerning the modified Uzawa type, see Bian and Meng (2004).

[7]Specifically, Meng (2006) shows that there exists at most one BGP equilibrium when the subjective discount rate (ρ) takes the following form: $\rho = q(C/Y) + p$, where C is average consumption and Y is average income (p and q are parameters).

[8]Palivos, Wang and Zhang (1997) provide necessary and sufficient conditions for the existence of balanced growth and asymptotically balanced growth paths in the AK model with the Uzawa-type time preference. In addition, as an example, they also consider time preference which depends on the ratio of private consumption to average capital in the economy (i.e. time preference is a function of both the variable taken as internal and that taken as external by the individual). However, they do not show that multiple BGP equilibria exist.

provide the conditions which ensure multiple BGP equilibria. In addition, in terms of a phase diagram, we clarify that the equilibrium path is indeterminate in the high-growth BGP equilibrium, while it is determinate in the low-growth BGP equilibrium. Thus, we find that there is a significant difference between the result of Meng (2006) and that of this paper. As mentioned above, in Meng (2006), there exists at most one BGP equilibrium (see also footnote 7). In this paper, on the other hand, there exist multiple BGP equilibria under certain conditions. Therefore, it turns out that the introduction of private investment for patience — the introduction of the variable taken as internal by the individual — plays a key role in this difference.

We then introduce a subsidy policy to private investment in future-oriented resources and examine the effect of this policy on an endogenous growth rate. In particular, we again focus on the case in which there exist multiple BGP equilibria and show that a change in the subsidy rate and an endogenous growth rate are negatively correlated in the high-growth BGP equilibrium, while they are positively correlated in the low-growth BGP equilibrium.

This paper is organized as follows. Section 2 explains endogenous time preference in this paper in detail, constructs a one-sector AK model with endogenous time preference, and derives the complete dynamic system. Section 3 provides the conditions for multiple BGP equilibria and examines the stability in terms of a phase diagram. Section 4 discusses a subsidy policy to private investment in future-oriented resources. The conclusion of this paper is presented in Section 5.

2 Model

2.1 Setup

In this paper, we consider a continuous-time, infinite-horizon, and one-sector growth model with inelastic labor supply. For simplicity, the total size of population is constant and normalized to unity. Furthermore, we employ the following production technology:

$$y(t) = Ak(t), \tag{1}$$

where A is a positive constant which reflects the level of the technology, y is per capita output, and k is per capita capital.

We assume that the economy admits a representative agent with preferences:

$$\int_0^\infty u(c(t)) \exp\{-z(t)\}\, dt. \tag{2}$$

The instantaneous utility function $u(\cdot)$ depends on private consumption, c. We assume that $u(\cdot)$ is twice continuously differentiable, strictly increasing, and strictly concave with respect to c. Moreover, we impose the Inada conditions on $u(\cdot)$: $\lim_{c\to 0} u'(c) = \infty$, $\lim_{c\to\infty} u'(c) = 0$.

In the objective function (2), z represents the endogenous time-preference rate, which is defined as follows:

$$z(t) \equiv \int_0^t \theta(\nu) d\nu,$$

where θ is the subjective discount rate. As mentioned in the Introduction, in this paper, we assume that the subjective discount rate depends not only on variables taken as external but also on a variable taken as internal by the individual. More specifically, the subjective discount rate takes the form

$$\theta(\nu) \equiv \theta\big(s(\nu), \bar{s}(\nu), \bar{c}(\nu)\big),$$

where s is private investment in future-oriented resources, $\bar{s}$ is average investment in future-oriented resources in the economy at large (investment externalities), and $\bar{c}$ is average consumption in the economy at large (consumption externalities).[9] We assume that $\theta(\cdot)$ satisfies the following conditions along the lines of Becker and Mulligan (1997) (in what follows, for readability, we omit ν):

$$\theta\big(s, \bar{s}, \bar{c}\big) > 0 \ , \quad \theta_1\big(s, \bar{s}, \bar{c}\big) < 0 \ , \quad \theta_{11}\big(s, \bar{s}, \bar{c}\big) > 0, \tag{3}$$

where subscripts of the subjective discount function indicate its partial derivatives with respect to the corresponding variables. Furthermore, we assume that

$$\theta_2\big(s, \bar{s}, \bar{c}\big) < 0. \tag{4}$$

Under (4), average investment in future-oriented resources captures the "spillover" effect of investment externalities because an increase in average investment in future-oriented resources raises the individual's lifetime utility.[10] Finally, we assume that

[9]We can assume that the subjective discount rate depends on private consumption as well. However, in this case, we will need quite complicated additional conditions to ensure that the Hamiltonian is concave. Thus, it will be quite difficult to conduct the analysis which satisfies such conditions. In addition, even if we introduce private consumption into time preference, the result will not change substantially compared to the result of this paper.

[10]For example, an increase in average investment in future-oriented resources (investment externalities) means that other agents get more information about the future. Under this circumstance, interactions with other agents may enhance the agent's knowledge about the future, which in turn lowers the agent's subjective discount rate and leads to an increase in the agent's lifetime utility.

the modified Uzawa type is also incorporated into the subjective discount function. Specifically, $\theta(\cdot)$ is an increasing function of average consumption in the economy:

$$\theta_3\big(s, \bar{s}, \bar{c}\big) > 0. \tag{5}$$

This assumption implies that a higher level of average consumption at time ν increases the subjective discount rate to the individual's utility at and after time ν. Thus, average consumption in the subjective discount rate captures the "jealousy (resp. admiration)" effect of consumption externalities if the instantaneous utility is positive (resp. negative), because an increase in average consumption reduces (resp. increases) the individual's lifetime utility.

In what follows, we consider the following instantaneous utility function:

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma}. \tag{6}$$

As the additional assumption, the instantaneous utility function $u(\cdot)$ is restricted to be nonnegative to ensure that an increase in private investment in future-oriented resources has a positive effect on the individual's lifetime utility under (3). Thus, we assume that $0 < \sigma < 1$.[11] Concerning the subjective discount function, we assume that

$$\theta\big(s(t), \bar{s}(t), \bar{c}(t)\big) = \beta\left(\frac{\bar{c}(t)}{s(t)^{1-\zeta}\bar{s}(t)^{\zeta}}\right) + \gamma, \tag{7}$$

where $\beta > 0$, $\gamma > 0$, and $0 < \zeta < 1$ due to (3) and (4).

2.2 Dynamic system

From the settings in Section 2.1, we can describe the maximization problem as follows:

$$\max \int_0^\infty \frac{c(t)^{1-\sigma}}{1-\sigma} e^{-z(t)} dt, \tag{8}$$

$$\text{s.t.} \quad \dot{k}(t) = Ak(t) - \delta k(t) - c(t) - s(t)\ , \quad \text{given } k(0) > 0, \tag{9}$$

$$\dot{z}(t) = \beta\left(\frac{\bar{c}(t)}{s(t)^{1-\zeta}\bar{s}(t)^{\zeta}}\right) + \gamma\ , \quad z(0) = 0. \tag{10}$$

This maximization problem is formulated as a pseudo planning that mimics the behavior of the competitive economy. In this problem, the planner is assumed to take the sequences of external effects, $\{\bar{c}(t), \bar{s}(t)\}_{t=0}^{\infty}$, as given. In the resource constraint

[11] Note that preferences exhibit "jealousy" under $0 < \sigma < 1$.

(9), $\delta \in (0,1)$ is the depreciation rate of capital. The resource constraint (9) implies that output is used for consumption, future-oriented resources and investment. Note that the price of future-oriented resources is normalized to one.

To derive the necessary conditions for an optimum, let us first set up the Hamiltonian, which takes the form

$$H = \frac{c(t)^{1-\sigma}}{1-\sigma} e^{-z(t)} + \lambda(t)\big[Ak(t) - \delta k(t) - c(t) - s(t)\big] - \mu(t)\left[\beta\left(\frac{\bar{c}(t)}{s(t)^{1-\zeta}\bar{s}(t)^{\zeta}}\right) + \gamma\right],$$

where $\lambda(t) > 0$ and $\mu(t) > 0$ are costate variables. In what follows, we drop time index from the endogenous variables. The first-order conditions are

$$c^{-\sigma} e^{-z} = \lambda, \tag{11a}$$

$$\mu\beta(1-\zeta)s^{\zeta-2}\bar{s}^{-\zeta}\bar{c} = \lambda, \tag{11b}$$

$$\lambda(A-\delta) = -\dot{\lambda}, \tag{11c}$$

$$\frac{c^{1-\sigma}}{1-\sigma} e^{-z} = -\dot{\mu}. \tag{11d}$$

The transversality conditions are given by

$$\lim_{t\to\infty} \lambda(t)k(t) = 0, \tag{12a}$$

$$\lim_{t\to\infty} \mu(t)z(t) = 0. \tag{12b}$$

Since it is assumed that the total size of population is constant and normalized to unity, the following conditions hold in equilibrium:

$$c = \bar{c}\,, \quad s = \bar{s}. \tag{13}$$

Using (9), (10), (11a)-(11d), and (13), we obtain the following dynamic system after some manipulation:[12]

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left\{(A-\delta) - \beta\left(\frac{c}{s}\right) - \gamma\right\}, \tag{14a}$$

$$\frac{\dot{s}}{s} = \frac{A-\delta}{2} - \frac{\beta(1-\zeta)}{2(1-\sigma)}\left(\frac{c}{s}\right)^2 + \frac{1}{2}\left(\frac{\dot{c}}{c}\right), \tag{14b}$$

$$\frac{\dot{k}}{k} = (A-\delta) - \left(\frac{c}{s}\cdot\frac{s}{k}\right) - \frac{s}{k}. \tag{14c}$$

[12] Concerning the derivations of (14a)-(14c), see Appendix.

We now introduce new variables, ψ and χ, which are defined as

$$\psi \equiv \frac{c}{s}\ ,\quad \chi \equiv \frac{s}{k}.$$

Since $\dot{\psi}/\psi = \dot{c}/c - \dot{s}/s$ and $\dot{\chi}/\chi = \dot{s}/s - \dot{k}/k$, from (14a)-(14c), we obtain the following complete dynamic system in terms of ψ and χ:

$$\frac{\dot{\psi}}{\psi} = \frac{\beta(1-\zeta)}{2(1-\sigma)}\psi^2 - \frac{\beta}{2\sigma}\psi + \frac{(A-\delta)(1-\sigma)-\gamma}{2\sigma} \equiv g(\psi), \tag{15a}$$

$$\frac{\dot{\chi}}{\chi} = (1+\psi)\chi - \frac{\beta(1-\zeta)}{1-\sigma}\psi^2 + \frac{\dot{\psi}}{\psi}. \tag{15b}$$

Note that the dynamic equation (15a) depends only on ψ, while the dynamic equation (15b) is affected by both ψ and χ.

3 BGP equilibrium and stability

In this section, we first show that there is the possibility of multiple BGP equilibria. Then, we provide the conditions which ensure multiple BGP equilibria. We finally analyze the stability of each BGP equilibrium in terms of a phase diagram.

3.1 BGP equilibrium

We consider a BGP equilibrium, which corresponds to $\dot{\psi} = 0$ and $\dot{\chi} = 0$. From (15a) and (15b), $\dot{\psi} = 0$ and $\dot{\chi} = 0$ immediately imply that

$$g(\psi) = 0, \tag{16a}$$

$$\chi = \left\{\frac{\beta(1-\zeta)}{1-\sigma}\right\}\frac{\psi^2}{1+\psi}. \tag{16b}$$

We define ψ_a as follows:

$$\psi_a \equiv \frac{1-\sigma}{2\sigma(1-\zeta)}\ (>0), \tag{17}$$

where ψ_a implies the axis of the graph of $g(\psi)$. The $\dot{\psi} = 0$ loci are given by the solutions of (16a):[13]

$$\psi_1 = \frac{1-\sigma}{\beta(1-\zeta)}\left\{\frac{\beta}{2\sigma} - \sqrt{\frac{\beta^2}{4\sigma^2} - \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{(1-\sigma)\sigma}}\right\}, \tag{18a}$$

$$\psi_2 = \frac{1-\sigma}{\beta(1-\zeta)}\left\{\frac{\beta}{2\sigma} + \sqrt{\frac{\beta^2}{4\sigma^2} - \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{(1-\sigma)\sigma}}\right\}. \tag{18b}$$

In order to ensure that ψ_1 and ψ_2 (the solutions of $g(\psi) = 0$) are real, we assume that the discriminant of $g(\psi) = 0$ is positive. The following inequality guarantees this:

$$\frac{1}{\beta}\left(A - \delta - \frac{\gamma}{1-\sigma}\right) < \frac{1}{4\sigma(1-\zeta)} \equiv \Delta. \tag{19}$$

In addition, to ensure that ψ_1 (the smaller solution of $g(\psi) = 0$) is positive, we assume that

$$A - \delta - \frac{\gamma}{1-\sigma} > 0. \tag{20}$$

Under (19) and (20), the $\dot{\psi} = 0$ loci ($\psi = \psi_1$ and $\psi = \psi_2$) are shown in Fig. 1.

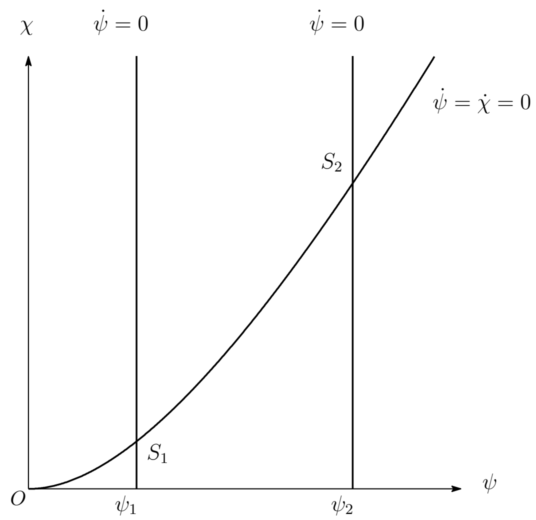

Fig. 1. $\dot{\psi} = 0$ loci and $\dot{\psi} = \dot{\chi} = 0$ locus

[13]Although $\psi = 0$ is also one of the $\dot{\psi} = 0$ loci, we ignore $\psi = 0$ because it is irrelevant to the analysis.

The expression (16b) implies the $\dot{\psi} = \dot{\chi} = 0$ locus. Thus, as shown in Fig. 1, a BGP equilibrium is determined by the intersection of (16b) with (18a) as well as that of (16b) with (18b). Therefore, we find that multiple BGP equilibria can exist.

3.2 Conditions for multiple BGP equilibria

In this subsection, we examine the conditions which ensure multiple BGP equilibria. We first assume that an endogenous growth rate is positive, which, from (14a), is guaranteed by

$$\psi < \frac{A - \delta - \gamma}{\beta} \equiv D. \tag{21}$$

Note that the domain of ψ is given by $(0, D)$. If this domain includes ψ_1 and ψ_2, then multiple BGP equilibria emerge. Thus, in order for multiple BGP equilibria to exist, the following inequality needs to be satisfied:

$$0 < \psi_1 < \psi_a < \psi_2 < D < \Delta, \tag{22}$$

where ψ_a, ψ_1, ψ_2, Δ and D are given in (17), (18a), (18b), (19) and (21), respectively. In what follows, we investigate the conditions which ensure (22). Before moving on to the analysis, we make the following assumption.

Assumption. *We impose the following constraints on σ and $(A - \delta)/\beta$:*

$$\frac{3}{4} < \sigma < 1, \tag{23}$$

$$\frac{1-\sigma}{1-\zeta} < \frac{A-\delta}{\beta} < \min\left\{ \frac{1}{4\sigma(1-\zeta)} \, , \, \frac{1-\sigma}{\sigma^2(1-\zeta)} \right\}. \tag{24}$$

Both (23) and (24) are needed for (22) to hold.[14] The roles of (23) and (24) are explained in the following analysis.

[14] Concerning (24), we find that

$$\frac{1}{4\sigma(1-\zeta)} < \frac{1-\sigma}{\sigma^2(1-\zeta)} \quad \text{if } \frac{3}{4} < \sigma < \frac{4}{5},$$

$$\frac{1}{4\sigma(1-\zeta)} = \frac{1-\sigma}{\sigma^2(1-\zeta)} \quad \text{if } \sigma = \frac{4}{5},$$

$$\frac{1}{4\sigma(1-\zeta)} > \frac{1-\sigma}{\sigma^2(1-\zeta)} \quad \text{if } \frac{4}{5} < \sigma < 1.$$

In addition, we refer to $(1-\sigma)/(1-\zeta) < (A-\delta)/\beta$ and $(A-\delta)/\beta < \min\{\cdot, \cdot\}$ as the first inequality and the second inequality, respectively.

Let us examine the conditions which ensure (22). From (23), the inequality,

$$\frac{1}{\beta}\left(A-\delta-\frac{\gamma}{1-\sigma}\right)<\frac{1}{\beta}(A-\delta-\gamma)\ \ (=D), \tag{25}$$

always holds.[15] In addition, (23) also guarantees that $\psi_2 < \Delta$. From the second inequality in (24), we see that

$$(D=)\ \frac{1}{\beta}(A-\delta-\gamma)<\frac{1}{4\sigma(1-\zeta)}\ \ (=\Delta). \tag{26}$$

Thus, from (25) and (26), (19) automatically holds. At this stage, under (20), (23) and (24), the following inequalities hold:

$$0<\psi_1<\psi_a<\psi_2<\Delta \ \ \text{and} \ \ D<\Delta.$$

Thus, we find that if $\psi_2 < D$, then (22) holds. Here, we assume that

$$(\psi_a=)\ \frac{1-\sigma}{2\sigma(1-\zeta)}<\frac{A-\delta-\gamma}{\beta}\ \ (=D). \tag{27}$$

Then the condition which guarantees that $\psi_2 < D$ is given by

$$\frac{A-\delta-\gamma}{\beta}>\sqrt{\left(\frac{1-\sigma}{1-\zeta}\right)\left(\frac{A-\delta}{\beta}\right)}. \tag{28}$$

Note that, from (23) and the first inequality in (24),

$$\frac{1-\sigma}{2\sigma(1-\zeta)}<\sqrt{\left(\frac{1-\sigma}{1-\zeta}\right)\left(\frac{A-\delta}{\beta}\right)}$$

always holds, so that (27) automatically holds when we assume (28). Therefore, we find that (22) holds under (20), (23), (24) and (28).

Finally, we write in a more concise way the conditions which ensure (22). From (20), we have

$$\gamma<(1-\sigma)(A-\delta).$$

Furthermore, from (28), we obtain

$$\gamma<\beta\left\{\frac{A-\delta}{\beta}-\sqrt{\left(\frac{1-\sigma}{1-\zeta}\right)\left(\frac{A-\delta}{\beta}\right)}\right\}. \tag{29}$$

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[15] Note that $D>0$ under (20) and (25).

Under the second inequality in (24), the following inequality holds:

$$\beta\left\{\frac{A-\delta}{\beta}-\sqrt{\left(\frac{1-\sigma}{1-\zeta}\right)\left(\frac{A-\delta}{\beta}\right)}\right\}<(1-\sigma)(A-\delta).$$

Thus, when we assume (29), both (20) and (28) are satisfied. Therefore, (23), (24) and (29) ensure (22). Summarizing the above analysis, we obtain the following proposition.

Proposition 1. *In the AK model, suppose that the instantaneous utility function and the subjective discount function are specified as (6) and (7), respectively. Then multiple BGP equilibria exist when the following conditions are satisfied:*

· *if* $3/4<\sigma<4/5$,

$$\frac{1-\sigma}{1-\zeta}<\frac{A-\delta}{\beta}<\frac{1}{4\sigma(1-\zeta)},$$

$$0<\gamma<\beta\left\{\frac{A-\delta}{\beta}-\sqrt{\left(\frac{1-\sigma}{1-\zeta}\right)\left(\frac{A-\delta}{\beta}\right)}\right\};$$

· *if* $4/5\leq\sigma<1$,

$$\frac{1-\sigma}{1-\zeta}<\frac{A-\delta}{\beta}<\frac{1-\sigma}{\sigma^2(1-\zeta)},$$

$$0<\gamma<\beta\left\{\frac{A-\delta}{\beta}-\sqrt{\left(\frac{1-\sigma}{1-\zeta}\right)\left(\frac{A-\delta}{\beta}\right)}\right\}.$$

3.3 Stability

Let us assume (23), (24) and (29) and consider the case where multiple BGP equilibria exist. In this subsection, we analyze the stability of each BGP equilibrium in terms of a phase diagram.

We first examine the dynamics of ψ. Under (19) and (20), the graph of $g(\psi)$ is shown in Fig. 2. From Fig. 2, we find that

$$g(\psi)>0 \quad \text{if} \quad \psi<\psi_1\ ,\quad \psi_2<\psi\ ;$$
$$g(\psi)<0 \quad \text{if} \quad \psi_1<\psi<\psi_2\ .$$

Thus, focusing on (15a), we see that

$$\dot{\psi}>0 \quad \text{if} \quad \psi<\psi_1\ ,\quad \psi_2<\psi\ ;$$
$$\dot{\psi}<0 \quad \text{if} \quad \psi_1<\psi<\psi_2\ .$$

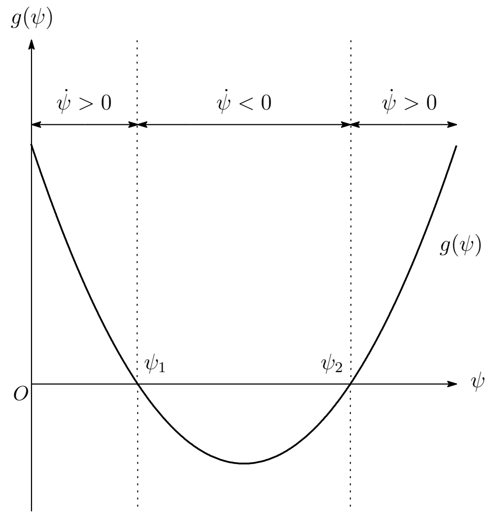

Fig. 2. Dynamics of ψ

We next consider the $\dot{\chi} = 0$ locus and the dynamics of χ. From (15b) and $\dot{\chi} = 0$, the $\dot{\chi} = 0$ locus is given as follows:[16]

$$\chi = \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} - \frac{\dot{\psi}}{\psi}(1+\psi)^{-1}. \tag{30}$$

Furthermore, as for the $\dot{\chi} = 0$ locus, the following expressions hold:

$$\chi = \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} - \frac{\dot{\psi}}{\psi}(1+\psi)^{-1} < \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} \quad \text{if} \ \ \dot{\psi} > 0 \ ;$$
$$\chi = \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} - \frac{\dot{\psi}}{\psi}(1+\psi)^{-1} > \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} \quad \text{if} \ \ \dot{\psi} < 0 \ .$$

In other words, the $\dot{\chi} = 0$ locus lies below (16b) in the area $\dot{\psi} > 0$, while it lies above (16b) in the area $\dot{\psi} < 0$. Concerning the dynamics of χ, we find that

$$\dot{\chi} > 0 \quad \text{if} \ \ \chi > \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} - \frac{\dot{\psi}}{\psi}(1+\psi)^{-1} \ ;$$
$$\dot{\chi} < 0 \quad \text{if} \ \ \chi < \left\{ \frac{\beta(1-\zeta)}{1-\sigma} \right\} \frac{\psi^2}{1+\psi} - \frac{\dot{\psi}}{\psi}(1+\psi)^{-1} \ .$$

[16] Although $\chi = 0$ is also the $\dot{\chi} = 0$ locus, we ignore $\chi = 0$ because it is irrelevant to the analysis.

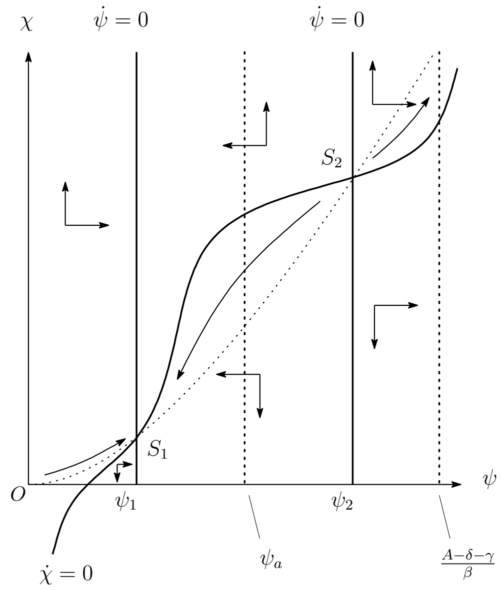

Fig. 3. Phase diagram

From the above analysis, under (23), (24) and (29), we obtain the phase diagram shown in Fig. 3. Noting that both ψ and χ are jumpable variables, we find that the equilibrium path is indeterminate in the high-growth BGP equilibrium (S_1), while it is determinate in the low-growth BGP equilibrium (S_2).

Proposition 2. *In the AK model, suppose that the instantaneous utility function and the subjective discount function are specified as (6) and (7), respectively, and that (23), (24) and (29) are satisfied. Then there exist multiple BGP equilibria, and the equilibrium path is indeterminate in the high-growth BGP equilibrium, while it is determinate in the low-growth BGP equilibrium.*

There is a significant difference between the result of Meng (2006) and that of this paper. Meng (2006) shows that there exists at most one BGP equilibrium in the AK model when the subjective discount rate (ρ) takes the following form:

$$\rho = q\left(\frac{C}{Y}\right) + p, \tag{31}$$

where C is average consumption, Y is average income, and both p and q are parameters (i.e. the subjective discount rate depends only on the variables taken as external by the individual). On the other hand, in this paper's setting, there exist multiple BGP equilibria under (23), (24) and (29).

We consider what causes such a difference between Meng's result and this paper's result. In this paper, we introduce private investment in future-oriented resources (the variable s) and assume that this endogenous variable affects time preference. Focusing on (14b), we find that introducing the variable s yields the term in $(c/s)^2$ (the second term on the right-hand side of (14b)).[17] Because of this quadratic term, multiple BGP equilibria emerge. On the other hand, under Meng's formulation (31), a quadratic term like $(c/s)^2$ in this paper does not exist. Thus, it turns out that the introduction of private investment in future-oriented resources — the introduction of the variable taken as internal by the individual — plays a key role in the difference between the result of Meng (2006) and that of this paper.[18]

4 Subsidy policy

In this section, we introduce a subsidy policy to private investment in future-oriented resources (e.g. education) and discuss the relationship between this policy and an endogenous growth rate. Suppose that the government levies lump-sum taxes on households and subsidizes their investments in future-oriented resources.

The budget constraint of a representative household becomes as follows:

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t) - (1-\tau)s(t) - T(t)\ , \quad \text{given } a(0) > 0, \qquad (32)$$

where a denotes the assets of the representative household, r is the rate of return on assets, and w is the wage earnings of the representative household. Furthermore, τ is the subsidy rate and T is the lump-sum tax. We assume that $0 < \tau < 1$ and that $T > 0$. The representative household maximizes (8) subject to (10) and (32).

In addition to the utility maximization conditions of the representative household, from (i) the profit maximization conditions of the representative firm ($r = A - \delta$ and $w = 0$), (ii) the government budget constraint ($T = \tau s$), and (iii) the market clearing condition ($a = k$), we obtain the following dynamic system in the same way

[17] If we do not introduce the endogenous variable s, then the term in $(c/s)^2$ does not arise.

[18] Note that we can obtain Proposition 1 and 2 even if there are no investment externalities ($\zeta = 0$).

as in Section 2:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left\{(A-\delta) - \beta\left(\frac{c}{s}\right) - \gamma\right\}, \tag{33a}$$

$$\frac{\dot{s}}{s} = \frac{A-\delta}{2} - \frac{\beta(1-\zeta)}{2(1-\tau)(1-\sigma)}\left(\frac{c}{s}\right)^2 + \frac{1}{2}\left(\frac{\dot{c}}{c}\right), \tag{33b}$$

$$\frac{\dot{k}}{k} = (A-\delta) - \left(\frac{c}{s}\cdot\frac{s}{k}\right) - \frac{s}{k}. \tag{33c}$$

Thus, from (33a)-(33c), the complete dynamic system finally becomes

$$\frac{\dot{\psi}}{\psi} = \frac{\beta(1-\zeta)}{2(1-\tau)(1-\sigma)}\psi^2 - \frac{\beta}{2\sigma}\psi + \frac{(A-\delta)(1-\sigma)-\gamma}{2\sigma} \equiv \tilde{g}(\psi), \tag{34a}$$

$$\frac{\dot{\chi}}{\chi} = (1+\psi)\chi - \frac{\beta(1-\zeta)}{(1-\tau)(1-\sigma)}\psi^2 + \frac{\dot{\psi}}{\psi}. \tag{34b}$$

Both ψ and χ are the same definitions as in Section 2. Let us denote the solutions of $\tilde{g}(\psi) = 0$ by $\tilde{\psi}_1$ and $\tilde{\psi}_2$:

$$\tilde{\psi}_1 = \frac{(1-\tau)(1-\sigma)}{\beta(1-\zeta)}\left\{\frac{\beta}{2\sigma} - \sqrt{\frac{\beta^2}{4\sigma^2} - \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{(1-\tau)(1-\sigma)\sigma}}\right\}, \tag{35a}$$

$$\tilde{\psi}_2 = \frac{(1-\tau)(1-\sigma)}{\beta(1-\zeta)}\left\{\frac{\beta}{2\sigma} + \sqrt{\frac{\beta^2}{4\sigma^2} - \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{(1-\tau)(1-\sigma)\sigma}}\right\}. \tag{35b}$$

We assume that the discriminant of $\tilde{g}(\psi) = 0$, denoted by d, is positive, and that $\tilde{\psi}_1 > 0$. In addition, we again focus on the case where multiple BGP equilibria exist.[19]

In what follows, we examine the effect of a change in the subsidy rate on an endogenous growth rate. From (33a), the endogenous growth rate is given by

$$g_i = \frac{1}{\sigma}\left\{(A-\delta) - \beta\tilde{\psi}_i - \gamma\right\}, \quad (i = 1, 2). \tag{36}$$

We assume that $g_i > 0$. Differentiating g_i with respect to τ, we have

$$\frac{\partial g_i}{\partial \tau} = -\frac{\beta}{\sigma}\frac{\partial \tilde{\psi}_i}{\partial \tau}. \tag{37}$$

[19] We omit the detailed analysis regarding the conditions for multiple BGP equilibria and the stability. We can analyze them in the same way as in Section 3.

Concerning $\partial\tilde{\psi}_i/\partial\tau$, we obtain

$$\frac{\partial\tilde{\psi}_1}{\partial\tau} = -\frac{1-\sigma}{\beta(1-\zeta)}\left[\frac{\beta}{2\sigma} - d^{-\frac{1}{2}}\left\{d + \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{2(1-\tau)(1-\sigma)\sigma}\right\}\right] > 0, \quad (38a)$$

$$\frac{\partial\tilde{\psi}_2}{\partial\tau} = -\frac{1-\sigma}{\beta(1-\zeta)}\left[\frac{\beta}{2\sigma} + d^{-\frac{1}{2}}\left\{d + \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{2(1-\tau)(1-\sigma)\sigma}\right\}\right] < 0, \quad (38b)$$

where

$$d \equiv \frac{\beta^2}{4\sigma^2} - \frac{\beta(1-\zeta)\{(A-\delta)(1-\sigma)-\gamma\}}{(1-\tau)(1-\sigma)\sigma} > 0.$$

Therefore, from (37), regarding the effect of a change in the subsidy rate on the endogenous growth rate g_i, we find that

$$\frac{\partial g_1}{\partial\tau} < 0\ , \quad \frac{\partial g_2}{\partial\tau} > 0.$$

Thus, a change in the subsidy rate and an endogenous growth rate are negatively correlated in the high-growth BGP equilibrium, while they are positively correlated in the low-growth BGP equilibrium.

Proposition 3. *In the AK model which has the instantaneous utility function and the subjective discount function specified as (6) and (7), respectively, suppose that the government levies lump-sum taxes on households and subsidizes their investments in future-oriented resources. Then under multiple BGP equilibria, a change in the subsidy rate and an endogenous growth rate are negatively correlated in the high-growth BGP equilibrium, while they are positively correlated in the low-growth BGP equilibrium.*

The intuition is as follows. Focusing on (34a), we obtain a graph like Fig. 2, so that, in terms of ψ, $\tilde{\psi}_1$ is stable and $\tilde{\psi}_2$ is unstable.[20] Here, under the linearized system around a BGP equilibrium, the relationship between $\dot{c}/c$ and $\dot{s}/s$ is illustrated in both Fig. 4 and Fig. 5, in which $\dot{c}/c$ and $\dot{s}/s$ are represented by CC and SS, respectively. In Fig. 4, which shows the high-growth BGP equilibrium, if the value of ψ is larger (resp. smaller) than $\tilde{\psi}_1$, then ψ converges to $\tilde{\psi}_1$ because $\dot{c}/c$ is smaller

[20] Note that, in terms of the complete dynamic system, the equilibrium path is indeterminate in the high-growth BGP equilibrium $(\tilde{\psi}_1, \tilde{\chi}_1)$, while it is determinate in the low-growth BGP equilibrium $(\tilde{\psi}_2, \tilde{\chi}_2)$.

(resp. greater) than $\dot{s}/s$. However, in Fig. 5, which represents the low-growth BGP equilibrium, if the value of ψ is larger (resp. smaller) than $\tilde{\psi}_2$, then ψ does not converge to $\tilde{\psi}_2$ since $\dot{c}/c$ is greater (resp. smaller) than $\dot{s}/s$. Thus, Fig. 4 and Fig. 5 are consistent with the stability of $\tilde{\psi}_1$ and that of $\tilde{\psi}_2$ mentioned at the beginning of this paragraph, respectively.

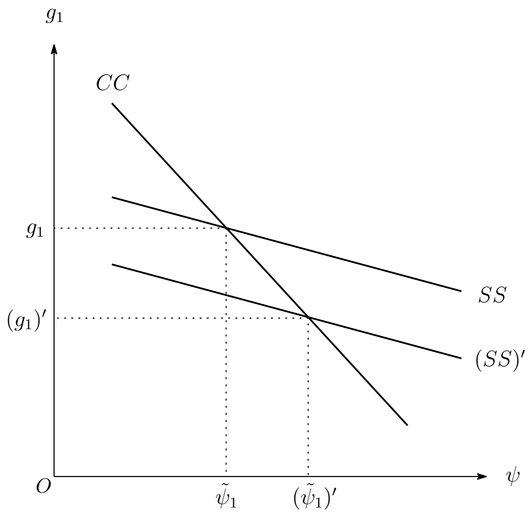

Fig. 4. The high-growth BGP equilibrium

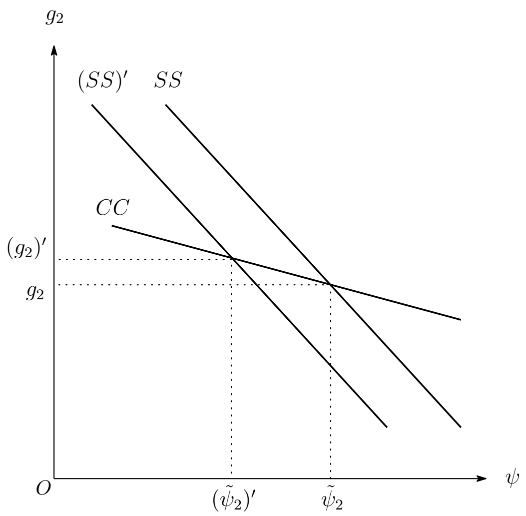

Fig. 5. The low-growth BGP equilibrium

Suppose that the economy is in a BGP equilibrium initially, and that the government raises the subsidy rate (τ). In this case, a rise in the subsidy rate shifts the lines of $\dot{s}/s$ in Fig. 4 and Fig. 5 from SS to $(SS)'$. This is because only $\dot{s}/s$ depends on the value of τ, which has a negative impact on $\dot{s}/s$ (see (33b)). In the high-growth BGP equilibrium, $\dot{c}/c > \dot{s}/s$ holds immediately after a rise in τ, so that the value of ψ $(= c/s)$ increases to $(\tilde{\psi}_1)'$.[21] This implies that an individual becomes less patient.[22] Because of this, savings are depressed, and the endogenous growth rate declines to $(g_1)'$. In the low-growth BGP equilibrium, on the other hand, immediately after a rise in τ, the value of ψ jumps to $(\tilde{\psi}_2)'$ because the low-growth BGP equilibrium is unstable in terms of ψ. Hence, an individual becomes more patient, so that savings increase and the endogenous growth rate rises to $(g_2)'$.

5 Conclusion

This paper has explored a one-sector AK model with endogenous time preference which depends not only on variables taken as external but also on a variable taken as internal by the individual. More specifically, we have assumed that time preference hinges on the following three factors: private investment in future-oriented resources; average investment in future-oriented resources in the economy at large (investment externalities); average consumption in the economy at large (consumption externalities).

Under this assumption, we have provided the conditions for multiple BGP equilibria and have shown that the equilibrium path is indeterminate in the high-growth BGP equilibrium, while it is determinate in the low-growth BGP equilibrium. Furthermore, we have then introduced a subsidy policy to private investment in future-oriented resources, and have clarified that a change in the subsidy rate and an endogenous growth rate are negatively correlated in the high-growth BGP equilibrium, while they are positively correlated in the low-growth BGP equilibrium.

In this paper, we have taken the result of Meng (2006) into consideration and have focused on whether the qualitative change — the emergence of multiple BGP equilibria — arises. Thus, from the result of this study, it has turned out that the introduction of private investment in future-oriented resources (the introduction of the variable taken as internal by the individual) plays a key role in generating

[21]Note that the high-growth BGP equilibrium is stable in terms of ψ.

[22]Recall that the subjective discount rate is an increasing function of ψ in equilibrium.

multiple BGP equilibria.

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Appendix

Derivations of (14a), (14b) and (14c)

We first derive (14a). Dividing time differentiation of (11a) by (11a) and using (10) and (13), we have

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left\{-\frac{\dot{\lambda}}{\lambda} - \beta\left(\frac{c}{s}\right) - \gamma\right\}. \tag{39}$$

In addition, from (11c), we obtain

$$\frac{\dot{\lambda}}{\lambda} = -(A-\delta). \tag{40}$$

Thus, substituting (40) into (39) yields (14a). Let us move on to the derivation of (14b). Applying (13) to (11b) and dividing time differentiation of (11b) by (11b), we see that

$$\frac{\dot{s}}{s} = -\frac{1}{2}\left(\frac{\dot{\lambda}}{\lambda}\right) + \frac{1}{2}\left(\frac{\dot{\mu}}{\mu}\right) + \frac{1}{2}\left(\frac{\dot{c}}{c}\right). \tag{41}$$

From (11a), (11b) and (13), we have

$$\mu = \frac{c^{-\sigma}e^{-z}}{\beta(1-\zeta)cs^{-2}}. \tag{42}$$

Furthermore, from (11d) and (42), $\dot{\mu}/\mu$ is given by

$$\frac{\dot{\mu}}{\mu} = -\frac{\beta(1-\zeta)}{1-\sigma}\left(\frac{c}{s}\right)^2. \tag{43}$$

Therefore, substituting (40) and (43) into (41), we obtain (14b). Finally, dividing (9) by k yields (14c).

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