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Abstract

In this paper, we study full collusion (total payoff maximization) in the repeated Bertrand duopoly with capacity constraints. Instead of a standard rationing rule, Efficient rule (E rule), we introduce a sales maximization rationing rule. Under this rule, when the demand of a firm with a lower price exceeds its capacity, the consumers who are willing to buy at that price are rationed to that firm according to their unwillingness to buy. Then, we investigate whether the full collusion can be sustained or not by an equilibrium under our rule. We have four main results. First, we find that unless each firm's capacity is too large, an asymmetric price pair maximizes one shot total payoffs and the maximum total payoff is strictly greater than the one under E rule. Second, we explicitly find a minimum discount factor under which the full collusion can be sustained along a simple path such that the firms alternate two asymmetric price pairs. Third, we find that there exists a range of capacity constraints within which the minimum discount factors above which the full collusion can be sustained are lower under our rule than under E rule. This implies that the payoff of the full collusion, which is greater than under E rule, can be sustained within a wider range of discount factors rather than under E rule. Fourth and finally, we show that there exists the interior optimal capacity which maximizes the total payoffs of the full collusion, and the total payoff is strictly greater than the profit of a monopolist with aggregate capacities. This implies that sufficiently patient firms intend to reduce their capacities to just the optimal level when they have extra capacities, and that each middle-size firm prefers to be independent, instead of being horizontally integrated.

Keywords: Repeated Bertrand oligopoly, Capacity constraints, Collusion, Sales maximization rule, Simple alternating path, Size of firm

1 Introduction

In this paper, we study full collusion (total payoff maximization) in the repeated Bertrand duopoly with capacity constraints. As is well known, the rationing rules have important roles in Bertrand model with capacity constraints. A standard rule is Efficient rule (E rule) by Levitan and Shubik (1972). Instead of E rule, we introduce a new rationing rule, a sales maximization rule (S rule) in this paper. We investigate whether the full collusion can be sustained or not.

Under both our rationing rule and E rule, consumers are split equally when both firms charge a same price. Thus, a difference between the two rationing rules appears when both firms charge different prices.

Suppose that two firms charge different prices and the demand of a firm with a lower price exceeds its capacity. Under our rule, the consumers who are willing to buy at the lower price, are rationed to the firm with the lower price according to their unwillingness to buy. Namely, the consumers who are less willing to buy are more likely to be rationed. After the capacity of the firm is exhausted, then the remaining consumers are rationed to the firm with a higher price. In this sense, our rule maximizes the total sales. Therefore, we call our rule a Sales maximization rule. Note that the consumers who are rationed to a firm with a lower price have lower Willingness to pay (WTP) rather than the consumers who are rationed to a firm with a higher price.

On the other hand, under E rule, the relation between the consumers' WTP and their priority of being rationed to a firm with a lower price is opposite to S rule. Suppose that two firms charge different prices and that the demand of a firm with a lower price exceeds its capacity. Under E rule, the consumers who are willing to buy at the lower price are rationed to the firm with the lower price according to their willingness to buy. Note that the consumers who are rationed to a firm with a lower price have higher WTP rather than the consumers who are rationed to a firm with a higher price.

Whether S rule is plausible or not depends on contexts of models considered. One of our interpretations is as follows. Suppose that the goods is luxury and consumers with higher opportunity costs of time have higher WTP. Also, suppose that consumers who intend to buy the goods from a firm with a lower price must spend time, for example, to search a lower price or form a line in front of the shop. Note that consumers with high WTP are unwilling to spend time of buying the goods due to high opportunity costs of time. Therefore, S rule might be plausible in this case.

In this paper, we mainly compare the full collusion under S rule with the one under E rule. We have four main results.

First, we show that a one shot full collusion price pair which maximizes the total profit must be asymmetric under S rule, unless each firm's capacity is too large.[1] This is strikingly different from E rule.[2] When each firm's capacity

[1]Indeed there exist only two one shot full collusion price pairs. The one is a price pair such that the role of firms in the other price pair is alternated.

[2]Under E rule, total profit maximization can be sustained under a symmetric price pair. See Proposition 4.

is not too large, the profits of one shot full collusion under S rule are strictly greater than under E rule. This is because more consumers are rationed to a firm with a higher price under S rule than under E rule at an asymmetric price pair, when each firm's capacity is small to some extent and as a result, the total profit increases. On the contrary, when the capacity is too large, at asymmetric price pairs, a firm with a higher price confronts relatively less consumers even under S rule. As a result, a price pair in which both firms charge the monopoly price without capacity constraints maximizes a one shot total payoff under both S rule and E rule, when each firm's capacity is sufficiently large.

Second, we investigate whether full collusion can be sustained under a *subgame perfect equilibrium* (SPE). As we have just seen, unlike E rule, there may exist multiple asymmetric one shot full collusion price pairs under S rule. Therefore, fully collusive paths, in which a one shot full collusion price pair is played each period, also have multiplicity. In this paper, we focus on simple alternating paths (SAPs). We define an SAP as follows; (*i*) Suppose that there exists no symmetric one shot full collusion price pair. Then we define an SAP as a path such that the two asymmetric one shot full collusion price pairs are played every period, with the firms charging the lower price by turns. (*ii*) Suppose that there exists a symmetric one shot full collusion price pair. Then we define an SAP as a path such that both firms charge the same price repeatedly.

We find a necessary and sufficient condition such that an SAP is an SPE path. That an SAP can be sustained under sufficiently large discount factor is analogous to folk theorem. On the other hand, we can apply Abreu (1986, 1988) to find a worst stick-and-carrot equilibrium and explicitly find a minimum discount factor above which an SAP can be sustained.

Third, we compare minimum discount factors of ours with those under E rule. A literature related to ours is Brock and Scheinkman (1985) (henceforth BS), which studies the repeated Bertrand oligopoly under E rule, and unveils relations between capacity constraints and minimum discount factors above which full collusion can be sustained.

Comparing with BS, we show that there exists a range of capacity constraints within which minimum discount factors for the full collusion are lower under S rule than under E rule. This implies that under S rule, the payoff of the full collusion, which is greater than under E rule, can be sustained within a wider range of discount factors rather than under E rule. However, this result is not true when each firm's capacity is very large. When each firm's capacity is very large, the full collusion path of ours and that of BS are the same. On the other hand, the minmax value of ours is strictly greater than that of BS. This is because a firm can charge a higher price to have relatively large sales when his rival charges a minmax price under S rule. The fact that the minmax value is larger under S rule means that we require a higher discount factor for sustaining full collusion under S rule.

Fourth and finally, we explicitly find the interior optimal capacity which maximizes the total payoffs of full collusion. We also show that the total payoff of two independent firms is strictly greater than the profit of a monopolist with aggregate capacities. These observations imply that sufficiently patient firms intend to reduce their capacities to just the optimal level and each middle-size firm prefers to be independent, instead of being horizontally integrated.

This paper proceeds as follows. In Section 2, we consider stage games. In Section 3, we analyze the full collusion in the repeated games, comparing S rule with E rule. In Section 4, we study a relation between capacity constraints and the size of a firm. Also, we give an interpretation of our model. Section 5 concludes the paper. The proofs of all propositions are in Appendix.

2 Stage game

In this section, we define our stage game. We consider Bertrand duopoly with capacity constraints. We assume that there exist two firms with same capacity constraints. Let k be each firm's capacity and suppose that $0 < k < 1$. The goods the firms sell is homogeneous. There is a continuum of consumers. The size of the consumers is mass 1. Each consumer has a unit demand and their Willingness to pay (WTP) is uniformly distributed on $[0, 1]$. Thus, for any $p \in [0, 1]$, the size of the consumers who are willing to buy at the price p is $D(p) = 1 - p$.

In Bertrand models with capacity constraints, rationing rules determine the profit of each firm. Although E rule is well known to be a standard rule, we introduce another rationing rule in this paper.

Our rationing rule is called Sales maximization rule (S rule). Suppose that firm i charges p_i and firm $j \neq i$ charges $p_j > p_i$. Then the consumers whose WTP is on $[p_i, 1]$ are willing to buy from firm i. Suppose that the capacity constraint binds; that is $D(p_i) = 1 - p_i > k$. According to S rule, the consumers whose WTP is on $[p_i, p_i + k]$ are rationed to firm i. Also, the remaining consumers whose WTP is on $[p_i + k, 1] \cap [p_j, 1]$ are rationed to firm j until the capacity is exhausted. Therefore, firm i's and firm j's sales are k and $\min\{k, 1 - p_i - k, 1 - p_j\}$, respectively.

Under S rule, thus the profit of firm i is as follows. When firm i and firm $j \neq i$ charge the prices p_i and p_j, respectively, firm i's profit is

$$r_i^S(p_i, p_j) = \begin{cases} p_i \cdot \min\{D(p_i), k\}, & \text{if } p_i < p_j \\ p_i \cdot \min\left\{\frac{D(p_i)}{2}, k\right\}, & \text{if } p_i = p_j \\ p_i \cdot \max\left\{0, \min\{D(p_j) - k, D(p_i), k\}\right\}. & \text{if } p_i > p_j \end{cases}$$

Also, let us define $\underline{v}^S = \inf_{p_j} \sup_{p_i} r_i^S(p_i, p_j)$. The following proposition describes the value and shows the existence of a price attaining the infimum.

Proposition 1.

Under S rule,

$$\underline{v}^S = k \cdot \frac{1 - 3k + \sqrt{5k^2 - 2k + 1}}{2},$$

and $\underline{v}^S = \max_{p_i} r_i^S(p_i, p_S^{\min})$, *where* $p_S^{\min} = \big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)/2$.

Note that $p_S^{\min}$ is a minmax price which grants at most a security level (the minmax value) to a rival firm. In this paper, we mainly pay our attentions to firms' full collusion. Thus, we define a one shot full collusion price pair as a price pair inducing the total profit maximization at the stage game.

We define $R^S(p_1, p_2) = r_1^S(p_1, p_2) + r_2^S(p_2, p_1)$. Now let (p_l, p_h) be a solution of $\max_{p_1, p_2} R^S(p_1, p_2)$ such that $p_2 \geqslant p_1$. The following proposition characterizes (p_l, p_h), together with $\pi_S = R^S(p_l, p_h)$, $\pi_{l,S} = r_1^S(p_l, p_h)$, and $\pi_{h,S} = \pi_S - \pi_{l,S}$.

Proposition 2.

If $k < 2/3$, *then we have a unique solution* (p_l, p_h). *On the other hand, if* $k \geqslant 2/3$, *then* (p_l, p_h) *has multiplicity.*

- *(i) If* $k \leqq 1/3$, *then* $p_l = 1-2k$ *and* $p_h = 1-k$. *In this case,* $\pi_{l,S} = (1-2k)k$, $\pi_{h,S} = (1-k)k$ *and thus* $\pi_S = 2k - 3k^2$.
- *(ii) If* $1/3 \leqq k < 2/3$, *then* $p_l = \frac{1-k}{2}$ *and* $p_h = \frac{1+k}{2}$. *In this case,* $\pi_{l,S} = \frac{1-k}{2} \cdot k$, $\pi_{h,S} = \frac{1+k}{2} \cdot \frac{1-k}{2} = \frac{1-k^2}{4}$ *and thus* $\pi_S = \frac{1+2k-3k^2}{4}$.
- *(iii) If* $k = 2/3$, *then* $\pi_S = 1/4$. *And either (a)* $(p_l, p_h) = (1/2, p)$ *such that* $p \geqslant 1/2$, *or (b)* $(p_l, p_h) = (1/6, 5/6)$.
- *(iv) If* $2/3 < k$, *then* $\pi_S = 1/4$. *And* $(p_l, p_h) = (1/2, p)$ *such that* $p \geqslant 1/2$.

We can observe that when the capacity constraint is tight to some extent, the one shot full collusion is such that both firms charge different prices. This is strikingly different from the result for E rule (see Proposition 4). An intuitive explanation of Proposition 2 is as follows. Suppose that a firm intends to charge a higher price than his rival. Thus, the firm can charge an appropriate price and serve the whole residual demands after his rival sells the goods. When the capacity constraint is tight to some extent, the profit of the firm which charges the price at which the whole remaining consumers are served after his rival's selling the goods is relatively large. However, the profit of the firm is relatively small when the capacity constraint is relaxed. Therefore, the one shot full collusion is such that both firms charge different prices when $k < 2/3$, and it is such that both firms charge the monopoly price without capacity constraints when $k > 2/3$.[3]

Next, we explain E rule. Suppose that the two firms charge different prices. Under E rule, the consumers are also willing to buy first from the firm with

[3]When $k = 2/3$, there exist both two types of one shot full collusion price pairs which we can see in the case in which $k < 2/3$ or the case in which $k > 2/3$.

the lower price. However, when the capacity constraint of that firm binds, the higher WTP a consumer has, the higher order of the priority of buying from that firm the consumer takes. Then the consumers are rationed to the firm with the lower price until the capacity of that firm is exhausted. Also, the remaining consumers whose WTP is above the higher price are rationed to the firm with the higher price. Recalling S rule, we can say that S rule is just opposite to E rule.

Now, we denote firm i's profit, $r_i^E(p_i, p_j)$ when firm i and firm $j \neq i$ charge prices p_i and p_j, respectively under E rule. We can write $r_i^E(p_i, p_j)$ as follows.

$$r_i^E(p_i, p_j) = \begin{cases} p_i \cdot \min\{D(p_i), k\}, & \text{if } p_i < p_j \\ p_i \cdot \min\left\{\frac{D(p_i)}{2}, k\right\}, & \text{if } p_i = p_j \\ p_i \cdot \max\left\{0, \min\{D(p_i) - k, k\}\right\}. & \text{if } p_i > p_j \end{cases}$$

Also, let us define $\underline{v}^E = \inf_{p_j} \sup_{p_i} r_i^E(p_i, p_j)$. The following proposition, which is quoted from BS, describes the value and shows the existence of a price attaining the infimum. The proof is omitted.

Proposition 3.

Under E rule,

(i) if $k \leqq 1/3$, *then*

$$\underline{v}^E = k \cdot (1 - 2k),$$

and $\underline{v}^E = \max_{p_i} r_i^E(p_i, p_E^{\min})$, *where* $p_E^{\min} = 1 - 2k$, *and*

(ii) if $1/3 < k$, *then*

$$\underline{v}^E = \frac{(1-k)^2}{4},$$

and $\underline{v}^E = \max_{p_i} r_i^E(p_i, p_E^{\min})$, *where* $p_E^{\min} = \frac{(1-k)^2}{4k}$.

Furthermore, there exists a Nash equilibrium (NE) in which the profit of each firm is exactly $\underline{v}^E$ *in both cases* (i) *and* (ii).

Note that $p_E^{\min}$ is a minmax price which grants at most a security level to a rival firm. Note also that under E rule, there exist multiple minmax prices in cases (i) and (ii). We focus on the price $p_E^{\min}$, which Lambson (1987) also focuses. Also, note that $p_E^{\min}$ is a maximal minmax price, as Lambson (1994) points out. In addition, for case (i), BS states that $(p_E^{\min}, p_E^{\min})$ is a pure *NE*. For case (ii), BS states that there exists a mixed *NE* without concretely giving an equilibrium. We can confirm that a profile in which each firm chooses a price on an interval $\left[\frac{(1-k)^2}{4k}, \frac{1-k}{2}\right]$ according to a distribution function $F(p) = \frac{\frac{(1-k)^2}{4} - pk}{p(1-p-2k)}$ is an *NE*.

Now, we define $R^E(p_1, p_2) = r_1^E(p_1, p_2) + r_2^E(p_2, p_1)$. Let $(p_{1,E}, p_{2,E})$ be a solution of $\max_{p_1,p_2} R^E(p_1, p_2)$ such that $p_2 \geqslant p_1$. The following proposition characterizes $(p_{1,E}, p_{2,E})$, together with $\pi_E = R^E(p_{1,E}, p_{2,E})$, $\pi_{1,E} = r_1^E(p_{1,E}, p_{2,E})$, and $\pi_{2,E} = \pi_E - \pi_{1,E}$.

Proposition 4.

If $k < 1/2$, then we have a unique solution $(p_{1,E}, p_{2,E})$. On the other hand, if $k \geqslant 1/2$, then $(p_{1,E}, p_{2,E})$ has multiplicity.

(i) If $k \leqq 1/4$, then $p_{1,E} = p_{2,E} = 1 - 2k$. In this case, $\pi_{1,E} = \pi_{2,E} = (1-2k)k$ and thus $\pi_E = (1-2k)2k$.

(ii) If $1/4 \leqq k < 1/2$, then $p_{1,E} = p_{2,E} = 1/2$. In this case, $\pi_{1,E} = \pi_{2,E} = 1/8$ and thus $\pi_E = 1/4$.

(iii) If $1/2 \leqq k$, then $\pi_E = 1/4$. And $(p_{1,E}, p_{2,E}) = (1/2, p)$ such that $p \geqslant 1/2$.

By Propositions 2 and 4, we see that when $k < 2/3$, the total profit of the one shot full collusion is greater under S rule than under E rule. We also note that when $2/3 \leqq k$, the total profits are the same under S rule and under E rule, and they are 1/4. This is because a firm which intends to charge a higher price than his rival confronts the relatively small size of consumers and as a result both firms charge the monopoly price in a one shot full collusion under S rule.

In the end of this section, we give the following proposition which states that the minmax value under S rule is strictly greater than that under E rule.

Proposition 5. *$\underline{v}^S > \underline{v}^E$ for any k.*

3 Repeated game

3.1 Repeated game

In this section, we consider the repeated game which has the stage game defined above. Both firms have common discount factor $\delta \in (0, 1)$. We assume perfect monitoring and thus, each firm can observe his rival's price in any period $t \geqslant 0$. We also assume that both firms can observe a realization of some public randomization devices (a random variable uniformly distributed on $[0, 1]$) and select continuation strategies on this basis at any period. Each firm observes a realization of randomization devices at the end of a period. Let $\omega(t)$ be the realization of randomization devices at the end of a period t.

Let $p_i(t)$ be the price of firm i in period t. For any $t \geqslant 1$, a history of period t is denoted by $h^t = \Big\{ \big(p_1(s), p_2(s), \omega(s)\big) \Big\}_{s=0}^{t-1}$, and let $h^0 = \{\emptyset\}$. For $t \geqslant 0$, let us denote the set of histories of period t by H^t. We define a pure strategy of firm i as $\sigma_i : \bigcup_{t=0}^{\infty} H^t \to p_i(t)$. Given a strategy profile, a stream of price profile is determined. We consider the expectation over the realization of public randomization devices. The payoff of firm i is

$$V^i = (1-\delta)\sum_{t=0}^{\infty} E\Big[r_i^S\big(p_i(t), p_j(t)\big)\Big].$$

Our solution concept is SPE.

3.2 SAPs and minimum discount factors

In this paper, we focus on fully collusive paths, along which both firms play full collusion price pairs every period. Note that there exist multiple full collusion paths, since a one shot full collusion price pair has multiplicity. Therefore, throughout this paper, we restrict our attentions to particular ones; we call them simple alternating paths (SAPs).

Note that if a symmetric full collusion price pair exists, the price pair is unique symmetric price pair. In this case, we define an SAP as the repetition of the price pair. If a symmetric price pair does not exist, then only two asymmetric full collusion price pairs (p_l, p_h) or (p_h, p_l) exist. In this case, we define an SAP as a cycle of the price pairs as follows.

$$\big\{(p_l, p_h),\ (p_h, p_l),\ (p_l, p_h), \cdots\big\}$$

or

$$\big\{(p_h, p_l),\ (p_l, p_h),\ (p_h, p_l), \cdots\big\}.$$

More specifically, we recall Proposition 2 and write an SAP as follows.

If $k \leqq 1/3$, then the SAPs have the following two forms.

$$\big\{(1-2k, 1-k),\ (1-k, 1-2k),\ (1-2k, 1-k), \cdots\big\}$$

or

$$\big\{(1-k, 1-2k),\ (1-2k, 1-k),\ (1-k, 1-2k), \cdots\big\}.$$

If $1/3 \leqq k < 2/3$, then the SAPs are

$$\Big\{\Big(\frac{1-k}{2}, \frac{1+k}{2}\Big), \Big(\frac{1+k}{2}, \frac{1-k}{2}\Big), \Big(\frac{1-k}{2}, \frac{1+k}{2}\Big), \cdots\Big\}$$

or

$$\Big\{\Big(\frac{1+k}{2},\frac{1-k}{2}\Big),\Big(\frac{1-k}{2},\frac{1+k}{2}\Big),\Big(\frac{1+k}{2},\frac{1-k}{2}\Big),\cdots\Big\}.$$

On the other hand, if $2/3 \leqq k$, then the SAP is unique and is as follows.

$$\Big\{\Big(\frac{1}{2},\frac{1}{2}\Big),\Big(\frac{1}{2},\frac{1}{2}\Big),\Big(\frac{1}{2},\frac{1}{2}\Big),\cdots\Big\}.$$

Recall that (p_l, p_h) has multiplicity when $2/3 \leqq k$ in Proposition 2. In this case, we focus on the price pair such that $p_l = p_h = 1/2$, because this price pair is easiest to sustain in the long-run relationships.

In the following Proposition 6, we study the necessary and sufficient condition such that an SAP can be sustained under an *NE*. For the following proposition, let us define $T_{l,S} = \sup_{p_1} r_1^S(p_1, p_h)$. More concretely,

$$T_{l,S} = \begin{cases} (1-k)k, & \text{if} \quad k \leqq 1/2 \\ 1/4. & \text{if} \quad 1/2 < k \end{cases}$$

Proposition 6.

(a) *Fix k. An SAP can be sustained under an NE if and only if*

$$\delta^2(T_{l,S} - \underline{v}^S) + \delta(\pi_{h,S} - \underline{v}^S) + \pi_{l,S} - T_{l,S} \geqslant 0. \qquad (C1)$$

(b) *For any k, there exists $\underline{\delta}(k) \in (0,1)$ such that $(C1)$ holds if and only if $\delta \geqslant \underline{\delta}(k)$. More specifically, $\underline{\delta}(k)$ is as follows.*

(i) *If $k \leqq 1/3$, then*

$$\underline{\delta}(k) = \frac{-k-1+\sqrt{5k^2-2k+1}+\sqrt{14k^2+8k+2-(10k+2)\sqrt{5k^2-2k+1}}}{2\big(k+1-\sqrt{5k^2-2k+1}\,\big)}.$$

(ii) *If $1/3 \leqq k \leqq 1/2$, then*

$$\begin{aligned}\underline{\delta}(k) =& \frac{1}{4\big(k^2+k-k\sqrt{5k^2-2k+1}\,\big)} \\ &\times \Big\{ -5k^2+2k-1+2k\sqrt{5k^2-2k+1} \\ &\quad + \sqrt{29k^4-28k^3+34k^2-4k+1-(4k^3+8k^2+4k)\sqrt{5k^2-2k+1}}\Big\}.\end{aligned}$$

(iii) If $1/2 \leqq k < 2/3$*, then*

$$\underline{\delta}(k) = \frac{1}{2\left(6k^2 - 2k + 1 - 2k\sqrt{5k^2 - 2k + 1}\right)} \times \Bigg\{ -5k^2 + 2k - 1 + 2k\sqrt{5k^2 - 2k + 1} + \sqrt{93k^4 - 92k^3 + 66k^2 - 20k + 5 - 12k(3k^2 - 2k + 1)\sqrt{5k^2 - 2k + 1}} \Bigg\}.$$

(iv) If $2/3 \leqq k$*, then*

$$\underline{\delta}(k) = \frac{1}{2\left(1 - 2k + 6k^2 - 2k\sqrt{5k^2 - 2k + 1}\right)}.$$

It is clear that Proposition 6 is also a necessary condition such that an SAP is an SPE path.

The following Proposition 7 is a main result. This proposition asserts that $(C1)$ is also a sufficient condition such that an SAP can be sustained under an SPE. That is, we can construct a worst stick-and-carrot equilibrium, which grants each firm to the exact value $\underline{v}^S$ for any $\delta \geqslant \underline{\delta}(k)$. In the worst stick-and-carrot equilibrium, at the initial period, both firms charge the minmax price mutually. Let us write the stage-game payoff of a firm at the initial period as $M = p_S^{\min} \cdot (1 - p_S^{\min})/2$.[4] Following the initial period, each firm behaves in the following manner. (a) The case in which $k < 2/3$; With an appropriate probability, both firms switch to the collusive behavior such that they choose the two SAPs with equal probability, using randomization devices, and henceforth they play the SAP. Also, both firms replay the worst stick-and-carrot equilibrium with the remaining probability. (b) The case in which $k \geqslant 2/3$; Both firms switch to the fully collusive path, which is the unique SAP, with an appropriate probability and replay the worst stick-and-carrot equilibrium with the remaining probability. Also, in the worst stick-and-carrot equilibrium, when some deviations happen, from the next period, the worst stick-and-carrot equilibrium is replayed with probability 1.

We have to verify that there exists an appropriate probability with which both firms switch to collusive behavior in the worst stick-and-carrot equilibrium. Interestingly, the existence is assured under $(C1)$.

Proposition 7.

An SAP can be sustained under an SPE if and only if $(C1)$ *holds.*

[4] We can confirm that $D(p_S^{\min}) < 2k$.

3.3 The comparison of S rule and E rule

In this subsection, we compare S rule with E rule. First, the following proposition is convenient for the comparison.

Proposition 8. *If $k \leqq 1/3$, then $\underline{\delta}(k)$ is increasing in k.*

Second, by Propositions 3 and 4, recall that under E rule, there exists a unique symmetric one shot full collusion price pair which maximizes the total profit at the stage game. Also, recall that there exists an *NE* in which the profit of each firm is exactly the minmax value $\underline{v}^E$ of the stage game. We define $\delta^E(k)$ as a minimum discount factor above which a full collusion in which a symmetric one shot full collusion price pair is played every period can be sustained by a trigger strategy which grants the minmax payoff $\underline{v}^E$ to a deviator after any deviation. BS explicitly finds $\delta^E(k)$. In this subsection, we compare $\underline{\delta}(k)$ with $\delta^E(k)$.

Note that if $k \leqq 1/4$, then we regard $\delta^E(k) = 0$, since the symmetric full collusion is the repetition of the one shot *NE*. For the remaining cases, $\delta^E(k)$ is as follows.

If $1/4 < k \leqq 1/3$, then $\delta^E(k) = 1/4k$. We have this by solving

$$\frac{1}{8} \geqslant (1-\delta)\frac{1}{2}k + \delta(1-2k)k.$$

If $1/3 \leqq k \leqq 1/2$, then

$$\delta^E(k) = \frac{4k-1}{2(4k-k^2-1)}.$$

We have this by solving

$$\frac{1}{8} \geqslant (1-\delta)\frac{1}{2}k + \delta\frac{(1-k)^2}{4}.$$

If $1/2 \leqq k$, then

$$\delta^E(k) = \frac{1}{2k(2-k)}.$$

We have this by solving

$$\frac{1}{8} \geqslant (1-\delta)\frac{1}{4} + \delta\frac{(1-k)^2}{4}.$$

The following Proposition 9 is another main result, which studies the relation between the rationing rules and the likelihood of the full collusion. It asserts that there exists a range of capacity constraints within which $\underline{\delta}(k) < \delta^E(k)$ holds.

Note that if $k \in (1/4, 1/3]$, then $\delta^E(k) = 1/4k$ is decreasing in k. Also, note that $\lim_{k\to 1/4} \delta^E(k) = 1$ and that $\delta^E(1/3) = 3/4$. On the other hand, if $k \in (0, 1/3]$, then $\underline{\delta}(k)$ is increasing in k by Proposition 8. Also, note that $\underline{\delta}(1/3) = \frac{-1+\sqrt{5+2\sqrt{2}}}{2} > 3/4$. Therefore, there exists a unique $\hat{k} \in (1/4, 1/3)$ such that $\underline{\delta}(\hat{k}) = \delta^E(\hat{k})$ holds. More concretely, $\hat{k}$ satisfies

$$\frac{-\hat{k} - 1 + \sqrt{5\hat{k}^2 - 2\hat{k} + 1} + \sqrt{14\hat{k}^2 + 8\hat{k} + 2 - (10\hat{k} + 2)\sqrt{5\hat{k}^2 - 2\hat{k} + 1}}}{2\left(\hat{k} + 1 - \sqrt{5\hat{k}^2 - 2\hat{k} + 1}\right)} = \frac{1}{4\hat{k}}.$$

Calculating this, we have $\hat{k} \in (0.294, 0.295)$.

Proposition 9.

(i) If $k \in (1/4, \hat{k}]$, then $\underline{\delta}(k) \leqq \delta^E(k)$. Also, $\underline{\delta}(k) = \delta^E(k)$ if and only if $k = \hat{k}$.

(ii) If $k \notin (1/4, \hat{k}]$, then $\delta^E(k) < \underline{\delta}(k)$.

Proposition 9 states that when a capacity constraint is such that $k \in (1/4, \hat{k})$, the total payoffs of the full collusion under S rule, which are strictly greater than under E rule, can be sustained within a wider range of discount factors under S rule rather than under E rule. In this sense, S rule is better for full collusion. Proposition 9(i) implies that when total capacities are greater than 50% but are smaller than 58% of an industry, S rule facilitates full collusion, since $\hat{k} \in (0.294, 0.295)$. When a capacity constraint is relaxed to $k \geqslant 2/3$, although the total payoff of the full collusion is the same under S rule and under E rule, the full collusion can be sustained within a wider range of discount factors under E rule than under S rule. In this sense, E rule is better for full collusion. When a capacity constraint is such that $k \in (0, 1/4] \cup (\hat{k}, 2/3)$, the full collusion can be sustained within a wider range of discount factors under E rule than under S rule. On the other hand, the total payoffs of the full collusion are strictly greater under S rule than under E rule. In this sense, S rule is better for full collusion as far as firms are sufficiently patient.

An intuitive explanation of Proposition 9 is that the increment of the total payoff under S rule might encourage the full collusion. Consider the case in which $1/4 < k \leqq 1/3$. Note that the difference between the value of the full collusion and the minmax value is relatively large because of S rule and thus, $\underline{\delta}(1/4)$ is relatively low. On the other hand, $\lim_{k\to 1/4} \delta^E(k) = 1$. This is because the difference between the value of the full collusion and the minmax value is very small under E rule, since they are the same at $k = 1/4$. Also, note that the temptation of deviating from full collusion is larger under S rule. As a result,

$\underline{\delta}(k)$ is increasing in k and $\delta^E(k)$ is decreasing in k. Therefore, there exists $\hat{k} \in (1/4, 1/3)$ such that if $k \in (1/4, \hat{k}]$, then $\underline{\delta}(k) \leqq \delta^E(k)$.

When the capacity constraint is relaxed to $1/3 \leqq k < 2/3$, the asymmetric price pair of the one shot full collusion induces the increment of the total payoff of the full collusion, however the relaxation of the capacity constraint offsets the effect. As a result, $\delta^E(k)$ is smaller than $\underline{\delta}(k)$. Moreover, when the capacity constraint is relaxed to $k \geqslant 2/3$, although the full collusion paths under S rule and under E rule are the same, the minmax payoff under S rule is strictly greater than under E rule. As a result, $\delta^E(k)$ is also smaller than $\underline{\delta}(k)$.

In addition, consider the case in which each firm has almost no capacity constraints. $(k \to 1)$. Note that an SAP is a path along which the monopoly price is charged by both firms, and that $p_S^{\min} \to 0$ and $p_E^{\min} \to 0$. Therefore, it is clear that $\underline{\delta}(k) \to 1/2$ and $\delta^E(k) \to 1/2$. That is, an advantage of E rule such that the minimum discount factor for full collusion is lower than under S rule vanishes as k goes to 1.[5]

4 Optimal capacity

In this section, we describe a relation between capacity constraints and the size of a firm. First, we find explicitly an interior optimal capacity which maximizes one shot total payoffs of one shot full collusion. We can easily confirm the following proposition by Proposition 2 and the proof is omitted.

Proposition 10. *It is $k = 1/3$ that maximizes one shot total payoffs of one shot full collusion.*

This proposition suggests that if both firms are sufficiently patient, they seem to prefer to be remaining middle-size (around $k = 1/3$) even by reducing their extra capacities when they have excessive capacities.

Second, we study the effect of a horizontal integration into a monopolist with aggregate capacities $2k$. Let us denote π_M by the payoff of a monopolist. Note that

$$\pi_M = \begin{cases} (1-2k)\cdot 2k, & \text{if} \quad k < 1/4 \\ 1/4. & \text{if} \quad k \geqslant 1/4 \end{cases}$$

We compare π_M with π_S and thus we have the following proposition.

Proposition 11. *For any $k < 2/3$, $\pi_S > \pi_M$. For any $k \geqslant 2/3$, $\pi_S = \pi_M$.*

This is because a monopolist cannot use price discrimination. Under S rule, the size of the consumers who are rationed to a firm with a higher price at an asymmetric price pair is relatively large, when $k < 2/3$. As a result, the total profit under an asymmetric one shot full collusion price pair increases. On the

[5] We can also observe that $\lim_{k\to 0} \underline{\delta}(k) = -\frac{1}{2} + \frac{\sqrt{5}}{2}$. This observation implies that the advantage of E rule such that the minimum discount factor for full collusion is lower than under S rule does not vanish as k goes to 0.

other hand, a firm which intends to charge a higher price than his rival confronts the relatively small size of consumers when $k \geqslant 2/3$, and as a result both firms charge the monopoly price in a one shot full collusion under S rule. Also, note that under E rule, there does not exist the advantage of price discrimination, since a one shot full collusion can be sustained under a symmetric price pair.

Propositions 10 and 11 suggest that sufficiently patient firms might intend to reduce their capacities to just the optimal level and each middle-size firm might prefer to be independent, instead of being horizontally integrated under S rule.

5 Concluding remarks

In this paper, we consider full collusion (total payoff maximization) in the repeated Bertrand duopoly with capacity constraints. As a rationing rule, we introduce a Sales maximization rule (S rule), instead of an Efficient rule (E rule). Our main findings are as follows. First, we show that under S rule, a one shot full collusion price pair which maximizes a stage-game payoff must be asymmetric and the maximum total payoff is strictly greater than the one under E rule, unless each firm's capacity is too large. Second, we explicitly find a minimum discount factor under which the full collusion can be sustained along a simple path such that the firms alternate two asymmetric price pairs. Third, we show that under S rule, the total payoffs of the full collusion, which are greater than the one under E rule, can be sustained within a wider range of discount factors, when capacity constraints are tight to some extent. On the contrary, the total payoffs of the full collusion under S rule and those under E rule are the same, and the full collusion can be sustained within a narrower range of discount factors under S rule than under E rule, when each firm's capacity is sufficiently large. Fourth and finally, we find explicitly the interior optimal capacity and suggest that sufficiently patient firms might intend to reduce their capacities to just the optimal level and each middle-size firm might prefer to be independent, instead of being horizontally integrated.

Appendix

Proof of Proposition 1.

Suppose that firm 2 charges the price $p_S^{\min} = (1 - 3k + \sqrt{5k^2 - 2k + 1}\,)/2$. Firm 1 charging a price $p_1 > p_S^{\min}$ confronts the demand, $\max\{0, \min\{1 - p_S^{\min} - k, 1 - p_1, k\}\}$. Since $0 < 1 - p_S^{\min} - k < k$,

$$r_1^S(p_1, p_S^{\min}) = \begin{cases} p_1(1 - p_S^{\min} - k), & \text{if} \quad p_S^{\min} < p_1 < p_S^{\min} + k \\ p_1(1 - p_1). & \text{if} \quad p_1 \geqslant p_S^{\min} + k \end{cases}$$

Note that

$$\begin{aligned} r_1^S(p_S^{\min}+k, p_S^{\min}) &= (1-p_S^{\min}-k)(p_S^{\min}+k) \\ &= \frac{(1-k+\sqrt{5k^2-2k+1}\,)(1+k-\sqrt{5k^2-2k+1}\,)}{4} \\ &= \frac{k(1-3k+\sqrt{5k^2-2k+1}\,)}{2} = kp_S^{\min}. \end{aligned}$$

Since $p_S^{\min}+k > 1/2$, $r_1^S(p_1, p_S^{\min}) < r_1^S(p_S^{\min}+k, p_S^{\min})$ for any $p_1 > p_S^{\min}+k$. Furthermore, for any $p_1 \in (p_S^{\min}, p_S^{\min}+k)$, $r_1^S(p_1, p_S^{\min}) < r_1^S(p_S^{\min}+k, p_S^{\min})$.

On the other hand, since $k < 1-p_S^{\min} < 2k$,

$$r_1^S(p_1, p_S^{\min}) = \begin{cases} p_S^{\min}\cdot\frac{1-p_S^{\min}}{2}, & \text{if} \quad p_1 = p_S^{\min} \\ p_1 k. & \text{if} \quad p_1 < p_S^{\min} \end{cases}$$

It is also clear that $r_1^S(p_1, p_S^{\min}) < r_1^S(p_S^{\min}+k, p_S^{\min})$ for any $p_1 \leqq p_S^{\min}$. To summarize, $p_S^{\min}+k$ is a unique best response to $p_S^{\min}$.

Second, consider the case in which firm 2 charges a price $p_2 < p_S^{\min}$. Thus, for sufficiently small $\epsilon > 0$, firm 1 charging the price $p_S^{\min}+k-\epsilon$ confronts the demand, $\max\{0, \min\{1-p_2-k, 1-p_S^{\min}-k+\epsilon, k\}\} = 1-p_S^{\min}-k+\epsilon$. Note that $r_1^S(p_S^{\min}+k-\epsilon, p_2) = (1-p_S^{\min}-k+\epsilon)(p_S^{\min}+k-\epsilon) > (1-p_S^{\min}-k)(p_S^{\min}+k) = kp_S^{\min}$ for sufficiently small $\epsilon > 0$, since $p_S^{\min}+k > 1/2$. Hence $\sup_{p_1} r_1^S(p_1, p_2) > kp_S^{\min}$.

Finally, consider the case in which firm 2 charges a price $p_2 > p_S^{\min}$. Since $1-p_S^{\min}-k > 0$, $r_1^S(p_S^{\min}+\epsilon, p_2) = (p_S^{\min}+\epsilon)\min\{1-p_S^{\min}-\epsilon, k\} = (p_S^{\min}+\epsilon)k > kp_S^{\min}$ for sufficiently small $\epsilon > 0$. Hence $\sup_{p_1} r_1^S(p_1, p_2) > kp_S^{\min}$.

To conclude, $kp_S^{\min} = \max_{p_1} r_1^S(p_1, p_S^{\min}) = \min_{p_2}\sup_{p_1} r_1^S(p_1, p_2) = \underline{v}^S$. □

Proof of Proposition 2.

First, we consider the case in which $k < 1/2$. Note that $R^S(p_l, p_h) \geqslant R^S(1-2k, 1-k) = (1-2k)k + (1-k)k = k(2-3k)$. Suppose that $p_l < 1-2k$. Note that $\pi_{l,S} < (1-2k)k$ and $\pi_{h,S} \leqq (1-k)k$ always holds, since $k < 1/2$ and $1-p_l \geqslant k$. This is a contradiction. Next, suppose that $p_l \geqslant 1-k$. Note that $\pi_{l,S} + \pi_{h,S} = p_l(1-p_l) \leqq (1-k)k < k(2-3k)$, since $k < 1/2$. This is a contradiction. Thus, we consider the case in which $1-2k \leqq p_l < 1-k$ in the following.

Note that $R^S(p_l, p_l+k) = p_l k + (p_l+k)(1-p_l-k)$, since $1-p_l-k \leqq k$. Suppose that $p_h > p_l+k$. Note that $p_l+k \geqslant 1-k > 1/2$, since $k < 1/2$. Thus, we have $\pi_{l,S}+\pi_{h,S} < p_l k + (p_l+k)(1-p_l-k)$, since $\pi_{h,S} = p_h(1-p_h) < (p_l+k)(1-p_l-k)$. This is a contradiction.

Suppose that $p_l < p_h < p_l+k$. Note that $\pi_{l,S}+\pi_{h,S} = p_l k + p_h(1-p_l-k) < R^S(p_l, p_l+k)$, since $\max\{0, \min\{1-p_l-k, 1-p_h, k\}\} = 1-p_l-k$. This is a contradiction.

Suppose that $p_h = p_l$. Since $p_l \geqslant 1 - 2k$, the total sales of both firms is $1 - p_l$. Thus note that $\pi_{l,S} + \pi_{h,S} = p_l(1 - p_l) < p_l k + (p_l + k)(1 - p_l - k)$. This is because $p_l k + (p_l + k)(1 - p_l - k) - p_l(1 - p_l) = k(1 - k - p_l) > 0$, since $p_l < 1 - k$. This is a contradiction.

Therefore, $\pi_S = \max_{1-2k \leqq p_l < 1-k} \left\{ p_l k + (p_l + k)(1 - p_l - k) \right\}$. We find that if $k \leqq 1/3$, then $p_l = 1 - 2k$ and $p_h = 1 - k$, since $\frac{1-k}{2} \leqq 1 - 2k$. In this case, $\pi_{l,S} = (1 - 2k)k$, $\pi_{h,S} = (1 - k)k$ and thus $\pi_S = 2k - 3k^2$. Also, we find that if $1/3 \leqq k (< 1/2)$, then $p_l = \frac{1-k}{2}$ and $p_h = \frac{1+k}{2}$, since $1 - 2k \leqq \frac{1-k}{2}$. In this case, $\pi_{l,S} = \frac{1-k}{2} \cdot k$, $\pi_{h,S} = \frac{1+k}{2} \cdot \frac{1-k}{2} = \frac{1-k^2}{4}$ and thus $\pi_S = \frac{1+2k-3k^2}{4}$.

Second we consider the case in which $k \geqslant 1/2$. Consider the price pair (p'_l, p'_h) such that $p'_h \geqslant p'_l \geqslant 1 - k$. Note that $\max_{1-k \leqq p'_l, p'_h} R^S(p'_l, p'_h) = R^S(1/2, 1/2) = 1/4$.

Next, consider the price pair (p'_l, p'_h) such that $1 - k > p'_l$ and $p'_l \leqq p'_h$. Note that $R^S(p'_l, p'_l + k) = p'_l k + (p'_l + k)(1 - p'_l - k)$. Suppose that $p'_h > p'_l + k$, then note that $R^S(p'_l, p'_h) = p'_l k + p'_h(1 - p'_h) < p'_l k + (p'_l + k)(1 - p'_l - k)$, since $p'_h(1 - p'_h) < (p'_l + k)(1 - p'_l - k)$. This is because $p'_l + k \geqslant 1/2$, since $k \geqslant 1/2$.

On the other hand, suppose that $p'_h < p'_l + k$. Then note that $R^S(p'_l, p'_h) < p'_l k + (p'_l + k)(1 - p'_l - k)$, by the same discussion of the case in which $k < 1/2$. Thus, we find that under the constraint that $p'_l < 1 - k$, $\max_{p'_l, p'_h} R^S(p'_l, p'_h) = R^S(\frac{1-k}{2}, \frac{1+k}{2}) = \frac{1+2k-3k^2}{4}$.

Therefore, $\pi_S = \max\{1/4, \frac{1+2k-3k^2}{4}\}$. We find that if $(1/2 \leqq) k < 2/3$, then $\pi_S = \frac{1+2k-3k^2}{4} > 1/4$ and $(p_l, p_h) = (\frac{1-k}{2}, \frac{1+k}{2})$. In this case, $\pi_{l,S} = \frac{1-k}{2} \cdot k$, $\pi_{h,S} = \frac{1+k}{2} \cdot \frac{1-k}{2} = \frac{1-k^2}{4}$. We also find that if $k = 2/3$, then $\pi_S = \frac{1+2k-3k^2}{4} = 1/4$ and either (a) $(p_l, p_h) = (1/2, p)$ such that $p \geqslant 1/2$, or (b) $(p_l, p_h) = (1/6, 5/6)$. In addition, we also note that if $2/3 < k$, then $\pi_S = 1/4 > \frac{1+2k-3k^2}{4}$ and $(p_l, p_h) = (1/2, p)$ such that $p \geqslant 1/2$.

By the above discussions, we have proven Proposition 2. □

Proof of Proposition 4.

We first show that $\pi_E = \max_p R^E(p, p)$. Suppose that $\pi_E = R^E(p'_l, p'_h)$ for some $p'_l < p'_h$. If $p'_l \geqslant 1 - k$, then $R^E(p'_l, p'_h) = R^E(p'_l, p'_l)$.

If $p'_h > 1 - k > p'_l$, then $R^E(p'_l, p'_h) = p'_l k \leqq R^E(p'_l, p'_l) = p'_l \cdot \min\{1 - p'_l, 2k\}$.

If $p'_h \leqq 1 - k$, then $R^E(p'_l, p'_h) = p'_l k + p'_h \cdot \min\{1 - p'_h - k, k\} < R^E(p'_h, p'_h) = p'_h \cdot \min\{1 - p'_h, 2k\}$. Therefore, $\pi_E = \max_p R^E(p, p)$.

Second, we find a solution of $\max_p R^E(p, p)$. Note that $R^E(p, p) = p \cdot \min\{1 - p, 2k\}$.

(*i*) If $k \leqq 1/4$, then $p_{1,E} = p_{2,E} = 1 - 2k$. In this case, $\pi_{1,E} = \pi_{2,E} = (1 - 2k)k$ and thus $\pi_E = (1 - 2k)2k$.

(*ii*) If $1/4 \leqq k < 1/2$, then $p_{1,E} = p_{2,E} = 1/2$. In this case, $\pi_{1,E} = \pi_{2,E} = 1/8$ and thus $\pi_E = 1/4$.

Finally, note that if $1/2 \leqq k$, then $R^E(1/2, 1/2) = R^E(1/2, p_2)$ for any $p_2 > 1/2$ and we show the following.

(iii) If $1/2 \leqq k$, then $\pi_E = 1/4$, and $p_{2,E} \geqslant p_{1,E} = 1/2$.

□

Proof of Proposition 5.

First, consider the case in which $k \leqq 1/3$. We have

$$\frac{2}{k}(\underline{v}^S - \underline{v}^E) = k - 1 + \sqrt{5k^2 - 2k + 1} = k - 1 + \sqrt{(k-1)^2 + (2k)^2} > 0.$$

Second, consider the case in which $1/3 < k < 1$. We have

$$\begin{aligned} 4(\underline{v}^S - \underline{v}^E) &= 4 \cdot k \cdot \frac{1 - 3k + \sqrt{5k^2 - 2k + 1}}{2} - 4 \cdot \frac{(1-k)^2}{4} \\ &= 2k\big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big) - (1-k)^2 \\ &= -7k^2 + 4k - 1 + 2k\sqrt{5k^2 - 2k + 1}. \end{aligned}$$

Note that $-7k^2 + 4k - 1 < 0$. Next, we have

$$\big(2k\sqrt{5k^2 - 2k + 1}\big)^2 - (7k^2 - 4k + 1)^2 = (1-k)(29k^3 - 19k^2 + 7k - 1).$$

Let $L(k) = 29k^3 - 19k^2 + 7k - 1$. Note that $L^{'}(k) = 87(k - \frac{19}{87})^2 + \frac{248}{87} > 0$, and that $L(1/3) = \frac{8}{27}$. Thus, $L(k) > 0$ if $1/3 < k < 1$. We have shown that $\underline{v}^S - \underline{v}^E > 0$. □

Proof of Proposition 6.

(a). Consider a following trigger strategy; Unless some deviations happen, the SAP is played. Once a firm deviates, a rival firm punishes a deviator by the repetition of charging the price $p_S^{\min}$. On the other hand, the deviator charges the price $p_S^{\min} + k$ every period. When both firms deviate at the same time, firm 1 charges the price $p_S^{\min} + k$, and firm 2 charges the price $p_S^{\min}$ henceforth.

Note that if $k < 2/3$, then charging p_h is a best response to charging p_l in the stage game. Thus, it is enough to consider the incentive of the firm which charges p_l at the initial period on the SAP. Recall that if $k \geqslant 2/3$, we focus on the price pair $(p_l, p_h) = (1/2,\ 1/2)$. In this case, it is also enough to consider the incentive of the firm which charges p_l at the initial period on the SAP by the symmetry.

Let the value of firm 1, which charges p_l at the initial period on the SAP be

$$V^1 = \frac{\pi_{l,S} + \delta\pi_{h,S}}{1 + \delta}.$$

Thus, the SAP can be sustained under an *NE* if and only if

$$V^1 \geqslant (1-\delta)T_{l,S} + \delta \underline{v}^S.$$

Multiplying $1+\delta$, we have

$$\delta^2(T_{l,S} - \underline{v}^S) + \delta(\pi_{h,S} - \underline{v}^S) + \pi_{l,S} - T_{l,S} \geqslant 0.$$

This is $(C1)$.

(b). We first prove the following lemma.

Lemma 1. $\pi_S - 2\underline{v}^S > 0$ *for any* k.

Note that $2\underline{v}^S = R^S(p_S^{\min}, p_S^{\min} + k)$ and $\pi_S - 2\underline{v}^S \geqslant 0$ for any k. Suppose that $\pi_S - 2\underline{v}^S = 0$. From Proposition 2, it is clear that if $k < 2/3$, then $p_S^{\min} = p_l$ must hold. However, if $k \leqq 1/3$, then we have $p_S^{\min} = \big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)/2 \neq 1 - 2k$. Also, if $1/3 \leqq k \leqq 2/3$, then we have $p_S^{\min} = \big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)/2 \neq (1-k)/2$. These are contradictions.

In addition, note that if $2/3 < k$, then $p_S^{\min} = 1/2$ must hold. However, if $2/3 \leqq k$, then we have $p_S^{\min} = \big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)/2 \neq 1/2$, since $-3k + \sqrt{5k^2 - 2k + 1} < 0$. This is a contradiction.

Also, note that if $k = 2/3$, then either $p_S^{\min} = (1-k)/2$ or $p_S^{\min} = 1/2$ must hold. However, these do not hold because of the previous arguments. This is a contradiction. Thus, $\pi_S - 2\underline{v}^S > 0$ for any k.

Second, note that

$$T_{l,S} = \begin{cases} (1-k)k, & \text{if} \quad k \leqq 1/2 \\ 1/4. & \text{if} \quad 1/2 < k \end{cases}$$

By Proposition 2, we also note that

$$\pi_{l,S} = \begin{cases} (1-2k)k, & \text{if} \quad k \leqq 1/3 \\ \frac{(1-k)k}{2}, & \text{if} \quad 1/3 \leqq k < 2/3 \\ 1/8, & \text{if} \quad 2/3 \leqq k \end{cases}$$

and that

$$\pi_{h,S} = \begin{cases} (1-k)k, & \text{if} \quad k \leqq 1/3 \\ \frac{1-k^2}{4}, & \text{if} \quad 1/3 \leqq k < 2/3 \\ 1/8. & \text{if} \quad 2/3 \leqq k \end{cases}$$

Note that $\pi_{l,S} - T_{l,S} < 0$. It is clear that if $\delta = 0$, then the left hand side of $(C1)$ is negative. Also, we note that if $\delta = 1$, then the left hand side of $(C1)$ is $\pi_S - 2\underline{v}^S > 0$ by Lemma 1. Therefore, there exists $\underline{\delta}(k) \in (0,1)$ such that $(C1)$

holds if and only if $\delta \geqslant \underline{\delta}(k)$. In addition, we explicitly find $\underline{\delta}(k)$ in the following manner.

(i) If $k \leqq 1/3$, then $(C1)$ is

$$f_i(\delta) = \delta^2\big(k+1-\sqrt{5k^2-2k+1}\,\big) + \delta\big(k+1-\sqrt{5k^2-2k+1}\,\big) - 2k \geqslant 0. \quad (1)$$

Thus, we have

$$\underline{\delta}(k) = \frac{-k-1+\sqrt{5k^2-2k+1}+\sqrt{14k^2+8k+2-(10k+2)\sqrt{5k^2-2k+1}}}{2\big(k+1-\sqrt{5k^2-2k+1}\,\big)}.$$

(ii) If $1/3 \leqq k \leqq 1/2$, then $(C1)$ is

$$\begin{aligned} f_{ii}(\delta) = \delta^2 \cdot 2k \cdot \big(k+1-\sqrt{5k^2-2k+1}\,\big) \\ + \delta\big(5k^2-2k+1-2k\sqrt{5k^2-2k+1}\,\big) + 2k^2 - 2k \geqslant 0. \end{aligned} \quad (2)$$

Thus, we have

$$\begin{aligned} \underline{\delta}(k) =& \frac{1}{4\big(k^2+k-k\sqrt{5k^2-2k+1}\,\big)} \\ &\times \Big\{ -5k^2+2k-1+2k\sqrt{5k^2-2k+1} \\ &+ \sqrt{29k^4-28k^3+34k^2-4k+1-(4k^3+8k^2+4k)\sqrt{5k^2-2k+1}} \Big\}. \end{aligned}$$

(iii) If $1/2 \leqq k < 2/3$, then $(C1)$ is

$$\begin{aligned} f_{iii}(\delta) = \delta^2\big(1-2k+6k^2-2k\sqrt{5k^2-2k+1}\,\big) \\ + \delta\big(1-2k+5k^2-2k\sqrt{5k^2-2k+1}\,\big) - 1 + 2k - 2k^2 \geqslant 0. \end{aligned} \quad (3)$$

Thus, we have

$$\begin{aligned} \underline{\delta}(k) =& \frac{1}{2\big(6k^2-2k+1-2k\sqrt{5k^2-2k+1}\,\big)} \\ &\times \Big\{ -5k^2+2k-1+2k\sqrt{5k^2-2k+1} \\ &+ \sqrt{93k^4-92k^3+66k^2-20k+5-12k(3k^2-2k+1)\sqrt{5k^2-2k+1}} \Big\}. \end{aligned}$$

(*iv*) If $2/3 \leqq k$, then $(C1)$ is

$$\begin{aligned} f_{iv}(\delta) &= \delta^2\Big\{2 - 4k\big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)\Big\} \\ &\qquad\qquad + \delta\Big(1 - 4k\big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)\Big) - 1 \\ &= \Big[2\Big\{1 - 2k\big(1 - 3k + \sqrt{5k^2 - 2k + 1}\big)\Big\}\delta - 1\Big](\delta + 1) \geqslant 0. \end{aligned}$$

Thus, we have

$$\underline{\delta}(k) = \frac{1}{2\big(1 - 2k + 6k^2 - 2k\sqrt{5k^2 - 2k + 1}\big)}.$$

□

Proof of Proposition 7.

We first prove the following Lemma 2.

Lemma 2. *If* $(C1)$ *holds, the following also holds.*

$$\delta(\pi_S - 2M) - 2(\underline{v}^S - M) \geqslant 0. \qquad (C2)$$

Let $D(\delta)$ be the difference between the left hand side of $(C1)$ and that of $(C2)$.

$$D(\delta) = \delta^2(T_{l,S} - \underline{v}^S) + \delta(\pi_{h,S} - \underline{v}^S - \pi_S + 2M) + \pi_{l,S} - T_{l,S} + 2(\underline{v}^S - M).$$

Note that $D(1) = \pi_{h,S} + \pi_{l,S} - \pi_S = 0$. Since

$$2(\underline{v}^S - M) = \frac{k\big(1 + k - \sqrt{5k^2 - 2k + 1}\big)}{2},$$

we have

$$\begin{aligned} D(0) &= \pi_{l,S} - T_{l,S} + 2(\underline{v}^S - M) \\ &= \begin{cases} k\big(1 - k - \sqrt{5k^2 - 2k + 1}\big)/2, & \text{if } k \leqq 1/3 \\ k\big(2k - \sqrt{5k^2 - 2k + 1}\big)/2, & \text{if } 1/3 \leqq k \leqq 1/2 \\ \big(-1 + 4k - 2k\sqrt{5k^2 - 2k + 1}\big)/4, & \text{if } 1/2 \leqq k < 2/3 \\ \Big\{-1 + 4k\big(1 + k - \sqrt{5k^2 - 2k + 1}\big)\Big\}\Big/8. & \text{if } 2/3 \leqq k \end{cases} \end{aligned}$$

We can easily confirm that if $k \leqq 1/2$, then $D(0) < 0$ holds, since $\sqrt{5k^2 - 2k + 1} = \sqrt{(1-k)^2 + (2k)^2}$. Consider the case in which $1/2 \leqq k$. Let us denote $H(k)$ by

$$H(k) = \Big(2k\sqrt{5k^2 - 2k + 1}\Big)^2 - (-1 + 4k)^2 = 20k^4 - 8k^3 - 12k^2 + 8k - 1.$$

We observe that $H(1/2) = 1/4$ and that if $1/2 \leqq k$, then $H^{'}(k) = 8(2k-1)(5k^2 + k - 1) \geqslant 0$. Thus, note that if $1/2 \leqq k < 2/3$, then $D(0) < 0$ holds. Also, we can confirm that if $2/3 \leqq k$, then $D(0) < 0$ holds, since $-1 + 4k - 2k\sqrt{5k^2 - 2k + 1} - \big\{-1 + 4k\big(1 + k - \sqrt{5k^2 - 2k + 1}\big)\big\} = 2k\big(\sqrt{5k^2 - 2k + 1} - 2k\big) > 0$.

Now note that $T_{l,S} - \underline{v}^S > 0$ by the definition of $T_{l,S}$ and the uniqueness of $p_S^{\min}$. To conclude, we show that $D(\delta) \leqq 0$, since $T_{l,S} - \underline{v}^S > 0$.

Second, fix k and $\delta \geqslant \underline{\delta}(k)$. Consider the following strategy profile $\sigma(\delta)$:

> Unless some deviations happen, both firms play an SAP. When some deviations happen, both firms switch to the worst stick-and-carrot equilibrium from the next period. In the worst stick-and-carrot equilibrium, both firms charge the minmax price mutually at the initial period. Next period, each firm behaves in the following manner. (a) The case in which $k < 2/3$; With the probability $\gamma(\delta)$, both firms switch to the collusive behavior such that both firms choose the two SAPs with equal probability, using randomization devices, and henceforth they play the SAP. Also, both firms replay the worst stick-and-carrot equilibrium with the remaining probability $1-\gamma(\delta)$. (b) The case in which $k \geqslant 2/3$; Both firms switch to the fully collusive path, which is the unique SAP, with the probability $\gamma(\delta)$ and replay the worst stick-and-carrot equilibrium with the remaining probability $1-\gamma(\delta)$. The probability $\gamma(\delta)$ is as follows.
>
> $$\gamma(\delta) = \frac{(1-\delta)(\underline{v}^S - M)}{\delta\big(\frac{\pi_S}{2} - \underline{v}^S\big)}.$$
>
> Also, in the worst stick-and-carrot equilibrium, both firms switch to the worst stick-and-carrot equilibrium with probability 1 after some deviations.

Note that $\delta(\frac{\pi_S}{2} - \underline{v}^S) > 0$ by Lemma 1. Also, note that if $(C2)$ holds, then $\gamma(\delta) \leqq 1$ and $\sigma(\delta)$ is well-defined. By the construction, the value of a firm in the worst stick-and-carrot equilibrium W is

$$W = (1-\delta)M + \delta\Big\{\gamma(\delta)\frac{\pi_S}{2} + \big(1 - \gamma(\delta)\big)W\Big\}.$$

We solve this by using the definition of $\gamma(\delta)$ to have

$$W = \frac{(1-\delta)M + \delta\gamma(\delta)\frac{\pi_S}{2}}{1 - \delta\big(1 - \gamma(\delta)\big)} = \frac{2\underline{v}^S\big(\frac{\pi_S}{2} - M\big)}{2\big(\frac{\pi_S}{2} - M\big)} = \underline{v}^S.$$

Therefore, a firm in the worst stick-and-carrot equilibrium indeed earns $\underline{v}^S$.

Thus, we have shown this proposition by Proposition 6 and Lemma 2.

$\square$

Proof of Proposition 8.

Consider $\underline{\delta}(k)$ of Proposition 6(b)(i). Note that $\underline{\delta}(k)$ is the solution of the following equation.

$$\frac{k+1+\sqrt{5k^2-2k+1}}{2k}\cdot f_i(\delta)$$
$$= \delta^2(-2k+2)+\delta(-2k+2)-k-1-\sqrt{5k^2-2k+1}=0,$$

where $f_i(\delta)$ is defined by (1). From this, we have

$$\begin{aligned}\underline{\delta}(k) &= \frac{2k-2+\sqrt{(-2k+2)^2-4(-2k+2)(-k-1-\sqrt{5k^2-2k+1})}}{2(-2k+2)} \\ &= \frac{-2(-k+1)+2\sqrt{(-k+1)(k+3+2\sqrt{5k^2-2k+1})}}{4(-k+1)}.\end{aligned}$$

Reformulating this, we have

$$\begin{aligned}\underline{\delta}(k) &= -\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{k+3+2\sqrt{5k^2-2k+1}}{1-k}} \\ &= -\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{k+3}{1-k}+2\sqrt{1+\left(\frac{2k}{1-k}\right)^2}}.\end{aligned}$$

From this, we see that $\underline{\delta}(k)$ is increasing in k when $k \leqq 1/3$. $\square$

Proof of Proposition 9.

First, it is clear that if $k \leqq 1/4$, then $\delta^E(k) < \underline{\delta}(k)$, since $\delta^E(k) = 0$ and $\underline{\delta}(k) > 0$.

Second, consider the case in which $1/4 < k \leqq 1/3$. Note that $\delta^E(k) = 1/4k$ is decreasing in k. We observe that

$$\lim_{k\to 1/4}\delta^E(k) = 1, \quad \delta^E(1/3) = 3/4.$$

Also, recall that $\underline{\delta}(k)$ is increasing in k by Proposition 8. We observe that

$$\underline{\delta}(1/3) = \frac{-1+\sqrt{5+2\sqrt{2}}}{2} > 3/4.$$

Therefore, we have shown that there exists $\hat{k} \in (1/4, 1/3)$ such that if $1/4 < k \leqq \hat{k}$, then $\underline{\delta}(k) \leqq \delta^E(k)$ holds and $\underline{\delta}(k) = \delta^E(k)$ if and only if $k = \hat{k}$. Also, we have shown that if $\hat{k} < k \leqq 1/3$, then $\delta^E(k) < \underline{\delta}(k)$.

Third, consider the case in which $1/3 \leqq k \leqq 1/2$. Note that $\delta^E(k) = \frac{4k-1}{2(4k-k^2-1)}$. Also, note that $\delta^E(k)$ is decreasing in k and that $\delta^E(1/3) = 3/4$. Thus, $\delta^E(k) \leqq 3/4$. Then, we have

$$\begin{aligned} f_{ii}\Big(\frac{3}{4}\Big) &= \frac{9k\big(k+1-\sqrt{5k^2-2k+1}\big)}{8} \\ &\quad + \frac{3\big(5k^2-2k+1-2k\sqrt{5k^2-2k+1}\big)}{4} + 2k^2 - 2k \\ &= \frac{55k^2-19k+6-21k\sqrt{5k^2-2k+1}}{8}, \end{aligned}$$

where $f_{ii}(\delta)$ is defined by (2).

We restrict our attentions to the numerator to show that $f_{ii}(3/4) < 0$. It is clear that $55k^2 - 19k + 6 > 0$. Also, we define

$$\begin{aligned} J_{ii}(k) &= \frac{1}{4}\Big\{(21k)^2(5k^2-2k+1) - (55k^2-19k+6)^2\Big\} \\ &= \frac{1}{4}\cdot(-820k^4 + 1208k^3 - 580k^2 + 228k - 36) \\ &= -205k^4 + 302k^3 - 145k^2 + 57k - 9. \end{aligned}$$

Note that $J''_{ii}(k) = -2460k^2 + 1812k - 290$, which is concave. Since $J''_{ii}(1/3) = 122/3 > 0$ and $J''_{ii}(1/2) = 1 > 0$, $J''_{ii}(k) > 0$ for any $k \in [1/3, 1/2]$. Since $J'_{ii}(k) = -820k^3 + 906k^2 - 290k + 57$, $J'_{ii}(1/3) = 827/27 > 0$ and $J_{ii}(1/3) = 206/81 > 0$, $J_{ii}(k) > 0$ for any $k \in [1/3, 1/2]$. Thus, $55k^2 - 19k + 6 - 21k\sqrt{5k^2-2k+1} < 0$ when $1/3 \leqq k \leqq 1/2$. Therefore, $\delta^E(k) < \underline{\delta}(k)$, since $f_{ii}(3/4) < 0$.

Fourth, consider the case in which $1/2 \leqq k \leqq 2/3$. Note that $\delta^E(k) = \frac{1}{2k(2-k)}$. Also, note that $\delta^E(k)$ is decreasing in k and $\delta^E(1/2) = 2/3$. Thus, $\delta^E(k) \leqq 2/3$. Then we have

$$\begin{aligned} f_{iii}\Big(\frac{2}{3}\Big) &= \frac{4\big(1-2k+6k^2-2k\sqrt{5k^2-2k+1}\big)}{9} \\ &\quad + \frac{2\big(1-2k+5k^2-2k\sqrt{5k^2-2k+1}\big)}{3} - 1 + 2k - 2k^2 \\ &= \frac{1-2k+36k^2-20k\sqrt{5k^2-2k+1}}{9}, \end{aligned}$$

where $f_{iii}(\delta)$ is defined by (3).

We restrict our attentions to the numerator to show that $f_{iii}(2/3) < 0$. It is clear that $1 - 2k + 36k^2 > 0$. Also, we define

$$J_{iii}(k) = (20k)^2(5k^2 - 2k + 1) - (1 - 2k + 36k^2)^2$$
$$= 704k^4 - 656k^3 + 324k^2 + 4k - 1.$$

Note that $J_{iii}^{''}(k) = 8448k^2 - 3936k + 648 > 0$. Since $J_{iii}^{'}(k) = 2816k^3 - 1968k^2 + 648k + 4$, $J_{iii}^{'}(1/2) = 188 > 0$ and $J_{iii}(1/2) = 44 > 0$, $J_{iii}(k) > 0$ for any $k \in [1/2, 2/3]$. Thus, $1 - 2k + 36k^2 - 20k\sqrt{5k^2 - 2k + 1} < 0$. Therefore, $\delta^E(k) < \underline{\delta}(k)$, since $f_{iii}(2/3) < 0$.

Finally, consider the case in which $2/3 < k$. Considering the incentives on the SAP under S rule and the full collusion path under E rule, respectively.

$$\frac{1}{8} \geqslant (1-\delta)\frac{1}{4} + \delta \cdot \underline{v}^S.$$
$$\frac{1}{8} \geqslant (1-\delta)\frac{1}{4} + \delta \cdot \underline{v}^E.$$

Thus, we have shown that $\delta^E(k) < \underline{\delta}(k)$, since $\underline{v}^S > \underline{v}^E$ by Proposition 5. □

Proof of Proposition 11.

Recalling Proposition 2, if $k < 1/4$, then we show that $\pi_M = (1 - 2k) \cdot 2k < \pi_S = (1 - 2k)k + (1 - k)k = 2k - 3k^2$. Also, if $1/4 \leqq k \leqq 1/3$, then we show that $\pi_M = 1/4 < \pi_S = 2k - 3k^2$. Also, if $1/3 \leqq k < 2/3$, then we show that $\pi_M = 1/4 < \pi_S = \frac{1+2k-3k^2}{4}$. Also, if $2/3 \leqq k$, then we show that $\pi_M = \pi_S = 1/4$.

□

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