KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE

OF

ECONOMIC RESEARCH

Discussion Paper No.1005

"Equilibrium Prices of the Market Portfolio in the CAPM
with Incomplete Financial Markets"

Chiaki Hara

October 2018

KYOTO UNIVERSITY

KYOTO, JAPAN

Equilibrium Prices of the Market Portfolio in the CAPM with Incomplete Financial Markets

Chiaki Hara*

Institute of Economic Research, Kyoto University

October 11, 2018   First Version: December 1998

Abstract

In the Capital Asset Pricing Model, we consider how introducing new assets will affect the prices of the existing ones. We prove that introducing new assets into financial markets increases the relative price of the market portfolio with respect to the risk-free bond if the elasticity of the marginal rates of substitution of the mean for standard deviation with respect to the latter is greater than one for every consumer; the relative price of the market portfolio decreases if the elasticity is less than one; and the relative price is left unchanged if the elasticity is equal to one.

1

1 Introduction

1.1 Overview of the Results

In this paper, we consider how introducing new assets into financial markets will affect the prices of the existing ones. Our exercise here is of comparative statics type, whereby we compare two equilibria, one of which is obtained with newly introduced assets and the other without them. For a reason that will soon become clear, we concentrate on the Capital Asset Pricing Model (CAPM), in which, by definition, all consumers believe in the same probability distribution over the states of nature, there is only one good, and their utility functions all depend only on the mean and standard deviation of random future consumptions. We further assume that:

- Consumption takes place only in one period

- There is no production

- No initial endowments for any asset by any consumer

In particular, by the "market portfolio," we mean an asset whose payout is perfectly correlated with the sum of all consumers' initial endowments; and, by the "risk-free bond," we mean the asset that pays one unit of the good with probability one.

The CAPM admits a very strong characterization of equilibrium asset prices, called the security market line. To be more specific, let's index the tradeable assets by i and denote the random payout of asset i by a_i and its equilibrium price by p_i. Denote by q_i the return of asset i (that is, $q_i = p_i^{-1} a_i$ and it is random) and by $\overline{q}_i$ its expected return (that is, $\overline{q}_i = E(q_i) = p_i^{-1} E(a_i)$). Denote by d the payout of the market portfolio and define p_d, q_d, and $\overline{q}_d$ to be its price, return, and expected return. The payout of the risk-free bond is denoted by $\mathbf{1}$. Denote its price and expected return by $p_{\mathbf{1}}$ and $\overline{q}_{\mathbf{1}}$, so that $\overline{q}_{\mathbf{1}} = p_{\mathbf{1}}^{-1}$. Then, at equilibrium, we must have

$$\overline{q}_i - \overline{q}_{\mathbf{1}} = \beta_i (\overline{q}_d - \overline{q}_{\mathbf{1}}) \tag{1}$$

for every asset i, where

$$\beta_i = \frac{C(q_i, q_d)}{V(q_d)},$$

which is called the beta of asset i and measures how much correlated the return of asset i is with that of the market portfolio. We emphasize here that the above equality holds

even when the asset markets are incomplete, as long as the market portfolio and the risk-free bond are available for trade.

It would give rise to unnecessary complications to analyze how the values in equality (1), such as $\bar{q}_i$, $\bar{q}_d$, and β_i, for existing assets are affected when some new assets are introduced, because they are all defined in terms of returns, and the returns are defined by dividing payouts by prices, the latter of which are endogenously determined at equilibrium. An equivalent condition of the security market line (1) in terms of the payouts a_i is that there exist a $t \in R_{++}$ and an $r \in R_{++}$ such that, for every asset i,

$$p_i = tE(a_i) - rC(a_i, d).^1 \tag{2}$$

Since we have assumed that there is no consumption in the first period, we can normalize asset prices p_i so that $t = 1$:

$$p_i = E(a_i) - rC(a_i, d). \tag{3}$$

We can then see that the case of $r = 0$ corresponds to the risk-neutral pricing and that a larger r means a larger risk premium and a larger Sharpe ratio. Moreover, in the CAPM, equilibrium asset prices can be uniquely specified by a single positive number r. The purpose of this paper is to find out under what conditions imposed on utility functions we can unambiguously sign the change in the value of r in equality (3) when new assets are introduced.

To state our main theorem, index the consumers by h and let, for each h,

$$
\begin{aligned}
U_h: \quad R_+ \times R &\rightarrow \quad R \\
(\sigma, \mu) &\mapsto \quad U_h(\sigma, \mu)
\end{aligned}
$$

be the utility function of consumer h over mean μ and standard deviation σ of random future consumptions. Besides smoothness and quasi-concavity, we assume that $D_1 U_h(\sigma, \mu) \leq 0$ and $D_2 U_h(\sigma, \mu) > 0$ for every $(\sigma, \mu) \in R_+ \times R$; and $D_1 U_h(\sigma, \mu) < 0$ if $\sigma > 0$, where D_1 denotes the partial derivative with respect to the first variable σ and D_2 denotes the partial derivative with respect to the second variable μ. That is, a higher mean and a lower standard deviation are always strictly preferred. Define

$$MRS_h(\sigma, \mu) = -\frac{D_1 U_h(\sigma, \mu)}{D_2 U_h(\sigma, \mu)}.$$

3

This marginal rate of substitution measures how much the mean of the consumption should be increased when the standard deviation is increased by one unit, in order for consumer h to enjoy the same utility level. Then the value

$$\frac{\sigma}{MRS_h(\sigma,\mu)} D_1 MRS_h(\sigma,\mu) \tag{4}$$

is the elasticity of the marginal rate of substitution with respect to standard deviation σ. This elasticity is thus greater than one if and only if a 1% increase in the standard deviation increases the marginal rate of substitution by more than 1%.

Suppose now that, in addition to the risk-free bond and the market portfolio (and possibly others), new assets are introduced into markets and they provide increased hedging opportunities with at least one consumer, in the sense that some portfolio of the new assets has non-zero covariance with his initial endowment. Our main result roughly says that if the elasticity (4) is larger than one at every (σ,μ) and for every consumer h, then, for every equilibrium without the new assets, there is an equilibrium with them at which the value of r in (3) is lower; if the elasticity is smaller than one, then the value of r is higher at some equilibrium with the new assets; and if the elasticity equals one, the value remains the same at some equilibrium with the new assets.

To see the applicability of the theorem, consider the following family of utility functions U_h over mean and standard deviation parameterized by $\tau_h > -1$ and $\delta_h > 0$:

$$U_h(\sigma,\mu) = \mu - \frac{\delta_h}{\tau_h + 2}\sigma^{\tau_h + 2}. \tag{5}$$

It is straightforward to check that

$$
\begin{aligned}
MRS_h(\sigma,\mu) &= \delta_h \sigma^{\tau_h + 1}, \\
D_1 MRS_h(\sigma,\mu) &= \delta_h(\tau_h + 1)\sigma^{\tau_h}, \\
\frac{\sigma}{MRS_h(\sigma,\mu)} D_1 MRS_h(\sigma,\mu) &= \tau_h + 1.
\end{aligned}
$$

Hence if $\tau_h > 0$, then introduction of new assets decreases the value of r; if $\tau_h < 0$, then it increases the value of r; and if $\tau_h = 0$, then the value of r remains constant. Recall that the utility function with $\tau_h = 0$, $U_h(\sigma,\mu) = \mu - (\delta_h/2)\sigma^2$, is obtained when a von-Neumann-Morgenstern (vNM for short) utility function exhibits the constant coefficient δ_h of absolute risk aversion and the random future consumptions follow Gaussian distributions. It is well known (and was proved by Oh (1990, 1996) and others) that,

4

in this case, the relative prices of existing assets among themselves are not affected by introduction of new assets. Our main theorem thus provides an alternative proof of this well known fact, but it shows more than that: in terms of elasticities of marginal rates of substitution between mean and standard deviation, the well known case is the critical case, above which introduction of new assets increases the price of the market portfolio and below which the former decreases the latter. In particular, the parameter τ_h in the functional form (5) measures the deviation from the case where the equilibrium price of the market portfolio is invariant to the financial market structures.

We should also note that, when it comes to evaluating the effect on the prices of existing assets by introduction of new assets, the marginal rates of substitution $MRS_h(\sigma, \mu)$, which were shown by Lajeri and Nielsen (2000) to represent degrees of absolute risk aversion over the choice between mean and standard deviation, are not, for themselves, a very helpful piece of information. What is of crucial importance is how much *in percentage* they will increase as the standard deviation increases. Indeed, in the above example, $MRS_h(\sigma, \mu)$ can be arbitrarily increased or decreased at every (σ, μ) by varying parameter δ_h even when keeping $\tau_h = 0$; but varying δ_h does not affect at all the sign in the change in the value of r when new assets are introduced.

Another example of the unitary elasticity case is

$$U_h(\sigma, \mu) = \mu - (\delta_h/2)(\sigma^2 + \mu^2)$$

with $\delta_h > 0$. This is obtained when a vNM utility function is a quadratic function $u_h(w) = w - (\delta_h/2)w^2$. It is then routine to show that the elasticity of the marginal rates of substitution is always equal to one. Our main theorem again provides an alternative proof of another well-known fact, which is the invariance of the relative prices of existing assets among themselves with quadratic utility functions.

A more general example is the case when U_h is a quadratic function of σ and μ,

$$U_h(\sigma, \mu) = c_h^0 \sigma^2 + c_h^1 \mu + c_h^2 \mu^2,$$

where c_h^0, c_h^1, and c_h^2 are constants. It is again routine to show that the elasticity of the marginal rates of substitution is always equal to one.

Our main result has a nice welfare implication of introduction of new assets. It is often argued that whether it is beneficial to consumers is often ambiguous, because the

5

negative pecuniary externality arising from the changes in the prices of existing assets may outweigh the benefit of enhanced risk-hedging opportunities. Suppose now that all consumers have utility functions whose elasticity is larger than one. Suppose also that one of them has an initial risky endowment which perfectly negatively correlated with the market portfolio. Since holding the market portfolio reduces the risk from his initial endowment, his portfolio at equilibrium must consist of a positive amount of the market portfolio and some amount of the risk-free bond. Since the market portfolio is assumed to be traded even before introduction of new assets, they do not enhance his risk-hedging opportunities in any essential way. According to our theorem, the price of the market portfolio goes up as a consequence of introducing new assets. We can thus conclude, with no formal calculation, that this consumer becomes worse off after introduction of new assets.

Our main result requires all consumers to have the elasticities of marginal rates of substitution greater than one, or all having the common elasticity equal to one, or all having elasticities less than one. It does not allow for some consumers to have elasticities greater than one and, at the same time, others to have elasticities less than one. It would be nice if we could establish the same sort of predictions on the directions of the change in r when the elasticities greater and less than one coexist. The natural candidate that aggregates different consumers' elasticities is the representative consumer's counterpart. We shall, however, show by means of an example that the representative consumer's elasticity may not provide a correct prediction. Indeed, it is completely possible that his elasticity is greater than or less than one, and yet the value of r may change in either direction depending on the types of newly introduced.

Our main theorem is not a comparative statics exercise presuming the existence of the two equilibria to be compared. Rather, when the existence of an equilibrium is assumed, it establishes the existence of another equilibrium having a particular property in relation to the first one. We can thus obtain the existence of an (arbitrary) equilibrium for a fixed collection of assets at no extra cost. The entire argument for our existence results depends on the security market line and the "mutual fund theorem." We shall also state and prove them in a way that is immediately applicable for the existence results.

1.2 Relationship with the Literature

Weil (1992) was the first to show that, under some assumptions on vNM utility functions, the equity premium puzzle of Mehra and Prescott (1985) can be partially solved by the incompleteness of asset markets. More specifically, he showed that if the consumers have the same vNM utility function exhibiting decreasing absolute risk aversion and decreasing prudence and if the risk-free bond and the market portfolio are the only assets available for trade, then the price of the market portfolio is lower than when the asset markets are complete, thereby justifying the observation by Mehra and Prescott that the price of the market portfolio is much lower than could be accounted for by a reasonable range of coefficients of relative risk aversion with complete financial markets. His model is different from ours in that there are two consumption periods; the consumers are ex-ante identical; their utility function may not depend only on mean and standard deviation (and thus does not satisfy the CAPM assumption); and, most importantly, only the two polar cases, the complete asset markets and the asset markets consisting only of the risk-free bond and the market portfolio, are compared. Our model, in particular, allows for two arbitrary asset markets, as long as one can be obtained by adding more assets to the other. Note that some economically interesting phenomena, such as some consumers getting worse off as a result of introducing new assets, do not occur when the comparison is restricted to the two polar cases. Thus Weil's result, on its own, does not really tell us whether there is anything intrinsic in the two polar cases when it comes to comparing equity premia. Our result however shows that there is nothing intrinsic in the polar cases.

The notion of prudence was given by Kimball (1990). It refers to the tendency that a consumer is willing to give up more of today's consumption when his consumption tomorrow is random than he is so when his consumption tomorrow is certain and equal to the expected value of the random consumption. Analogously to the Arrow-Pratt coefficient of absolute risk aversion, the coefficient of prudence can be defined as the ratio of the second and third derivatives. The notion of decreasing prudence can then be given. The equivalent, differential condition of decreasing prudence involves up to the fourth derivatives and was much earlier used by Chipman (1973), when combined with the assumption that the consumptions follow Gaussian distributions, to obtain the concavity of the derived utility function over the mean and *variance*.[2]

Note that equality (3) is equivalent to

$$p_i = E\left(((1+r)\mathbf{1} - rd)\,a_i\right),$$

which is, in turn, equivalent to saying that the state price density (or pricing kernel) is equal to $(1+r)\mathbf{1} - rd$. Note that the variance of this random variable, $V\left((1+r)\mathbf{1} - rd\right)$, is analogous to the volatility of the state price density, a variable of interest in Hansen and Jaganathan (1990). It is an increasing function of r. Our main result thus says that the volatility of the state price density decreases as new assets are introduced into financial markets if all consumers have elasticities of marginal rates of substitution of mean for standard deviation greater than one; and it increases if they all have the elasticities less than one.[3]

Oh (1990, 1996) obtained the security market line and the mutual fund theorem in the CAPM with incomplete financial markets. In particular, he allowed for the case where some consumers' initial endowments cannot be represented as any linear combinations of the traded assets. He (and his predecessors referred to in his papers) also proved that introducing new assets does not change the equilibrium prices of the existing ones if the consumers have quadratic vNM utility functions or if they have negative exponential vNM utility functions and consumptions follow Gaussian distributions. Our comparative statics result not only covers these invariance conditions but predicts the direction of changes in the prices of the market portfolio according to the elasticities of the marginal rates of substitution. Dana (1999) and Hens and Loeffler (1996) established the existence of an equilibrium in the CAPM with complete markets based on the intermediate value theorem, without appealing to the fixed point theorem. Underlying this approach are the security market line and the mutual theorem, because they reduce the task of finding an equilibrium to one of solving a single equation by a single unknown. The fact that the intermediate value theorem is sufficient for the existence proof is quite important for our comparative statics result, because the result is of global nature and also has an order structure, to be discussed in Section 4. We note in passing that this comparative statics exercise based on the intermediate value theorem is a very special case of monotone comparative statics of Milgrom and Shannon (1994). Dana (1999), Hens and Loeffler (1996), Hens, Laitenberger, and Loeffler (2000) also provided sufficient conditions for the uniqueness of an equilibrium, which are closely related with the condition by Lajeri and Nielsen (2000) for decreasing (or increasing) risk aversion.

Detemple and Selden (1991) provided a general equilibrium model similar to but different from the CAPM, in which there are the risk-free bond and a stock (which thus equals the market portfolio) initially traded in markets, and introduction of an option on the stock increases the stock price. Our result, however, allows for arbitrary assets to be introduced and still unambiguously predicts the direction of the changes in the prices of the market portfolio.

Hara (2011) proved that if there are only finitely many states, S in number, then, regardless of the consumers' preferences or initial endowments, there is a sequence of S assets such that if those S assets are introduced into markets one by one in the order of the sequence, then:

- The asset markets eventually become complete;

- every time a new asset is introduced, the prices of the previously introduced ones are not changed at all.

In view of this result, we can say that the main result of this paper depends crucially on the assumption that the risk-free bond and the market portfolio are always available for trade. While not getting into details, we can point out that the proof method of this paper and Hara (2011) are quite different. This paper uses the intermediate value theorem, while Hara (2011) uses the zero intersection property of a continuous section on a vector bundle, which is even stronger than the fixed-point property.

Elul (1999) showed that if the markets are sufficiently incomplete and there are not too many types of consumers, then it is generically possible to introduce a new security that leads to a Pareto-improving equilibrium allocation. A key step in his proof is to show that there generically exists a non-redundant asset whose introduction does not change the prices of any existing assets. This property does not hold in our framework. The genericity condition he used refers to a suitably defined set of utility functions and initial endowments, in which utility functions depending only on mean and standard deviation constitute a negligible set.

After some earlier versions of this paper were written, Koch-Medina and Wenzel-burger (2018) published some results that are also included in this paper. Among others, they claimed that their comparative statics result (Proposition 8 of their paper) extended our comparative statics result (Theorem 4 of this paper) to the case in which

the market portfolio is non-tradeable. In Section 6, we articulate the sense in which our comparative statics result is extended, and give an example in which it is not extended in the sense we deem appropriate. The example, indeed, shows that introducing a new asset increases, rather than decreases, the risk premium and the Sharpe ratio even when the representative consumer has a utility function (5) with $\tau_h > 0$.

1.3   Organization of the Paper

Section 2 formulates the model. Section 3 establishes the existence and characterization of an equilibrium in the CAPM, that is, the security market line and the mutual fund theorem. These results are concerned with the case where a collection of assets is fixed. Section 4 is the central part of this paper. It provides the comparative statics result regarding introduction of new assets are introduced and an intuitive account on it. Section 5 shows that it is impossible to use the utility function of the representative consumer to predict the direction of changes in the prices of the market portfolio induced by introduction of new assets. Section 6 concludes, mentioning the possibility of extending the results in this paper to other versions of the CAPM. Appendix A contains the proof of the characterizations of an equilibrium the CAPM. Appendix B proves the existence of an equilibrium in the CAPM. Appendix C proves our main result.

2   The Model

The uncertainty of the economy is described by a probability space Ω. There is only one physical good available in every state and the commodity space X is taken to be the L^2 space over Ω. For simplicity of exposition, throughout this paper, we identify an element of the L^2 space, which is defined to be an equivalent class of random variables that are equal to one another with probability one, with a random variable itself in the equivalent class. The mean $E : X \to R$, variance $V : X \to R$, standard deviation $S : X \to R$, and covariance $C : X \times X \to R$ are defined in the standard way.

Each *consumer*, indexed $h \in \{1, \cdots, H\}$, has a utility function $U_h : R_+ \times R$ over the standard deviations and the means of random consumptions. We assume that U_h is twice continuously differentiable, strictly quasi-concave, and satisfies $D_1 U_h(\sigma, \mu) \leq 0$ and $D_2 U_h(\sigma, \mu) > 0$ for every $(\sigma, \mu) \in R_+ \times R$; and $D_1 U_h(\sigma, \mu) < 0$ if $\sigma > 0$. Define

$W_h : X \to R$ by $W_h(x_h) = U_h(S(x_h), E(x_h))$ for every $x_h \in X$. Then W_h assigns the utility level he obtains from a random consumption $x_h \in X$. The initial endowment vector of consumer h is denoted by $d_h \in X$. In the language of finance, U_h represents the consumer's attitude towards risk and d_h represents his initial risk exposure, which he would like to hedge against by participating in market transactions.

For simplicity of exposition, we define a consumer's utility maximization problem and an equilibrium of asset markets directly in terms of market spans and state price functions. A *market span* is a linear subspace M of X, to be understood as the linear subspace spanned by the traded assets; the vectors on M are thus understood as representing the payouts of portfolios. If $M \neq X$, then not all risks can be hedged by asset trades, in which case we say that the asset markets are *incomplete*. A *state price function* is a real-valued linear function $p : M \to R$ on a given market span M, to be understood as coinciding with the arbitrage-free prices of the payouts of portfolios, where R denotes the set of real numbers. The utility maximization problem of consumer h is then:

$$\text{Max}_{x_h \in X} \quad W_h(x_h),$$
$$\text{s.t.} \quad x_h - d_h \in M, \tag{6}$$
$$p(x_h - d_h) \leq 0.$$

Note that the linearity of M and p means that there are no transaction costs or short-sales constraints.

We say that a state price function p and a consumption allocation $(x_h^*)_{h \in \{1, \cdots, H\}}$ constitute an *equilibrium* for the market span M if, for every h, x_h^* is a solution to the above maximization problem and $\sum_{h=1}^{H} x_h^* = \sum_{h=1}^{H} d_h$.

3 CAPM with a Fixed Market Span

In this section we present the existence and characterization of an equilibrium when a single market span is fixed. This is an intermediate step toward our main theorem, where we compare equilibria for two market spans.

We start off with the characterization result, which consists of the security market line and the mutual fund theorem, and then move on to the existence result. Although both results are more or less well known, we here present them, as well as their proofs, for two reasons. The first one is internal consistency of our exposition: The characterization

results form the basis for the subsequent analysis. The second one is that the existence result covers the case of incomplete asset markets, which has its own interest because Dana (1996) and Hens and Loeffler (1996) assumed complete markets.

3.1 Characterization

We first introduce some notation. Denote by $\mathbf{1}$ the element of X that takes value 1 at every $\omega \in \Omega$. We can think of this as representing the payout of the *risk-free bond*. Write $d = \sum_{h=1}^{H} d_h \in X$. This is nothing but the aggregate initial endowment, but we call it the *market portfolio*, following the tradition of the CAPM. For each subspace M of X, denote by orth$M$ the orthogonal complement of M, by π_M the orthogonal projection from X onto M, and by P_M the set of all state price functions defined on M. Let $N = \{x \in X | E(x) = 0\}$. Then $\pi_N(d)$ is the "de-meaned" market portfolio. Denote by $\mathcal{M}$ the set of all market spans that contain $\mathbf{1}$ and d.

The following characterization theorem is essentially due to Oh (1990, 1996).

Proposition 1 *Let $M \in \mathcal{M}$ and suppose that $p \in P_M$ and $(x_h^*)_{h \in \{1, \cdots, H\}} \in X^H$ constitute an equilibrium for M.*

1. Assume that $S(d) > 0$. Then there exist a $t \in R_{++}$ and an $r \in R_{++}$ such that

$$p(m) = E((t\mathbf{1} - r\pi_N(d))m) \tag{7}$$

for all $m \in M$.

2. For every h, there exist an $a_h^* \in R_+$ and a $b_h^* \in R$ such that

$$x_h^* = a_h^* \pi_N(d) + b_h^* \mathbf{1} + \pi_{\text{orth}M}(d_h).$$

Part 1 of this proposition provides the security market line. It implies that the state price density is a strictly positive combination of the risk-free bond $\mathbf{1}$ and the negative of the de-meaned market portfolio, $-\pi_N(d)$. Note that the relative prices of assets on N is invariant to the choice of M. If $p(d) \neq 0$, then, for those $m \in M$ with $p(m) \neq 0$, equality (7) can be equivalently written in the return form as:

$$p(m)^{-1}E(m) - p(\mathbf{1})^{-1} = \beta(m)(p(d)^{-1}E(d) - p(\mathbf{1})^{-1}), \tag{8}$$

where

$$\beta(m) = \frac{C(p(m)^{-1}m, p(d)^{-1}d)}{V(p(d)^{-1}d)},$$

which is the beta of the asset m. Equality (8) says that the expected rate of return of the asset in excess of the risk-free rate of return must be proportional to the covariance between its rate of return and the rate of return on the market portfolio d.

The condition that t be positive is equivalent to saying that the risk-free bond must have a positive price. The condition that r be positive is equivalent to saying that the de-meaned market portfolio $\pi_N(d)$ must have a negative price. This is equivalent to $p(d)^{-1}E(d) > p(\mathbf{1})^{-1}$ if $p(d) > 0$: hence the presence of the equity premium. When the expected rates of return $p(m)^{-1}E(m)$ is regarded as a linear function of beta $\beta(m)$, the positivity of r is also equivalent to saying that this linear function has a positive slope.

Part 2 of Proposition 1 is nothing but the mutual fund theorem, with a modification due to incomplete asset markets. It says that every consumer's equilibrium consumption must consist of three terms. The first one is made of the risk-free bond and the second one is made of the market portfolio. Note that everyone holds a non-negative amount of the market portfolio, while some consumers may take a negative amount of (that is, sell short of) the risk-free bond. The third term represents the initial endowment risk that a consumer cannot hedge by trading in the asset markets; this would be zero were the asset markets to be complete. Recall that a payout $x \in M$ is on the mean-variance efficient frontier in terms of returns if and only if there exist an $a \in R_+$ and $b \in R$ such that $x = a\pi_N(d) + b\mathbf{1}$ (under the assumption that the required mean b is sufficiently large so that if $E(x) \geq b$ then $p(x) > 0$ for every $x \in M$). Hence our mutual fund theorem says that, even with incomplete asset markets, once subtracted by the unhedgeable endowment risk, every consumer's equilibrium consumption is on the mean-variance efficient frontier.

A proof of Proposition 1 is given in Appendix A.

3.2 Existence

Proposition 2 *For every $M \in \mathcal{M}$, there exists an equilibrium for M.*

It is worthwhile to note that, regardless of the dimension of the market span M, the proposition can be proved by the intermediate value theorem; the fixed point theorem, which is the basic tool to equilibrium existence theorems, is not necessary. The reason can be easily found in Proposition 1. All state price functions that can arise at equilibrium are parameterized by two variables, t and r. Since they are homogeneous of

degree zero, we can take $t = 1$ and all relevant prices can be parameterized by a single parameter r. One the other hand, the consumers' utility maximization problem involves two variables, a_h and b_h, for which the market clearing conditions are $\sum_{h=1}^{H} a_h^* = 1$ and $\sum_{h=1}^{H} b_h^* = E(d)$. By Walras law, the first market clearing condition implies the second. Hence, in the CAPM, searching for an equilibrium is nothing but satisfying a single equation $\sum_{h=1}^{H} a_h^* = 1$ by choosing a single unknown r. The intermediate value theorem is therefore sufficient for this task. The only caveat here is that, with the unbounded consumption set X, those values of r under which the aggregate demand function is well defined may not constitute an interval. We thus need to be careful about on which interval we apply the theorem. All of these are done in Appendix B.

4   Comparative Statics with Variable Market Spans

In this section, we present our main theorem regarding the effect of introducing new assets on the price (and thus the expected rate of return) on the market portfolio. To begin, define $MRS(\cdot|U_h) : R_{++} \times R \to R_{++}$ by

$$MRS(\sigma, \mu|U_h) = -\frac{D_1 U_h(\sigma, \mu)}{D_2 U_h(\sigma, \mu)}.$$

This marginal rate of substitution measures how much the mean of the consumption should be increased when the standard deviation is increased by one unit, in order to keep the consumer on the same utility level as before. Note that $MRS(\cdot|U_h)$ is defined only for strictly positive standard deviations (and hence takes positive values) and continuously differentiable.

Definition 3 We say that U_h has *elastic marginal rates of substitution* (EMRS for short) if, for every $(\sigma, \mu) \in R_{++} \times R$,

$$\frac{\sigma}{MRS(\sigma, \mu|U_h)} D_1 MRS(\sigma, \mu|U_h) > 1.$$

We say that U_h has *inelastic marginal rates of substitution* (IMRS for short) if, for every $(\sigma, \mu) \in R_{++} \times R$,

$$\frac{\sigma}{MRS(\sigma, \mu|U_h)} D_1 MRS(\sigma, \mu|U_h) < 1.$$

We say that U_h has *unitarily elastic marginal rates of substitution* (UMRS for short)

14

if, for every $(\sigma, \mu) \in R_{++} \times R$,

$$\frac{\sigma}{MRS_h(\sigma, \mu | U_h)} D_1 MRS(\sigma, \mu | U_h) = 1.$$

The definition should be clear. The left hand side is the elasticity of the marginal rates of substitution with respect to standard deviations, when the mean is fixed. If the elasticity is larger than one, then a 1% increase in the standard deviation increases the marginal rate of substitution by more than 1%, in which case we say that U_h has elastic marginal rate of substitution. Inelastic and unitarily elastic marginal rates of substitutions are defined analogously.

Assume that $S(d) > 0$ and define $\widehat{d} = S(d)^{-1} \pi_N(d)$. This is the "normalized" market portfolio with zero mean and unit standard deviation. According to Part 1 of Proposition 1, for every $M \in \mathcal{M}$, every equilibrium price function is a positive scalar multiple of a $p \in P_M$ for which there exists an $r \in R_{++}$ such that $p(m) = E((\mathbf{1} - r\widehat{d})m)$ for every $m \in M$. So, for each $r \in R_{++}$, define $\varphi(r) \in P_X$ by $\varphi(r)(m) = E((\mathbf{1} - r\widehat{d})m)$. Note that $\varphi(r)(d) = E(d) - rS(d)$. Hence, if $r' < r$, then the relative price of the market portfolio with respect to the risk-free bond is higher when the state price function equals $\varphi(r')$ than when it equals $\varphi(r)$. In other words, the risk premium is lower with r' than with r. Define $P_M^* = \{r \in R_{++} | \varphi(r) \text{ is an equilibrium state price function for } M\}$.

Theorem 4 *Assume that $S(d) > 0$ and that $D_1 U_h(0, \mu) = 0$ for every $h \in \{1, \cdots, H\}$ and $\mu \in R$. Let $M \in \mathcal{M}$, $L \in \mathcal{M}$, and $M \subset L$. Assume also that there exists an h such that $S(\pi_{\mathrm{orth}M}(d_h)) > S(\pi_{\mathrm{orth}L}(d_h))$.*

1. If every U_h has EMRS, then, for every $r^M \in P_M^*$, there exists an $r^L \in P_L^*$ such that $r^M > r^L$.

2. If every U_h has EMRS, then, for every $r^L \in P_L^*$, there exists an $r^M \in P_M^*$ such that $r^M > r^L$.

3. If every U_h has IMRS, then, for every $r^M \in P_M^*$, there exists an $r^L \in P_L^*$ such that $r^M < r^L$.

4. If every U_h has IMRS, then, for every $r^L \in P_L^*$, there exists an $r^M \in P_M^*$ such that $r^M < r^L$.

5. If every U_h has UMRS, then $P_M^* = P_L^*$.

The first assumption of the theorem, $S(d) > 0$, is indispensable because $E|M \in P_M^* \cap P_L^*$ if $S(d) = 0$, which is shown in Appendix B. The next assumption, $D_1 U_h(0, \mu) = 0$

for every $h \in \{1, \cdots, H\}$ and $\mu \in R$, implies that every consumer has a positive demand for the market portfolio at every equilibrium. This assumption is satisfied if the U_h are derived from a vNM utility function in the expected utility form. The expansion from M to L is of course interpreted as a consequence of introduction of new assets. The inequality $S(\pi_{\mathrm{orth}M}(d_h)) > S(\pi_{\mathrm{orth}L}(d_h))$ is equivalent to the existence of an $z \in L \backslash M$ for which $C(z, d_h) \neq 0$. Hence the existence of an h with $S(\pi_{\mathrm{orth}M}(d_h)) > S(\pi_{\mathrm{orth}L}(d_h))$ means that the new assets have indeed enhanced the risk-hedging opportunities for some consumer. Part 1 then claims that if every U_h has EMRS, then, for every equilibrium before the introduction of the new assets, there exists an equilibrium after the introduction at which the price of the market portfolio is higher. Part 2 claims that, with EMRS, for every equilibrium after the introduction, there exists an equilibrium before the introduction at which the price of the market portfolio is lower. Parts 1 and 2 are equivalent if the equilibria for M and L are both unique. If both P_M^* and P_L^* are compact, then Part 1 is equivalent to $\min P_M^* > \min P_L^*$ and Part 2 is equivalent to $\max P_M^* > \max P_L^*$. In other words, the *interval* of equilibrium prices, $[\min P_M^*, \max P_M^*]$, is a strictly decreasing function of the market span M with respect to the standard order $\geq$ on R. The symmetric interpretation can be given to Parts 3 and 4. Part 5 says that the set of the equilibrium price functions are not affected by market spans under the assumption of UMRS.

Although the proof of Theorem 4 is given in Appendix C, it will be helpful to give an informal account on it, ignoring certain technical points to be taken care of in Appendix C. We concentrate on Part 1.

It is shown in Appendix B that the utility maximization problem is reduced to a two-variable one:[4]

$$
\begin{aligned}
\mathrm{Max}_{(a_h, b_h) \in R^2} \quad & W_h(a_h \widehat{d} + b_h \mathbf{1} + \pi_{\mathrm{orth}M}(d_h)), \\
\mathrm{s.t.} \quad & -r a_h + b_h \leq -r C(d_h, \widehat{d}) + E(d_h).
\end{aligned}
\tag{9}
$$

We can interpret this utility maximization problem as the one where the utility function is

$$
U_h^M(a_h, b_h) \equiv W_h(a_h \widehat{d} + b_h \mathbf{1} + \pi_{\mathrm{orth}M}(d_h)),
\tag{10}
$$

which is subordinated to M, the budget is derived from the initial endowment $(C(d_h, \widehat{d}), E(d))$, which is *not* subordinated to M, and the price vector is $(-r, 1)$. Denote by $(a_h^M(r), b_h^M(r)) \in R^2$ the solution to the above maximization problem. To prove Part 1, it suffices to show

that $a_h^M(r^M) \le a_h^L(r^M)$ for every h, with strict inequality for some h. In fact, then, $\sum_{h=1}^H a_h^M(r^M) < \sum_{h=1}^H a_h^L(r^M)$. Since $\varphi(r^M)$ is an equilibrium price function for M, $\sum_{h=1}^H a_h^M(r^M) = S(d)$ and the inequality implies that there is an excess demand for the market portfolio at $\varphi(r^M)$ when the market span is L. On the other hand, we must have $\sum_{h=1}^H a_h^L(0) = 0$, because the consumers do not demand the market portfolio in the absence of risk premium. So there is an excess supply of the market portfolio at $\varphi(0)$. By the intermediate value theorem, there must exist an $r^L \in (0, r^M)$ such that $\sum_{h=1}^H a_h^L(r^L) = S(d)$. Part 1 of Theorem 4 would then follow.

The inequality $a_h^M(r^M) \le a_h^L(r^M)$ follows if the indifference curve going through $(a_h^M(r^M), b_h^M(r^M))$ becomes less steep as the market span expands from M to L; and this is the point where the assumption of EMRS is used. In fact, the slope of the indifference curve going through any (a_h, b_h) is

$$MRS(a_h, b_h | U_h^M) = -\frac{D_1 U_h^M(a_h, b_h)}{D_2 U_h^M(a_h, b_h)},$$

which is the marginal rate of substitution of U_h^M that measures how much of the risk-free bond should be increased when the normalized market portfolio is increased by one in order to keep the consumer on the same utility level as before, in the presence of the unhedgeable initial endowment risk $\pi_{\mathrm{orth}M}(d_h)$. The smaller $MRS(a_h, b_h | U_h^M)$, the flatter the indifference curve at (a_h, b_h). Write $\theta_h^M = S(\pi_{\mathrm{orth}M}(d_h))$. Then $E(a_h \hat{d} + b_h \mathbf{1} + \pi_{\mathrm{orth}M}(d_h)) = b_h$ and $S(a_h \hat{d} + b_h \mathbf{1} + \pi_{\mathrm{orth}M}(d_h)) = (a_h^2 + (\theta_h^M)^2)^{1/2}$ because $S(\hat{d}) = 1$ and $C(\hat{d}, \pi_{\mathrm{orth}M}(d_h)) = 0$. Thus $U_h^M(a_h, b_h) = U_h((a_h^2 + (\theta_h^M)^2)^{1/2}, b_h)$. Since the derivative of the function $a_h \mapsto (a_h^2 + (\theta_h^M)^2)^{1/2}$ equals $a_h(a_h^2 + (\theta_h^M)^2)^{-1/2}$, the chain rule differentiation implies that

$$MRS(a_h, b_h | U_h^M) = a_h \frac{MRS((a_h^2 + (\theta_h^M)^2)^{1/2}, b_h | U_h)}{(a_h^2 + (\theta_h^M)^2)^{1/2}}. \tag{11}$$

The same expression can be obtained for L. Since $M \subset L$, $\theta_h^M \ge \theta_h^L$. If $\theta_h^M = \theta_h^L$, then

$$MRS(a_h, b_h | U_h^M) = MRS(a_h, b_h | U_h^L)$$

and thus $a_h^M(r^M) = a_h^L(r^M)$. If $\theta_h^M > \theta_h^L$, where such an h indeed exists by the assumption of the theorem, then $(a_h^2 + (\theta_h^M)^2)^{1/2} > (a_h^2 + (\theta_h^L)^2)^{1/2}$. Note that the square root appears in two places on the right hand side of equality (11). First, as the denominator of the fraction. Second, as the σ-variable for the function $MRS(\cdot | U_h)$ in the numerator

of the fraction. The whole fraction decreases if the elasticity of $MRS(\cdot|U_h)$ with respect to σ is greater than one, which is nothing but our EMRS assumption. We thus obtain

$$MRS(a_h, b_h|U_h^M) > MRS(a_h, b_h|U_h^L), \tag{12}$$

that is, the indifference curve becomes flatter as the market span expands. Therefore $a_h^M(r^M) > a_h^L(r^M)$. It is important to note that the condition that $MRS(\cdot|U_h)$ is an increasing function of σ is not sufficient to guarantee inequality (12). The reason is that when a_h increases by one unit, the standard deviation of the consumption $a_h\widehat{d} + b_h\mathbf{1} + \pi_{\mathrm{orth}M}(d_h)$ increases only by $a_h(a_h^2 + (\theta_h^M)^2)^{-1/2}$, which is less than one if $\theta_h^M > 0$ and a decreasing function of θ_h^M. To guarantee that the indifference curve becomes flatter as θ_h^M decreases, therefore, we need to assume that the decrease in $MRS_h((a_h^2 + (\theta_h^M)^2)^{1/2}, b_h|U_h)$ caused by the decrease in $(a_h^2 + (\theta_h^M)^2)^{1/2}$ is sufficiently large in absolute value. This is what is done by our EMRS assumption. In the language of contract theory, we can say that the parameter θ_h^M generates the single-crossing property among the utility functions U_h^M for various $M \in \mathcal{M}$. The consequence of this property is the monotonicity of the interval $[\min P_M^*, \max P_M^*]$.

Note finally that the unhedgeable parts of initial endowments, $\pi_{\mathrm{orth}M}(d_h)$, plays the same role in the proof as the background risks of Ross (1981) and Pratt and Zeckhauser (1987). The difference is that we applied the zero correlation property, while Ross used mean-preserving spreads and Pratt and Zeckhauser assumed stochastic independence.

5 Representative Consumer

In this section we show by means of an example that when some consumers have EMRS and others IMRS, the equilibrium price of the market portfolio may increase or decrease by the introduction of new assets into financial markets, depending on the payout structure of the newly introduced assets. This example also shows that, while the elasticity of marginal rate of substitution of mean for standard deviation is a well defined concept even for the representative consumer, it cannot be used to predict the directions of changes in the equilbrium prices of the market portfolio.

To be specific, we consider the parameteric family of utility functions that appeared in the introduction (5):

$$U_h(\sigma, \mu) = \mu - \frac{\delta_h}{\tau_h + 2} \sigma^{\tau_h + 2}. \tag{13}$$

Since the utility function is quasi-linear with respect to mean, the equilibrium is unique and equilibrium allocation is a solution to the optimization problem of maximizing the (non-weighted) sum $\sum W_h(x_h)$ subject to $x_h - d_h \in M$ for every h and $\sum x_h = \sum d_h$. Denote the value function of this maximization problem with $M = X$ by W.

It is not difficult to show that W depends only on mean and standard deviation, and quasi-linear with respect to mean. We can thus write

$$W(x) = E(x) - u(S(x))$$

for some increasing and convex function u defined over R_+. Thus, if we define a utility function U over mean μ and standard deviation σ by $U(\sigma, \mu) = \mu - u(\sigma)$, then $W(x) = U(S(x), E(x))$ and, thus, U represents the representative consumer's utility in terms of mean and standard deviation. Denote the marginal rate of substitution derived from U by MRS, then it depends on σ but not on μ and satisfies $MRS(\sigma) = u'(\sigma)$. Thus

$$\frac{\sigma}{MRS(\sigma)} MRS'(\sigma) = \frac{u''(\sigma)\sigma}{u'(\sigma)}, \tag{14}$$

that is, the elasticity of the marginal rate of substitution of U equals the Arrow-Pratt measure of relative risk aversion of u, except that there is no -1 multiplied, because u is convex. The same can be said of for the individual consumers' utility functions U_h. We can thus apply part 2 of Corollary 7 and part 2 of Proposition 15 of Hara, Huang, and Kuzmics (2007) to show that ERC is a decreasing function, starting the maximum τ_h and converging to the minimum τ_h. This implies that if the standard deviation of the market portfolio is small, then (14) takes a value close to the maximum τ_h, and if the standard deviation of the market portfolio is large, then (14) takes a value close to the minimum τ_h.

Now consider the following example. There are four states and four consumers. Let $\hat{d}$, y^1, y^2 constitute an orthonormal basis of N. Their initial endowments are

$$
\begin{aligned}
d_1 &= c(\hat{d} + y^1), \\
d_2 &= c(\hat{d} - y^1), \\
d_3 &= c(\hat{d} + y^2), \\
d_4 &= c(\hat{d} - y^2),
\end{aligned}
$$

where c is a positive constant. The market portfolio equals $c\hat{d}$. Consumers $h = 1, 2$ have the same value of τ_h, which is greater than one. Consumers $h = 3, 4$ have the same value of τ_h, which is less than one. Assume that the market portfolio and the risk-free bond are initially the only asset available for trade.

If c is small, then the representative consumer's elasticity is greater than one; if c is large, then the representative consumer's elasticity is less than one. If we predicted the direction of changes in the equilibrium prices of the market portfolio based on the elasticity, then we would conclude that the introduction of new assets will decrease the price of the market portfolio when c is small, and it will increase the price if c is large.

However, if an asset with payout y^1 is introduced, only the first two consumers will trade it; hence, by $\tau_h > 1$, the price of the market portfolio goes up. On the other hand, if an asset with payout y^2 is introduced, only the last two consumers will trade it; hence, by $\tau_h < 1$, the price of the market portfolio goes down. This is regardless of the values of c. Thus the prediction based on the representative consumer's elasticity is incorrect.

6   Conclusion

We have established sufficient conditions in terms of utility functions under which introducing new assets increases or decreases the risk premium at equilibrium in the CAPM. Those conditions are sufficient to derive the direction of changes in the risk premium unambiguously regardless of what the new assets under consideration are like. It is also noteworthy that these conditions are on the elasticity of the marginal rates of substitution between mean and standard deviation, not on the marginal rate of substitution itself.

There are a couple of possible directions of future research. Although we assumed that the market portfolio and the risk-free bond are available for trade from the beginning, there are many situations of practical importance where this is not really the

case. One is the case where the bond pays in nominal amounts (in dollars, say) and not indexed to inflation. Another is where the traded assets reflects the random dividends of the firms and the consumers as a whole suffer from labor income risk which is not reflected by the firms' dividends. It is therefore interesting to see whether the same elasticity conditions remain to be sufficient for our comparative statics result when we no longer assume that the market portfolio and the risk-free bond are both in the market span. We will argue in the sequel that if either of the risk-free bond or the market portfolio is not traded, then anything like the conditions in Theorem 4, which restrict the elasticities with respect to standard deviations alone of the marginal rates of substitution is not sufficient to unambiguously sign the change in the prices of the market portfolio.

First, it is easy to extend Proposition 1 to the case where the market portfolio or the risk-free bond (or both) is not traded. Namely:

Let M be a linear subspace of X. Suppose that $p \in P_M$ and $(x_h^*)_{h \in \{1, \cdots, H\}} \in X^H$ constitute an equilibrium for M and that, for every h, there exists an $x_h \in X$ such that $x_h - d_h \in M$ and $W_h(x_h) > W_h(x_h^*)$. Then:

1. If $S(d) > 0$, then there exist a $t \in R_{++}$ and an $r \in R_{++}$ such that

$$p(m) = E((t\mathbf{1} - r\pi_N(d))m) \qquad (15)$$

for all $m \in M$.

2. For every h, there exist an $a_h^* \in R_+$ and a $b_h^* \in R$ such that

$$x_h^* = a_h^* \pi_{N \cap M}(d) + b_h^* \pi_M(\mathbf{1}) + \pi_{\text{orth} M}(d_h). \qquad (16)$$

Since the state price function p is homogeneous of degree zero, we can assume without loss of generality that $t = 1$. We then investigate how the value of r would be affected when the market span is expanded from M. We, however, need to take some additional care for this task. Note that $E((\mathbf{1} - r\pi_N(d))m) = E((\pi_M(\mathbf{1}) - r\pi_{N \cap M}(d))m)$ for every $m \in M$. Hence, if $\pi_M(\mathbf{1})$ and $\pi_{N \cap M}(d)$ are linearly dependent, then a continuum of values of r is consistent with the given equilibrium. This implies that a deviation from the original value of r as a consequence of an expanded market span may have no economic significance. In any attempt to extend our comparative statics result (Theorem 4), therefore, it is necessary to assume that $\pi_M(\mathbf{1})$ and $\pi_{N \cap M}(d)$ are linearly independent.

This assumption implies two things: First, $\pi_M(\mathbf{1}) \neq 0$, which is equivalent to $M \not\subseteq N$, that is, there must be an asset with non-zero expected payout. Second, $\pi_{N \cap M}(d)$ cannot be a scalar multiple of $\pi_M(\mathbf{1})$, which is equivalent to saying that both the risk-free bond and the market portfolio cannot be best hedged on M by the same portfolio.

It is clear that there is no obvious extension of our main comparative statics result to the case where $\mathbf{1} \notin M$: the marginal rates of substitution between $\pi_{N \cap M}(d)$ and $\pi_M(\mathbf{1})$ have no immediate relation with the marginal rates of substitution between mean μ and standard deviation σ.

No obvious extension is available when $\mathbf{1} \in M$ but $d \notin M$ either, as we now show with an example of a single-consumer economy. Let the commodity space X be spanned by three elements $\mathbf{1}$, d_0, and d_1, where d_0 and d_1 have zero means, unit variances, and zero covariance. In other words, $\{\mathbf{1}, d_0, d_1\}$ is an orthonormal basis of X. Suppose that the single consumer's utility function U_1 satisfies (5), where $\delta_1 = 1$ but τ_1 is arbitrary. His initial endowment vector is d_1 (or it can be d_1 plus a positive multiple of $\mathbf{1}$ to guarantee its positive expected return), which is also the market portfolio. Let $\eta \in]0, 1[$ and define $d_\eta = (1 - \eta^2)^{1/2} d_0 + \eta d_1$. Then $E(d_\eta) = 0$, $V(d_\eta) = 1$, and $C(d_1, d_\eta) = \eta$. Let M be the linear subspace of X that is spanned by $\mathbf{1}$ and d_η, and L be the linear subspace that is spanned by $\mathbf{1}$, d_η, and d_1. Then, both M and L contain the risk-free bond $\mathbf{1}$; L contains the market portfolio d_1 but M does not; M is included in L; and $L = X$, that is, the asset markets are complete with market span L.

Since the market portfolio d_1 is contained in the market span L, the set P_L^* can be defined as in Section 4. On the other hand, the corresponding set for M needs to be carefully defined, because d_1 is not contained in M. Here, we define P_M^* as the set of $r \in R_{++}$ such that the state price function defined by $E((\mathbf{1} - r d_\eta)m)$ is an equilibrium state price function for M. We have chosen d_η, not d_1, in the definition of the state price functional because r is then equal to the market price of risk on M. More precisely, when the state price function is $E((\mathbf{1} - r d_\eta)m)$, the highest Sharpe ratio that can be attained by the portfolios on M is equal to r, but, when the state price function is $E((\mathbf{1} - r d_1 m)$, the highest Sharpe ratio that can be attained by the portfolios on M is equal to ηr. Thus the coefficient r in the former gives the correct market price of risk.

We now claim that $P_M^* = \{\eta\}$ and $P_L^* = \{1\}$, that is, writing $r^M = \eta$ and $r^L = 1$, we have $r^M < r^L$ regardless of the value of τ_1. If Theorem 4 were valid for this comparison,

then, according to its part 1, we would have $r^M > r^L$ whenever $\tau_1 > 0$. However, we in fact have $r^M < r^L$. This shows that Theorem 4, without additional assumptions, cannot be extended to the case where the market portfolio is outside the market span, contrary to the claim by Koch-Medina and Wenzelburger (2018). The difference in the claims is due to the fact that they consider the change in background risks that would not affect the proxy of the market portfolio (d_η in M and d_1 in L in our case), as they considered an individual change in background risks, while in our setting the change in the background risks arises only from a change in market spans, and a change in market span typically changes the proxy of the market portfolio, if not on the market span, which was not taken into consideration in their analysis.

The proof of $P_M^* = \{\eta\}$ and $P_L^* = \{1\}$ is easy. First, note that the solution to the single consumer's utility maximization problem (6) coincides with d_1, and that $D_1 U(S(d_1), E(d_1)) = -1$ and $D_2 U(S(d_1), E(d_1)) = 1$. Second, for every $\alpha \in R$ close to 0, $S(d_1 + \alpha d_\eta) = (1 + 2\eta\alpha + \alpha^2)^{1/2}$ and $E(d_1 + \alpha d_\eta) = 0$. Their derivatives with respect to α evaluated at $\alpha = 0$ are equal to η and 0. Third, since $(-1, 1) \cdot (\eta, 0) = -\eta$, the equilibrium price of d_η is equal to $-\eta$, which implies that $r^M = \eta$. We can similarly show that $r^L = 1$. The proof also shows the logic behind our example: the proxy of the market portfolio becomes a better one as the market span expands; and since the single consumer consumes the market portfolio, the better the proxy, the lower its price. Since the market price of risk is nothing but the price of the proxy, multiplied by -1, the market price increases as the market span expands, regardless of the single consumer's utility function.

It would be very nice if we could include the non-random first-period consumption in our model and find out conditions under which how the relative price between the first-period consumption and the risk-free bond is affected by introduction of new assets: this is the way Weil (1992) considered his own "risk-free rate puzzle", in addition to the equity premium puzzle. This task, however, seems a rather difficult one when dealing with arbitrary market spans, because we can no longer apply the intermediate value theorem for a comparative statics result; this is, in turn, because there are two relative prices involved, that between the first-period consumption and the risk-free bond and that between the risk-free bond and the market portfolio. To extend our results to the case with the first-period consumption, therefore, it will be necessary to assume

that the utility function is separable among the first-period consumptions, the mean of the second-period consumption, and the standard deviation of the second-period consumption; and also that the aggregate demand function has the gross substitute property.

7 Appendix A: Proof of the Characterization Result

We prove Proposition 1 in this appendix. The following lemma is the crucial implication of the assumption that the consumers' preferences depend only on mean and standard deviation.

Lemma 5 *Let $M \in \mathcal{M}$, $p \in P_M$, and $x_h^* \in X$. Suppose that $S(x_h^*) > 0$ and x_h^* is a solution to the utility maximization problem (6). Then there exist a $t_h \in R_{++}$ and an $r_h \in R_{++}$ such that $p(m) = E((t_h \mathbf{1} - r_h \pi_N(x_h^*))m)$ for every $m \in M$.*

Proof of Lemma 5. By $S(x_h^*) > 0$ and the chain rule differentiation, W_h is differentiable at x_h^* and

$$DW_h(x_h^*)(x) = E((D_2 U_h(S(x_h^*), E(x_h^*))\mathbf{1} + 2D_1 U_h(S(x_h^*), E(x_h^*))\pi_N(x_h^*))x)$$

for every $x \in X$. By the first-order necessary condition for a maximum, there must exist a $\lambda_h > 0$ such that

$$p(m) = \lambda_h DW_h(x_h^*)(m)$$

for every $m \in M$. By defining $t_h = \lambda_h D_2 U(S(x_h^*), E(x_h^*))$ and $r_h = -2\lambda_h D_1 U(S(x_h^*), E(x_h^*))$, we complete the proof. ∎

Proof of Proposition 1. 1. Define $\mathcal{H} = \{h \in \{1, \cdots, H\} | S(x_h^*) > 0\}$. Since $\sum_{h=1}^{H} x_h^* = d$ and $S(d) > 0$, $\mathcal{H}$ is non-empty. For each $h \in \mathcal{H}$, let t_h and r_h be as in Lemma 5. Then

$$r_h^{-1}p(m) = E((r_h^{-1}t_h \mathbf{1} - \pi_N(x_h^*))m).$$

Since $\sum_{h \in \mathcal{H}} \pi_N(x_h^*) = \pi_N(d)$, if we add both sides of the above equality over $h \in \mathcal{H}$, then

$$\left(\sum_{h \in \mathcal{H}} r_h^{-1}\right) p(m) = E\left(\left(\left(\sum_{h \in \mathcal{H}} r_h^{-1}t_h\right) \mathbf{1} - \pi_N(d)\right) m\right).$$

By defining $t = (\sum_{h \in \mathcal{H}} r_h^{-1})^{-1}(\sum_{h \in \mathcal{H}} r_h^{-1} t_h)$ and $r = (\sum_{h \in \mathcal{H}} r_h^{-1})^{-1}$, we complete the proof of Part 1.

2. Suppose first that $S(d) = 0$. Define $x_h = E(x_h^*)\mathbf{1} + \pi_{\text{orth}M}(d_h)$. Then $x_h - d_h \in M$ and $\sum_{h=1}^{H} x_h = \sum_{h=1}^{H} d_h$. Moreover, $E(x_h) = E(x_h^*)$ and $S(x_h) \leq S(x_h^*)$, with strict inequality if $x_h \neq x_h^*$. Thus $W_h(x_h) \geq W_h(x_h^*)$, with strict inequality if $x_h \neq x_h^*$. The first welfare theorem, applied to the market span M, therefore implies that $x_h^* = x_h = E(x_h^*)\mathbf{1} + \pi_{\text{orth}M}(d_h)$ for every h. By letting $a_h^* = 0$ and $b_h^* = E(x_h^*)$, we establish Part 1 for the case of $S(d) = 0$.

Suppose next that $S(d) > 0$. For those h with $S(x_h^*) = 0$, we have $\pi_{\text{orth}M}(d_h) = 0$ and thus the claim is true with $a_h^* = 0$. So take an h with $S(x_h^*) > 0$. By Part 1 and Lemma 5, $E((t\mathbf{1} - r\pi_N(d))m) = E((t_h\mathbf{1} - r_h\pi_N(x_h^*))m)$ for every $m \in M$. Hence

$$\pi_M(t\mathbf{1} - r\pi_N(d)) = \pi_M(t_h\mathbf{1} - r_h\pi_N(x_h^*)). \tag{17}$$

Since $\mathbf{1} \in M$, $\pi_M(\mathbf{1}) = \mathbf{1}$. Since $\pi_N(d) \in M$, $\pi_M(\pi_N(d)) = \pi_N(d)$. Thus, together with $\pi_M \circ \pi_N = \pi_{M \cap N}$, equality (17) is equivalent to

$$\pi_{M \cap N}(x_h^*) = r_h^{-1} r \pi_N(d) + r_h^{-1}(t_h - t)\mathbf{1}.$$

Since $M \cap N$ is the orthogonal complement in M of the line spanned by $\mathbf{1}$, $\pi_M(x_h^*) = \pi_{M \cap N}(x_h^*) + c_h\mathbf{1}$ for some $c_h \in R$. Hence

$$\pi_M(x_h^*) = a_h^* \pi_N(d) + b_h^* \mathbf{1}, \tag{18}$$

where $a_h^* = r_h^{-1} r$ and $b_h^* = r_h^{-1}(t_h - t) + c_h$. Since $x_h^* - d_h \in M$, $\pi_{\text{orth}M}(d_h) = \pi_{\text{orth}M}(x_h^*)$. Thus $\pi_M(x_h^*) = x_h^* - \pi_{\text{orth}M}(d_h)$. Hence, by equality (18), $x_h^* = a_h^* r \pi_N(d) + b_h^* \mathbf{1} + \pi_{\text{orth}M}(d_h)$. $\blacksquare$

8   Appendix B: Proof of the Existence Result

In this appendix, we establish two important lemmas, Lemmas 8 and 9. These are powerful enough to be used in the proof of our main theorem in Section 4, as well as to immediately imply the mere existence of an equilibrium, stated as Proposition 2.

Let $M \in \mathcal{M}$. If $S(d) = 0$, then it is easy to check that the state price function $E|M \in P_M$ and the consumption allocation $(E(d_h)\mathbf{1} + \pi_{\text{orth}M}(d_h))_{h \in \{1, \cdots, H\}} \in X^H$

constitute an equilibrium. In the rest of this appendix, therefore, we thus assume that $S(d) > 0$. We define $\widehat{d} = S(d)^{-1}\pi_N(d)$. This is the normalized market portfolio with zero mean and unit standard deviation. According to Part 1 of Lemma 5, to establish existence of an equilibrium, we can restrict our attention to those $p \in P_M$ for which there exists an $r \in [0,1]$ such that $p(m) = E((1-r)\mathbf{1} - r\widehat{d})m)$. Part 2 of the lemma implies that we can restrict our attention to those consumption bundles $x_h \in X$ such that $x_h = a_h\widehat{d} + b_h\mathbf{1} + \pi_{\text{orth}M}(d_h)$ for some $(a_h, b_h) \in R^2$. For such p and x_h, we have $p(x_h - d_h) = -r(a_h - C(d_h, \widehat{d})) + (1-r)(b_h - E(d_h))$. Hence the utility maximization problem (6) is equivalent to

$$\text{Max}_{(a_h,b_h)\in R^2} \quad W_h(a_h\widehat{d} + b_h\mathbf{1} + \pi_{\text{orth}M}(d_h)), \tag{19}$$
$$\text{s.t.} \quad -ra_h + (1-r)b_h \leq -rC(d_h, \widehat{d}) + (1-r)E(d_h).$$

Since W_h is strictly quasi-concave, this maximization problem has at most one solution. For each h, let T_h^M be the set of those r for which the maximization problem (19) has a solution and, for each $r \in T_h^M$, denote the solution by $(a_h^M(r), b_h^M(r)) \in R^2$. We have thus defined a function $a_h^M : T_h^M \to R_+$.

Lemma 6 *1.* $0 \in T_h^M$ *and* $1 \notin T_h^M$.

2. $a_h^M(0) = 0$.

3. The graph $\{(r, a_h) \in [0,1] \times R_+ | \ r \in T_h^M \ \text{and} \ a_h^M(r) = a_h\}$ *is a closed subset of* $[0,1] \times R_+$

4. If $(r^n)_n$ *is a sequence in* T_h^M *converging to an* $r \in [0,1]\backslash T_h^M$, *then* $a_h^M(r^n) \to \infty$.

5. If (r^n) *is a sequence in* T_h^M *converging to an* $r \in T_h^M$, *then the set* $\{a_h^M(r^n)| \ n \in \{1,2,\cdots\}\}$ *is bounded.*

6. a_h^M *is continuous.*

7. T_h^M *is an open subset of* $[0,1]$.

Proof of Lemma 6. 1. If $r = 0$, then $(a_h, b_h) = (0, E(d_h))$ solves (19). Hence $0 \in T_h^M$. If $r = 1$, then b_h can always be increased without violating the budget constraint, which implies that there is no solution. Hence $1 \notin T_h^M$.

2. This follows from the assertion above that $(a_h, b_h) = (0, E(d_h))$ solves (19) when $r = 0$.

3. This is standard.

4. This is an immediate consequence of Part 3 and the non-negativity of a_h^M.

5. Suppose not. Then, without loss of generality, we can assume that $a_h^M(r^n) \to \infty$. Suppose that (a^n, b^n) satisfies the budget constraint of (19) when $r = r^n$ and $(a^n, b^n) \to (a_h^M(r), b_h^M(r))$. Define

$$(\alpha^n, \beta^n) = (|a_h^M(r^n) - a^n| + |b_h^M(r^n) - b^n|)^{-1}(a_h^M(r^n) - a^n, b_h^M(r^n) - b^n) \in R^2.$$

We can assume the sequence $((\alpha^n, \beta^n))_n$ in R^2 converges, say, to (α, β). Note that $|\alpha| + |\beta| = 1$. By quasi-concavity,

$$W_h((\alpha^n + a^n)\widehat{d} + (\beta^n + b^n)\mathbf{1} + \pi_{\text{orth}M}(d_h)) \geq W_h(a^n\widehat{d} + b^n\mathbf{1} + \pi_{\text{orth}M}(d_h)).$$

Taking the limits of both sides, we obtain

$$W_h((\alpha^n + a_h^M(r))\widehat{d} + (\beta^n + b_h^M(r))\mathbf{1} + \pi_{\text{orth}M}(d_h)) \geq W_h(a_h^M(r)\widehat{d} + b_h^M(r)\mathbf{1} + \pi_{\text{orth}M}(d_h)),$$

which implies that $(\alpha + a_h^M(r), \beta + b_h^M(r))$ is also a solution. But this is a contradiction. The proof of Part 5 is thus completed.

6. This follows from Parts 3 and 5.

7. Suppose not. Then there exists a sequence (r^n) in $[0,1] \backslash T_h^M$ that converges to an $r \in T_h^M$. Suppose (a^n, b^n) satisfies the budget constraint of (19) when $r = r^n$ and $(a^n, b^n) \to (a_h^M(r), b_h^M(r))$. Then, by $r^n \notin T_h^M$, there exists an $(\widehat{a}^n, \widehat{b}^n)$ such that

$$W_h(\widehat{a}^n\widehat{d} + \widehat{b}^n\mathbf{1} + \pi_{\text{orth}M}(d_h)) > W_h(a^n\widehat{d} + b^n\mathbf{1} + \pi_{\text{orth}M}(d_h)).$$

Moreover, we can take $(\widehat{a}^n, \widehat{b}^n)$ such that $|\widehat{a}^n - a^n| + |\widehat{b}^n - b^n| \geq 1$. Define

$$(\alpha^n, \beta^n) = (|\widehat{a}^n - a^n| + |\widehat{b}^n - b^n|)^{-1}(\widehat{a}^n - a^n, \widehat{b}^n - b^n) \in R^2.$$

We can assume the sequence $((\alpha^n, \beta^n))_n$ in R^2 converges, say, to (α, β). Note that $|\alpha| + |\beta| = 1$. By quasi-concavity,

$$W_h((\alpha^n + a^n)\widehat{d} + (\beta^n + b^n)\mathbf{1} + \pi_{\text{orth}M}(d_h)) \geq W_h(a^n\widehat{d} + b^n\mathbf{1} + \pi_{\text{orth}M}(d_h)).$$

Taking the limits of both sides, we obtain

$$W_h((\alpha^n + a_h^M(r))\widehat{d} + (\beta^n + b_h^M(r))\mathbf{1} + \pi_{\text{orth}M}(d_h)) \geq W_h(a_h^M(r)\widehat{d} + b_h^M(r)\mathbf{1} + \pi_{\text{orth}M}(d_h)),$$

which implies that $(\alpha + a_h^M(r), \beta + b_h^M(r))$ is also a solution. But this is a contradiction. The proof of Part 7 is thus completed. ∎

Define $T^M = \cap_{h=1}^H T_h^M$. Then T^M consists of those $r \in [0,1]$ for which the aggregate demand $\sum_{h=1}^H a_h^M(r)$ is well defined. So we define the aggregate demand function $a^M : T^M \to R_+$ by $a^M(r) = \sum_{h=1}^H a_h^M(r)$. For each $r \in T^M$, r corresponds an equilibrium state price function if and only if $a^M(r) = S(d)$.

Lemma 7 *1. $0 \in T^M$ and $1 \notin T^M$.*

  2. $a^M(0) = 0$.

  3. If $(r^n)_n$ is a sequence in T^M converging to an $r \in [0,1]\backslash T^M$, then $a(r^n) \to \infty$.

  4. a^M is continuous.

  5. T^M is an open subset of $[0,1]$.

Proof of Lemma 7. These are all immediate consequences of Parts 1, 2, 4, 6, and 7 of Lemma 6. ∎

Lemma 8 *Let $r \in T^M$ and $b^M(r) < S(d)$. Then there exists an $r^* \in (r,1) \cap T^M$ such that $a^M(r^*) = S(d)$.*

Lemma 9 *Let $r \in [0,1]$. Assume either that $[0,r] \subseteq T^M$ and $b^M(r) > S(d)$ or that $[0,r] \nsubseteq T^M$. Then there exists an $r^* \in (0,r) \cap T^M$ such that $a^M(r^*) = S(d)$.*

Proof of Lemma 8. By Part 5 of Lemma 7, there exists an $\bar{r} \in]r,1]$ such that $[r,\bar{r}[\subseteq T^M$ and $\bar{r} \notin T^M$. By Part 3 of Lemma 7, there exists an $r' \in [r,\bar{r}[$ such that $a^M(r') > S(d)$. By Part 4 of Lemma 7, we can apply the intermediate value theorem to a^M on $[r,r']$ to show that there is an $r^* \in]r,r'[$ such that $a^M(r^*) = S(d)$. The proof is thus completed. ∎

Proof of Lemma 9. Suppose first that $[0,r] \subseteq T^M$ and $b^M(r) > S(d)$. Then, by Parts 2 and 4 of Lemma 7, we can apply the intermediate value theorem to a^M on $[0,r]$ to show that there is an $r^* \in]0,r[$ such that $a^M(r^*) = S(d)$.

Suppose next that $[0,r] \nsubseteq T^M$. Then, by Part 5 of Lemma 7, there exists an $\bar{r} \in]0,r]$ such that $[0,\bar{r}[\subseteq T^M$ and $\bar{r} \notin T^M$. By Part 3 of Lemma 7, there exists an $r' \in]0,\bar{r}[$ such that $a^M(r') > S(d)$. By Parts 2 and 4 of Lemma 7, we can apply the intermediate value theorem to a^M on $[0,r']$ to show that there is an $r^* \in]0,r'[$ such that $a^M(r^*) = S(d)$. The proof is thus completed. ∎

Proof of Proposition 2. This follows from Part 2 of Lemma 7 and Lemma 8. ∎

9 Appendix C: Proof of the Comparative Statics Result

In this appendix, we give a formal proof of Theorem 4. It is more or less along the same lines as the intuitive explanation immediately after the theorem, except for three points. First, we use the same normalization for state price functions, $p(m) = E(((1-r)\mathbf{1}-r\widehat{d})m)$ for $r \in [0,1]$, as in Appendix B; the notation therein will be used here as well. Second, we take into consideration the possibility that the set T^L of the price functions for which the aggregate demands are well defined may not be an interval. Third, we prove here that flatter indifference curves imply higher demands for the market portfolio.

Lemma 10 *If $D_1 U_h(0, \mu) = 0$ for every $\mu \in R$, then $a_h^M(r) > 0$ for every $M \in \mathcal{M}$ and $r \in T_h^M \cap]0, 1[$.*

Proof of Lemma 10. Since $D_1 U_h(0, \mu) = 0$, the induced utility function U_h^M defined in (10) is differentiable at $(0, \mu)$ and $DU_h^M(0, \mu) = (0, D_2 U_h(0, \mu))$ for $\mu \in R$. On the other hand, the first-order condition for the modified maximization problem (9) is that there exists an $\lambda_h > 0$ such that $DU_h^M(a_h^M(r), b_h^M(r)) = \lambda_h(-r, 1-r)$. Hence, if $r > 0$, then we must have $a_h^M(r^M) > 0$. ∎

Proof of Theorem 4. 1. If $[0, r^M] \not\subseteq T^L$, then Part 1 can easily be established by applying Lemma 9 to $M = L$ and $r = r^M$. So suppose that $[0, r^M] \subseteq T^L$. Then $r^M \in T_h^M \cap T_h^L$ for every h. By Lemma 10, $a_h^M(r^M) > 0$ for every h. Hence

$$S(a_h^M(r^M)\widehat{d} + b_h^M(r^M)\mathbf{1} + \pi_{\text{orth}M}(d_h)) \geq S(a_h^M(r^M)\widehat{d} + b_h^M(r^M)\mathbf{1} + \pi_{\text{orth}L}(d_h)) > 0$$

for every h, with strict inequality for some h. Writing $\theta_h^M = S(\pi_{\text{orth}M}(d_h))$ and similarly for θ_h^L, we obtain

$$(a_h^M(r^M)^2 + (\theta_h^M)^2)^{1/2} \geq (a_h^M(r^M)^2 + (\theta_h^L)^2)^{1/2} > 0$$

for every h, with strict inequality for some h. Hence $MRS((a_h^M(r^M)^2 + (\theta_h^M)^2)^{1/2}, b_h^M(r^M)|U_h)$ and $MRS((a_h^M(r^M)^2 + (\theta_h^L)^2)^{1/2}, b_h^M(r^M)|U_h)$ are both defined. By the first-order condition for a maximum,

$$\frac{r^M}{1 - r^M} = MRS(a_h^M(r^M), b_h^M(r^M)|U_h^M).$$

The right hand side equals $MRS((a_h^M(r^M)^2 + (\theta_h^M)^2)^{1/2}, b_h^M(r^M)|U_h)$. By the assumption of EMRS,

$$MRS((a_h^M(r^M)^2 + (\theta_h^M)^2)^{1/2}, b_h^M(r^M)) \geq MRS((a_h^M(r^M)^2 + (\theta_h^L)^2)^{1/2}, b_h^M(r^M))$$

29

for every h, with strict inequality for some h. The right hand side equals $MRS(a_h^M(r^M), b_h^M(r^M)|U_h^L)$. Hence

$$\frac{r^M}{1-r^M} \geq MRS(a_h^M(r^M), b_h^M(r^M)|U_h^L)$$

for every h, with strict inequality for some h. It is then standard to check that

$$a_h^M(r^M) \leq a_h^L(r^M)$$

for every h, with strict inequality for some h. Thus, by summing over h, we obtain $S(d) = a^M(r^M) < a^L(r^M)$. Lemma 9 now establishes Part 1.

2. We first prove that $r^L \in T_h^M$ for every h. By Lemma 10, $a_h^L(r^L) > 0$ for every h and the assumption of EMRS implies that

$$\frac{r^L}{1-r^L} \leq MRS(a_h^L(r^L), b_h^L(r^L)|U_h^M) \tag{20}$$

for every h, with strict inequality for some h. If the equality holds, then $r^L \in T_h^M$ and $a_h^M(r^L) = a_h^L(r^L)$. So suppose that the strict inequality holds and define $r_h \in]0,1[$ so that $r_h = MRS(a_h^L(r^L), b_h^L(r^L)|U_h^M)$. Define also a polyhedral subset K_h of R^2 as the area bounded by three inequalities:

$$
\begin{aligned}
a_h &\geq 0, \\
-r^L a_h + (1-r^L) b_h &\leq -r^L C(d_h, \widehat{d}) + (1-r^L) E(d_h), \\
-r_h a_h + (1-r_h) b_h &\geq -r_h a_h^L(r^L) + (1-r_h) b_h^L(r^L).
\end{aligned}
$$

Then $(a_h^L(r^L), b_h^L(r^L)) \in K_h$ and K_h is compact because $r^L < r_h$. So let (a_h^*, b_h^*) be a solution to the maximization problem

$$
\begin{aligned}
\text{Max}_{(a_h, b_h) \in R^2} \quad & U_h^M(a_h, b_h), \\
\text{s.t.} \quad & (a_h, b_h) \in K_h.
\end{aligned}
$$

Note that $U_h^M(a_h^*, b_h^*) \geq U_h^M(a_h^L(r^L), b_h^L(r^L))$. We now show that (a_h^*, b_h^*) is also the solution to the maximization problem (9). In fact, let $(a_h, b_h) \in R^2$ and $-r^L a_h + b_h \leq -r^L C(d_h, \widehat{d}) + (1-r^L) E(d_h)$. If $a_h \geq 0$ and $U_h^M(a_h, b_h) \geq U_h^M(a_h^L(r^L), b_h^L(r^L))$, then $-r_h a_h + (1-r_h) b_h \geq -r_h C(d_h, \widehat{d}) + (1-r_h) E(d_h)$ and hence $U_h^M(a_h, b_h) \leq U_h^M(a_h^*, b_h^*)$. If $a_h < 0$, then $-r^L(-a_h) + (1-r^L) b_h \leq -r^L C(d_h, \widehat{d}) + (1-r^L) E(d_h)$. Thus, by the preceding result, $U_h^M(-a_h, b_h) \leq U_h^M(a_h^*, b_h^*)$. Since $U_h^M(a_h, b_h) = U_h^M(-a_h, b_h)$, $U_h^M(-a_h, b_h) \leq U_h^M(a_h^*, b_h^*)$. We have therefore shown that (a_h^*, b_h^*) is also the solution to the maximization problem (9). Hence $r^L \in T_h^M$ for every h.

Hence $a_h^M(r^L)$ is defined and we can show that (20) implies that

$$a_h^M(r^L) \leq a_h^L(r^L)$$

for every h, with strict inequality for some h. Thus, by summing over h, we obtain $a^M(r^L) < a^L(r^L) = S(d)$. Lemma 8 now establishes Part 2.

3. This can be proved in the same way as for Part 2.

4. This can be proved in the same way as for Part 1.

5. The assumption of UMRS implies that if $r^M \in P_M^*$, then

$$MRS(a_h^M(r^M), b_h^M(r^M)|U_h^M) = MRS((a_h^M(r^M), b_h^M(r^M)|U_h^L).$$

Hence $a_h^L(r^M) = a_h^M(r^M)$ and $r^M \in P_L^*$. Thus $P_M^* \subseteq P_L^*$. The reversed inclusion can also be shown in the same way. ∎

10   Footnotes

1. By plugging $a_i = \mathbf{1}$ and $a_i = d$ and solving the system of two linear equations, we can find that $t = p_{\mathbf{1}}$ and $r = (p_{\mathbf{1}}E(d) - p_d)V(d)^{-1}$. The equivalence of the two representations of the security market line is stated in Duffie (1987).

2. The concavity in term of the mean and standard deviation follows immediately from that of the vNM utility function.

3. Recall that there are multiple equivalent state price densities in incomplete markets. All that we mean here is that if we use those spanned by the market portfolio and the risk-free bond, the volatility will change in the directions predicted by our theorem. The volatility may change in opposite directions if some state prices to be compared are spanned by these two assets.

4. The price functions under consideration in the present context are of the form $E((\mathbf{1} - r\widehat{d})m)$ for $r \in R_{++}$, while the prices function in Appendix B are of the form $E(((1 - r)\mathbf{1} - r\widehat{d})m)$ for $r \in [0, 1]$. The difference is trivial, so we use the same symbol $(a_h^M(r), b_h^M(r))$ to denote the solutions to the two maximization problems.

References

[1] J. S. Chipman, The ordering of portfolios in terms of means and variance, *Review of Economic Studies*, **40** (1973), 167-190.

[2] R.-A. Dana, Existence, uniqueness and determinacy of equilibrium in C.A.P.M. with a riskless asset, *Journal of Mathematical Economics*,.**32** (1999), 167-175.

[3] J. Detemple and L. Selden, A general equilibrium analysis of option and stock market interactions, *International Economic Review*, **32** (1991), 279-303.

[4] D. Duffie, Security Markets: Stochastic Models, 1987, Academic Press, San Diego.

[5] R. Elul, Welfare-improving financial innovation with a single good, *Economic Theory*, 13 (1999), 25-40.

[6] C. Hara, Pareto improvement and agenda control of sequential financial innovations, *Journal of Mathematical Economics*, **47** (2011), 336-345.

[7] C. Hara, J. Huang, and C. Kuzmics, Representative consumer's risk aversion and efficient risk-sharing rules, manuscript, *Journal of Economic Theory*, **137**, (2007), 652-672.

[8] Th. Hens and A. Loeffler, Existence and uniqueness of equilibria in the CAPM with a riskless asset, manuscript, August 1996.

[9] Th. Hens, J. Laitenberger, and A. Loeffler, Two remarks on the uniqueness of equilibria in the CAPM, *Journal of Mathematical Economics*, **37**, (2002), 123-132.

[10] M. S. Kimball, Precautionary saving in the small and large, *Econometrica*, **58** (1990), 53-73.

[11] P. Koch-Medina and J. Wenzelburger, Equilibria in the CAPM with non-tradeable endowments, *Journal of Mathematical Economics*, **75**, (2018), 93-107.

[12] F. Lajeri and L. T. Nielsen, Parametric characterizations of risk aversion and prudence, *Economic Theory*, **15** (2000), 469-476.

[13] R. Mehra and E. C. Prescott, The equity premium: a puzzle, *Journal of Monetary Economics*, **15** (1985), 145-161.

[14] P. Milgrom and C. Shannon, Monotone comparative statics, Econometrica; **62** (1994), 157-180.

[15] G. Oh, A general equilibrium with incomplete asset markets approach to the capital asset pricing model, manuscript, Department of Economics, Yale University, October 1990.

[16] G. Oh, Some results in the CAPM with non-traded endowments, *Management Science*, **42** (1996), 286-293.

[17] J. W. Pratt and R. J. Zeckhauser, Proper risk aversion, *Econometrica*, **55** (1987), 143-154.

[18] S. A. Ross, Some stronger measures of risk aversion in the small and the large with applications, *Econometrica*, **49** (1981), 621-638.

[19] P. Weil, Equilibrium asset prices with undiversifiable labor income risk, *Journal of Economic Dynamics and Control*, **16** (1992), 769-790.