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"Rejection prices and an auctioneer with non-monotonic utility"

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Rejection prices and an auctioneer with non-monotonic utility*

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Abstract

This paper considers an auctioneer who has a non-monotonic utility function with a unique maximizer. The auctioneer is able to reject all bids over some amount by using rejection prices. We show that the optimal rejection price for such an auctioneer is lower than and equal to that maximizer in first-price and second-price sealed-bid auctions, respectively. Further, in each auction we characterize a necessary and sufficient condition that by using the optimal rejection price not only the auctioneer but also bidders can be better off, compared to a standard auction. Finally, we find that the auctioneer strictly prefers a first-price sealed-bid auction if he is risk-averse when his revenue is lower than the maximizer or if the distribution of revenues which are lower than the maximizer in a standard first-price sealed-bid auction is first-order stochastic dominant over the one in a standard second-price sealed-bid auction.

Keywords: Auction, Rejection prices, Non-monotonic utility

JEL Classification: D44, D82

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1 Introduction

In standard auction theory, it is assumed that an auctioneer's utility monotonically increases with money. In this paper, contrary to the standard assumption, we consider an auctioneer with non-monotonic utility.

Here is a real-life example.[1] Nowadays, many of China's local governments have introduced a mechanism in land auctions which is similar to the trading curb in stock markets. Under this mechanism, the government will set a highest selling price for each residential land and end the land auction if the bids go above that price. The news reported that why the local governments introduce the mechanism is to control surging land costs that have been driving up home prices. On the one hand, in China, the land auctions are a major source of fiscal income for the local governments.[2] On the other hand, as a part of home prices, an extravagant land price may aggravate the real estate bubble in China. Therefore, the local governments would like to keep the land price within a reasonable range.

Two features are important in this example. First, the auctioneer (the local governments in China) has a non-monotonic utility function with a unique maximizer (interior). This is because the land price imposes a negative externality which increases with money on his utility. Hence, the utility of the auctioneer increases with money while the externality is small but decreases while it is large enough. Second, the auctioneer may refuse all bids over some amount (in this paper we call it a rejection price). Being afraid of the negative externality, the auctioneer would like to refuse the bid which is higher than the rejection price in order to maximize his expected utility.

In fact, such a kind of auctioneer is not uncommon in reality. In a large auction which sells high-stake objects, the auctioneer may also have a non-monotonic utility function. The worth of a good is large compared to the wealth of a bidder in a large auction. Thus winners can declare bankruptcy if the good is worth less than expected. Therefore similar to the local governments in China, the auctioneer is also concerned about high hammer prices. For example, in the 1996 radio frequency spectrum auction in the U.S., the winning bids totaled 10.2 billion dollars which were almost three times as high as the prices in previous spectrum auctions. Unfortunately, many of the winners declared bankruptcy and the auctioneer raised only 400 million dollars in the next three years.[3]

Another example is a thief disposing stolen goods or a referee controlling a sport game in exchange for bribes. Of course, it is advantageous for the auctioneer (the thief or the referee) to get more income. However the illegality of his act will easily be exposed as the transaction price rises, and he will be in prison once it is exposed. Hence the auctioneer also has a non-monotonic utility function with a unique maximizer and may also dislike high transaction prices.

[1] http://www.reuters.com/article/us-china-property-nanjing/chinas-nanjing-to-introduce-price-cap-in-land-auctions-idUSKCN0YI0VL

[2] The Property Law of the People's Republic of China passed in 2007 codified that land is owned by collectivities or by the state.

[3] See Zheng (2001) and Board (2007).

The goal of this paper is to study how an auctioneer who has a non-monotonic utility function with a unique maximizer can use the rejection price to increase expected utility in first-price and second-price sealed-bid auctions. We also investigate whether using the rejection price can also increase bidders' expected utilities or not. And we further examine which auction does an auctioneer with non-monotonic utility prefer.

In our model, an auctioneer announces publicly a rejection price before a given first-price or second-price sealed-bid auction starts. Subsequently, n risk-neutral bidders bid for an object. At the end of bidding, the bidder with the highest bid among the bids which do not exceed the rejection price, gets the object and pays the highest and second-highest one among the bids which do not exceed the rejection price in the first-price and second-price sealed-bid auctions, respectively.

For a given first-price or second-price sealed-bid auction with a rejection price, our approach is first showing an equilibrium bidding behavior which is based on a natural equilibrium bidding behavior in a standard first-price or second-price sealed-bid auction.[4] And then we show the optimal rejection price by assuming that the auctioneer believes that all bidders will follow the equilibrium bidding behavior if he announces a rejection price.

We begin the analysis by considering equilibrium bidding behavior in a second-price sealed-bid auction with a rejection price in an environment with independently and identically distributed private values. Since there is no incentive for a bidder to bid higher than his private value (suffer a loss) or the rejection price (be rejected), the optimal strategy for the bidder is bidding the lower one between his private value and the rejection price. We then study the optimal rejection price, or utility maximization, from the perspective of the auctioneer. We show that the optimal rejection price is just the unique maximizer. Further, we obtain the same conclusion even in the case where bidders' private values follow different distributions and/or bidders are not risk-neutral. Clearly, the equilibrium under such a mechanism is inefficient, since the bidder who values the object most may lose if multiple bids are equal to the rejection price.

Perhaps more surprisingly, we find that under the optimal rejection price for the auctioneer, all bidders may be better off. In general, under any rejection price all bidders will be better off if and only if the bidder with the maximum value is better off. Therefore, not only the auctioneer but also all bidders will be better off by using the optimal rejection price, if and only if the bidder with the maximum value is better off. Namely, the optimal rejection price results in a Pareto improvement.

We also analyze a first-price sealed-bid auction with a rejection price. By assuming the equilibrium bidding strategy is non-decreasing, we find that in the equilibrium bidding strategy there exists a jump point below which bidders bid the same as the standard model and above which bidders bid the rejection

[4] Note that standard first-price and second-price sealed-bid auctions can be regarded as first-price or second-price sealed-bid auctions with infinite rejection prices, respectively.

price. Why does the jump point exist? Consider the lowest value among all values with which a bidder bids the rejection price. Note that the equilibrium winning probability is discontinuous at that value, since the probability of tie at the rejection price is not zero. Clearly, if the equilibrium bidding strategy were continuous, the one whose value is lower than that value but sufficiently close to it, would improve his payoff by bidding the rejection price. So the equilibrium bidding strategy should be discontinuous at that value.

Similar to the second-price sealed-bid auction, the optimal rejection price could be the unique maximizer. But due to the discontinuous bidding strategy, with a lower rejection price bidders are more likely to bid it in a first-price sealed-bid auction than in a second-price sealed-bid auction, i.e., the auctioneer have much more probability of receiving the rejection price. Hence the optimal rejection price will be lower than the unique maximizer if a loss by receiving the lower rejection price can be negligible. And we also find that it makes a Pareto improvement to the standard model if and only if the bidder with the maximum value can be better off.

Finally, we study the preferences of the auctioneer over the two auctions. In standard auction theory, Matthews (1979) and Waehrer et al. (1998) show that a risk-averse auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction. Similarly, we prove that with the optimal rejection price the auctioneer also strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction if he is risk-averse when his revenue is lower than the maximizer. Further, for a given rejection price, we find that the distribution function of the auctioneer's payoff in a first-price sealed-bid auction is first-order stochastic dominant over that in a second-price sealed-bid auction for some distribution functions of bidders' values. Thus, the auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction as well. And we also find the case where the auctioneer strictly prefers a second-price sealed-bid auction to a first-price sealed-bid auction.

Our paper is similar to both the literature on buy prices and the literature on bid caps (ceiling prices). Within the former literature, Budish and Takeyama (2001) study a simple model with two bidders and two types, and find that a risk-neutral auctioneer gains by augmenting his auction with a buy price when bidders are risk-averse. Hidvegi et al. (2006) and Inami (2011) extended these results to a model with n bidders where bidders' types are continuously distributed and discretely distributed, respectively. Mathews (2003) and Mathews and Katzman (2006) are particularly related to our work. These papers study a risk-averse auctioneer facing risk-neutral bidders in an auction with a temporary buy price. Such an auctioneer can gain from augmenting his auction with a buy price and this option may result in a Pareto improvement compared to a standard auction.

Within the literature on bid caps (ceiling prices) which is the most similar conception to rejection prices, Chowdhury (2008) analyzes a simple second-price auction with independent private values where the bidders may potentially collude. An optimal policy which includes both a reserve price and an efficient ceiling price prevents collusion. Gavious et al. (2002) considered bid caps in symmetric all-pay auctions and showed that using a bid cap when the bidders have concave cost functions decreases the average bid. They proved that if the bidders have convex cost functions and there is a sufficiently large number of bidders, then the auctioneer, who wishes to maximize the average bid, might benefit from fixing a bid cap. Sahuguet (2006) studied asymmetric all-pay auctions with private values and showed that capping the bids is profitable for an auctioneer who wants to maximize the sum of the bids.

Our paper is different in two important ways from the literature mentioned above. First, in a second-price sealed-bid auction with a rejection price we assume the payment rule which is the same as ceiling prices that bidders pay follows a standard second-price sealed-bid auction's rule, whereas in a second-price sealed-bid auction with a buy price the winner pays the buy price if he bids it. Second, the most frequently studied rationales for auctions with a buy price or a ceiling price are based on bidders' risk attitudes. Consequently, these mechanisms incentivize bidders to bid aggressively to maximize auctioneer's utility. On the contrary, we assume bidders are risk-neutral and focus our attention on an auctioneer whose utility is non-monotonic and has a unique maximizer. Such an assumption induces the auctioneer to restrain bids to maximize his utility. In particular, in a second-price sealed-bid auction, the equilibrium bidding strategy and the optimal rejection price are the same regardless of bidders' risk-attitudes.

This paper is organized as follows. In section 2, we present the basic setting and assumptions. In section 3, we study a second-price sealed-bid auction with a rejection price. In section 4, we analyze a first-price sealed-bid auction with a rejection price. In section 5, we compare the two auctions. In section 6, we discuss a more general non-monotonic utility. Section 7 concludes.

2 Model

There is one indivisible object for sale, and n potential risk-neutral bidders are bidding for the object. Bidder $i \in N = \{1, 2, ..., n\}$ assigns a private value of v_i to the object and v_i is independently and identically distributed on some interval $[\underline{v}, \bar{v}] \subset [0, \infty)$ according to an increasing distribution function $F(\cdot)$. It is assumed that $F(\cdot)$ admits a continuous density $f(\cdot) = F'(\cdot) > 0$ and has full support.

In this paper, we consider that the auctioneer has a non-monotonic utility function with a unique maximizer. We assume that $u(\cdot)$ is strictly increasing on the interval $[0, v^*]$ and is strictly decreasing on the interval (v^*, ∞), where $v^* \in (\underline{v}, \bar{v})$ denotes the unique maximizer. For tractability, we assume that the utility function $u(\cdot)$ is continuous on the interval $[0, \infty)$. Note that in the case $v^* \leq \underline{v}$, the auctioneer is unwilling to participate in the auction since his utility is strictly decreasing. And in the case $v^* \geq \bar{v}$, the utility function is just same with the standard assumption. Hence, it is not interesting to study these two cases.

Before a given k^{th}-price sealed-bid auction starts, for $k \in \{\mathrm{I}, \mathrm{II}\}$, the auctioneer can announce a rejection price $R_k \in (\underline{v}, \bar{v})$. That is, the auctioneer rejects bids above R_k. At the end of the auction, the bidder who gives the highest bid among the bids which are not rejected wins the object and pays the k^{th}-highest one among the bids which are not rejected. We assume that if there is a tie the object goes to each winning bidder with equal probability, i.e., when l bids tie, each winning bidder gets the object with probability $1/l$, for any $l \in N$.

In a given k^{th}-price sealed-bid auction with a rejection price R_k, we let b_i denote bidder i's bid for every $i \in N$ and $b = (b_1, b_2, ..., b_n)$ denote the bid profile. Notice that we assume $\underline{v} \geq 0$, so that no bidder would bid a negative amount. So we can let $b_i \mathbf{1}_{\{b_i \leq R_k\}}$ denote bidder i's bid after screening by a rejection price R_k, for every $i \in N$ and $k \in \{\mathrm{I}, \mathrm{II}\}$, where[5]

$$\mathbf{1}_{\{b_i \leq R_k\}} = \begin{cases} 1 & \textit{if } b_i \leq R_k \\ -1 & \textit{if } b_i > R_k. \end{cases}$$

Since the bids which are higher than the rejection price become minus after screening, the bidder who submits such a bid must lose in the auction. And we let $l(b_i) = |\{b_j | b_j = b_i, j = 1, 2, ..., n\}|$ denote the number of bids which are equal to b_i. For every $i \in N$ and $k \in \{\mathrm{I}, \mathrm{II}\}$, we let $u_k(v_i, b)$ denote the utility of the bidder i with the bid profile b. More concretely,

$$u_{\mathrm{I}}(v_i, b) = \begin{cases} (v_i - b_i)/l(b_i) & \textit{if } b_i \mathbf{1}_{\{b_i \leq R_{\mathrm{I}}\}} \geq \max\limits_{j \neq i} \{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{I}}\}}, 0\} \\ 0 & \textit{if } b_i \mathbf{1}_{\{b_i \leq R_{\mathrm{I}}\}} < \max\limits_{j \neq i} \{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{I}}\}}, 0\}. \end{cases}$$

in a first-price sealed-bid auction with a rejection price R_{I}, and

$$u_{\mathrm{II}}(v_i, b) = \begin{cases} (v_i - \max\limits_{j \neq i} \{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\})/l(b_i) & \textit{if } b_i \mathbf{1}_{\{b_i \leq R_{\mathrm{II}}\}} \geq \max\limits_{j \neq i} \{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\} \\ 0 & \textit{if } b_i \mathbf{1}_{\{b_i \leq R_{\mathrm{II}}\}} < \max\limits_{j \neq i} \{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\} \end{cases}$$

in a second-price sealed-bid auction with a rejection price R_{II}.

3 Second-price sealed-bid auction with rejection prices

In this section, we study a second-price sealed-bid auction with a rejection price R_{II}. In this paper, we consider extending a natural equilibrium bidding strategy in a standard second-price sealed-bid auction to our model. Then the bidding strategy in a second-price auction with a rejection price is straightforward.

Proposition 1. *Equilibrium strategies in a second-price sealed-bid auction with a rejection price R_{II} are given by $\beta_{\mathrm{II}}(v_i) = \min\{v_i, R_{\mathrm{II}}\}$.*

[5] If we assume a standard indicator function, a bid which equals zero and a bid which is higher than the rejection price cannot be distinguished after screening.

Proof. Consider bidder $i \in N$ and suppose that $p_i = \max_{j \neq i}\{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\}$ is the highest competing bid after screening by the rejection price. By bidding $\beta_{\mathrm{II}}(v_i)$, bidder i will win if $\beta_{\mathrm{II}}(v_i) > p_i$ and not if $\beta_{\mathrm{II}}(v_i) < p_i$.

Suppose, however, that he bids an amount $b_i < \beta_{\mathrm{II}}(v_i)$. If $\beta_{\mathrm{II}}(v_i) > b_i \geq p_i$, then he still wins and his profit is $v_i - p_i$. If $p_i > \beta_{\mathrm{II}}(v_i) > b_i$, he still loses. However, if $\beta_{\mathrm{II}}(v_i) \geq p_i > b_i$, then he loses, whereas if he had bid $\beta_{\mathrm{II}}(v_i)$, he would have made a non-negative profit. Thus, bidding less than $\beta_{\mathrm{II}}(v_i)$ can never increase his profit. Suppose, on the contrary, his bid b_i is higher than $\beta_{\mathrm{II}}(v_i)$. Clearly, there is no incentive for him to bid higher than R_{II} (be rejected). On the one hand, if $R_{\mathrm{II}} \geq b_i \geq p_i > \beta_{\mathrm{II}}(v_i)$, then he will suffer a loss. On the other hand, he is indifferent between bidding b_i and $\beta_{\mathrm{II}}(v_i)$ in the case that $R_{\mathrm{II}} \geq b_i > \beta_{\mathrm{II}}(v_i) \geq p_i$. Thus it is also not profitable to bid higher than $\beta_{\mathrm{II}}(v_i)$. □

Proposition 1 states that any bidder bids the lower one between his value and the rejection price. In this paper, we only focus on this equilibrium bidding strategy. Now we examine what effect such a rejection price has on the expected utility of the auctioneer. We let $U_{\mathrm{II}}(R_{\mathrm{II}})$ denote the expected utility of the auctioneer in a second-price sealed-bid auction with a rejection price R_{II}. The following proposition implies that it is optimal for the auctioneer to use the rejection price and tell the truth, i.e., the auctioneer chooses the optimal rejection price which equals the unique maximizer.

Proposition 2. (Truth-telling for auctioneer) *In a second-price sealed-bid auction with a rejection price R_{II}, the unique optimal rejection price is $R^*_{\mathrm{II}} = v^*$.*

Proof. The expected utility of the auctioneer with a rejection price R_{II} can be calculated as follows:

$$\begin{aligned} U_{\mathrm{II}}(R_{\mathrm{II}}) =& [1 - F^n(R_{\mathrm{II}}) - n(1 - F(R_{\mathrm{II}}))F^{n-1}(R_{\mathrm{II}})]u(R_{\mathrm{II}}) \\ &+ n\int_{\underline{v}}^{R_{\mathrm{II}}} (1 - F(t))u(t)dF^{n-1}(t). \end{aligned}$$

So

$$\begin{aligned} U_{\mathrm{II}}(v^*) - U_{\mathrm{II}}(R_{\mathrm{II}}) =& [1 - F^n(v^*) - n(1 - F(v^*))F^{n-1}(v^*)]u(v^*) \\ &- [1 - F^n(R_{\mathrm{II}}) - n(1 - F(R_{\mathrm{II}}))F^{n-1}(R_{\mathrm{II}})]u(R_{\mathrm{II}}) \\ &+ n\int_{R_{\mathrm{II}}}^{v^*} (1 - F(t))u(t)dF^{n-1}(t). \end{aligned}$$

Notice that $u(\cdot)$ is strictly increasing on $[\underline{v}, v^*]$, then for any $R_{\rm II} \in [\underline{v}, v^*)$,

$$\begin{aligned} U_{\rm II}(v^*) - U_{\rm II}(R_{\rm II}) > & [1 - F^n(v^*) - n(1 - F(v^*))F^{n-1}(v^*)]u(v^*) \\ & - [1 - F^n(R_{\rm II}) - n(1 - F(R_{\rm II}))F^{n-1}(R_{\rm II})]u(R_{\rm II}) \\ & + n \int_{R_{\rm II}}^{v^*} (1 - F(t))u(R_{\rm II})dF^{n-1}(t) \\ = & [1 - F^n(v^*) - n(1 - F(v^*))F^{n-1}(v^*)][u(v^*) - u(R_{\rm II})] \\ > & 0. \end{aligned}$$

And since $u(\cdot)$ is strictly decreasing on $(v^*, \bar{v}]$, for any $R_{\rm II} \in (v^*, \bar{v}]$,

$$\begin{aligned} U_{\rm II}(v^*) - U_{\rm II}(R_{\rm II}) > & [1 - F^n(v^*) - n(1 - F(v^*))F^{n-1}(v^*)]u(v^*) \\ & - [1 - F^n(R_{\rm II}) - n(1 - F(R_{\rm II}))F^{n-1}(R_{\rm II})]u(R_{\rm II}) \\ & - n \int_{v^*}^{R_{\rm II}} (1 - F(t))u(v^*)dF^{n-1}(t) \\ = & [1 - F^n(R_{\rm II}) - n(1 - F(R_{\rm II}))F^{n-1}(R_{\rm II})][u(v^*) - u(R_{\rm II})] \\ > & 0. \end{aligned}$$

Thus the unique optimal rejection price is $R_{\rm II}^* = v^*$. $\square$

Remark 1. *Proposition 2 holds even if $u(\cdot)$ is discontinuous on the interval $[\underline{v}, \bar{v}]$. The reason is as follows. If $u(\cdot)$ is discontinuous, the only problem is whether the integral $n \int_{\underline{v}}^{R_{\rm II}} (1 - F(t))u(t)dF^{n-1}(t)$ exists or not. Notice that $u(\cdot)$ is monotonic both on the interval $[\underline{v}, v^*]$ and the interval $[v^*, \bar{v}]$, therefore $u(\cdot)$ is integrable on the interval $[\underline{v}, \bar{v}]$. Since $f(\cdot)$ is also integrable on the interval $[\underline{v}, \bar{v}]$, so the integral exists.*

Proposition 2 implies that it is optimal for the auctioneer to reject the bids over the maximizer. Before we give an explanation, let $F_{\rm II}(v; R_{\rm II})$ denote the distribution function of the auctioneer's revenue v in a second-price sealed-bid auction with a rejection price $R_{\rm II}$. Then it is easy to see that for any rejection price $R_{\rm II} \in [\underline{v}, \bar{v}]$,

$$F_{\rm II}(v; R_{\rm II}) = \begin{cases} nF^{n-1}(v) - (n-1)F^n(v) & \text{if } R_{\rm II} > v \geq \underline{v} \\ 1 & \text{if } v = R_{\rm II}. \end{cases}$$

Then $F_{\rm II}(v; R'_{\rm II})$ is first-order stochastic dominant over $F_{\rm II}(v; R''_{\rm II})$ for any $R'_{\rm II} > R''_{\rm II}$. Thus if $R_{\rm II} < v^*$, the auctioneer prefers the rejection price v^* to the rejection price $R_{\rm II}$, since $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, v^*]$. If $R_{\rm II} > v^*$, notice that $F_{\rm II}(v; R_{\rm II}) = F_{\rm II}(v; v^*)$ for any $v \in [\underline{v}, v^*)$ and $u(\cdot)$ is strictly decreasing on the interval $[v^*, \bar{v}]$, therefore the auctioneer still prefers the rejection price v^*.

We, therefore, obtain the optimal rejection price. Note that the auction becomes inefficient[6] with a rejection price, since the object may not end up in the hands of the bidder who values it the most ex post if there are more than two bidders whose values are higher than the rejection price. In spite of this inefficiency, the next proposition shows a condition that is necessary and sufficient to guarantee all bidders' ex post expected utilities can be improved with a rejection price, compared to a standard second-price sealed-bid auction. To state our next result, we need to introduce the notion of improvement and some additional notations.

Definition 1. (Pareto improvement for bidders) *A k^{th}-price sealed-bid auction with a rejection price R_k makes a Pareto improvement for bidders to a standard k^{th}-price sealed-bid auction, if the ex post expected utilities of bidders with any values are improved and the ex post expected utilities of bidders with at least one value are strictly improved, where $k \in \{\mathrm{I}, \mathrm{II}\}$.*

We let $u_{\mathrm{II}}(v; R_{\mathrm{II}})$ denote the ex post expected utility of a bidder with value $v \in [\underline{v}, \bar{v}]$ in a second-price sealed-bid auction with a rejection price R_{II}. With Proposition 1, we can calculate that[7]

$$u_{\mathrm{II}}(v; R_{\mathrm{II}}) = \begin{cases} \int_{\underline{v}}^{v} F^{n-1}(t)dt & \text{if } R_{\mathrm{II}} > v \geq \underline{v} \\ \int_{\underline{v}}^{R_{\mathrm{II}}} F^{n-1}(t)dt + \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(R_{\mathrm{II}})(v - R_{\mathrm{II}}) & \text{if } \bar{v} \geq v \geq R_{\mathrm{II}}. \end{cases}$$

It is easy to see that $u_{\mathrm{II}}(v; \bar{v}) = \int_{\underline{v}}^{v} F^{n-1}(t)dt$ denotes the ex post equilibrium expected utility by a bidder with value v in a standard second-price sealed-bid auction.

Proposition 3. *Using a rejection price R_{II} makes a Pareto improvement for bidders to a standard second-price sealed-bid auction if and only if $u_{\mathrm{II}}(\bar{v}; R_{\mathrm{II}}) \geq u_{\mathrm{II}}(\bar{v}; \bar{v})$, namely*

$$\int_{R_{\mathrm{II}}}^{\bar{v}} \left(\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(R_{\mathrm{II}}) - F^{n-1}(t)\right)dt \geq 0.$$

Proof. Let $\Delta_{\mathrm{II}}(v) = u_{\mathrm{II}}(v; R_{\mathrm{II}}) - u_{\mathrm{II}}(v; \bar{v})$, then

$$\Delta_{\mathrm{II}}(v) = \begin{cases} 0 & \text{if } R_{\mathrm{II}} > v \geq \underline{v} \\ \int_{R_{\mathrm{II}}}^{v} \left(\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(R_{\mathrm{II}}) - F^{n-1}(t)\right)dt & \text{if } \bar{v} \geq v \geq R_{\mathrm{II}}. \end{cases}$$

[6] In fact, the efficiency which is defined in standard auction theory does not apply in our model. In standard auction theory, if the object ends up in the hands of the bidder who values it the most ex post, the welfare of a standard auction will be maximized. But in our model, if the auctioneer's revenue is larger than v^*, the welfare of our model may not be maximized since the auctioneer's utility is decreasing in his revenue in this case.

[7] The probability that a bidder with value $v \in [R_{\mathrm{II}}, \bar{v}]$ wins the object is $\frac{1-F^n(R_{\mathrm{II}})}{n(1-F(R_{\mathrm{II}}))} = \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(R_{\mathrm{II}})$.

So it suffices to show that $\Delta_{\mathrm{II}}(v) \geq 0$ for any $v \in [R_{\mathrm{II}}, \bar{v}]$ and $\Delta_{\mathrm{II}}(v) > 0$ for some $v \in [R_{\mathrm{II}}, \bar{v}]$ if and only if $u_{\mathrm{II}}(\bar{v}; R_{\mathrm{II}}) \geq u_{\mathrm{II}}(\bar{v}; \bar{v})$. We take the derivative with respect to v, then for any $v \in [R_{\mathrm{II}}, \bar{v}]$

$$\Delta'_{\mathrm{II}}(v) = \frac{1}{n} \sum_{i=1}^{n} F^{n-i}(R_{\mathrm{II}}) - F^{n-1}(v)$$

and

$$\Delta''_{\mathrm{II}}(v) = -(n-1)F^{n-2}(v)f(v) < 0,$$

i.e., $\Delta_{\mathrm{II}}(v)$ is strictly concave on the interval $[R_{\mathrm{II}}, \bar{v}]$. Thus $\Delta_{\mathrm{II}}(v) > \min\{\Delta_{\mathrm{II}}(R_{\mathrm{II}}), \Delta_{\mathrm{II}}(\bar{v})\}$ for any $v \in (R_{\mathrm{II}}, \bar{v})$. Notice that $\Delta_{\mathrm{II}}(R_{\mathrm{II}}) = 0$, thus $\Delta_{\mathrm{II}}(v) > 0$ for any $v \in (R_{\mathrm{II}}, \bar{v})$ if and only if

$$\Delta_{\mathrm{II}}(\bar{v}) = u_{\mathrm{II}}(\bar{v}; R_{\mathrm{II}}) - u_{\mathrm{II}}(\bar{v}; \bar{v}) = \int_{R_{\mathrm{II}}}^{\bar{v}} (\frac{1}{n} \sum_{i=1}^{n} F^{n-i}(R_{\mathrm{II}}) - F^{n-1}(t))dt \geq 0.$$

□

Proposition 3 implies that for a given rejection price if the ex post expected utility of the bidder with value $\bar{v}$ can be improved, all bidders' ex post expected utilities can be Pareto improved. Consider a bidder with value $\bar{v}$ and an event that there are m bidders whose values are higher than the rejection price R_{II} in the rest, where $m = 1, 2, ..., n-1$. In the equilibrium, the bidder wins with probability $\frac{1}{m+1}$ and his payment is R_{II}. It can be regarded as that the bidder wins with probability 1 but he needs to pay $\bar{v} - \frac{\bar{v}-R_{\mathrm{II}}}{m+1}$. Notice that in a standard second-price sealed-bid auction if bidders' values which are higher than R_{II} are uniformly distributed and the event that there are m other bidders whose values are higher than R_{II} happens, he also wins with probability 1 and his payment is $\int_{R_{\mathrm{II}}}^{\bar{v}} td(\frac{t-R_{\mathrm{II}}}{\bar{v}-R_{\mathrm{II}}})^m = \bar{v} - \frac{\bar{v}-R_{\mathrm{II}}}{m+1}$. And if all the other bidders' values are lower than R_{II}, he gets the same ex post expected utilities both in a standard second-price sealed-bid auction and in a second-price sealed-bid auction with a rejection price R_{II}. Therefore his ex post expected utility can be written as

$$u_{\mathrm{II}}(\bar{v}; R_{\mathrm{II}}) = \int_{\underline{v}}^{R_{\mathrm{II}}} F^{n-1}(t)dt + \int_{R_{\mathrm{II}}}^{\bar{v}} ((1 - F(R_{\mathrm{II}}))\frac{t - R_{\mathrm{II}}}{\bar{v} - R_{\mathrm{II}}} + F(R_{\mathrm{II}}))^{n-1}dt.$$

Compared to $u_{\mathrm{II}}(\bar{v}; \bar{v}) = \int_{\underline{v}}^{\bar{v}} F^{n-1}(t)dt$, the following corollary is straightforward.

Corollary 1. *Using a rejection price R_{II} makes a Pareto improvement for bidders to a standard second-price sealed-bid auction if for any $v \in [R_{\mathrm{II}}, \bar{v}]$*

$$F(v) \leq (1 - F(R_{\mathrm{II}}))\frac{v - R_{\mathrm{II}}}{\bar{v} - R_{\mathrm{II}}} + F(R_{\mathrm{II}}).$$

Proof. By Proposition 3, it suffices to show that $u_{\mathrm{II}}(\bar{v}; R_{\mathrm{II}}) \geq u_{\mathrm{II}}(\bar{v}; \bar{v})$. By the condition,

$$u_{\mathrm{II}}(\bar{v}; R_{\mathrm{II}}) - u_{\mathrm{II}}(\bar{v}; \bar{v}) = \int_{R_{\mathrm{II}}}^{\bar{v}} [((1-F(R_{\mathrm{II}}))\frac{t - R_{\mathrm{II}}}{\bar{v} - R_{\mathrm{II}}} + F(R_{\mathrm{II}}))^{n-1} - F^{n-1}(t)]dt \geq 0.$$

□

Corollary 1 implies that using a rejection price R_{II} makes a Pareto improvement for bidders to a standard second-price sealed-bid auction if the distribution function $F(\cdot)$ first-order stochastic dominates a distribution function which is the same as $F(\cdot)$ on the interval $[\underline{v}, R_{\mathrm{II}}]$ and in which bidders' values which are higher than R_{II} are uniformly distributed.

Based on Propositions 2 and 3, we can show that using the optimal rejection price can result in a Pareto improvement, compared to a standard second-price sealed-bid auction. Before showing the result, we give the definition of the Pareto improvement.

Definition 2. (Pareto improvement) *A k^{th}-price sealed-bid auction with a rejection price R_k makes a Pareto improvement to a standard k^{th}-price sealed-bid auction if it makes a Pareto improvement for bidders and the expected utility of the auctioneer is improved, where $k \in \{\mathrm{I}, \mathrm{II}\}$.*

Theorem 1. *Using the optimal rejection price $R^*_{\mathrm{II}} = v^*$ makes a Pareto improvement to a standard second-price sealed-bid auction if and only if*

$$\int_{v^*}^{\bar{v}} \Big(\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v^*) - F^{n-1}(t)\Big)dt \geq 0.$$

Proof. Clearly, due to the Proposition 3, the ex post expected utilities of bidders are improved by the optimal rejection price. Thus, we only need to show the expected utility of the auctioneer is improved. Note that $U_{\mathrm{II}}(\bar{v}) = n\int_{\underline{v}}^{\bar{v}}(1-F(t))u(t)dF^{n-1}(t)$ denotes the auctioneer's expected utility in a standard second-price sealed-bid auction. And we have $U_{\mathrm{II}}(v^*) > U_{\mathrm{II}}(\bar{v})$ in Proposition 2. □

Corollary 2. *Using the optimal rejection price $R^*_{\mathrm{II}} = v^*$ makes a Pareto improvement to a standard second-price sealed-bid auction if for any $v \in [v^*, \bar{v}]$*

$$F(v) \leq (1 - F(v^*))\frac{v - v^*}{\bar{v} - v^*} + F(v^*).$$

Proof. This corollary holds by Corollary 1 and Theorem 1. □

3.1 General second-price sealed-bid auction with rejection prices

In standard auction theory, the conclusions of a second-price sealed-bid auction such as the equilibrium bidding strategies do not depend on bidders' risk-attitudes. Therefore, in this subsection, we investigate a more general second-price sealed-bid auction with a rejection price and show that the rejection prices work regardless of bidders' risk-attitudes.

Specifically, this subsection assumes that bidder $i \in N$ assigns a value of v_i to the object and each v_i is independently distributed on some interval $[\underline{v}, \bar{v}] \subset$

$[0,\infty)$ according to an increasing distribution function $F_i(\cdot)$. It is assumed that $F_i(\cdot)$ admits a continuous density $f_i(\cdot) = F_i'(\cdot) > 0$ and has full support. We let $u_i(\cdot)$ denote the utility function of bidder i. We use the same notation in the original version besides what we mentioned above. Therefore, we assume the utility of bidder i who bids b_i with value v_i, for every $i \in N$, is

$$u_i(v_i, b) = \begin{cases} \hat{u}_i(v_i - \max\limits_{j\neq i}\{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\})/l(b_i) & \textit{if } b_i \mathbf{1}_{\{b_i \leq R_{\mathrm{II}}\}} \geq \max\limits_{j\neq i}\{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\} \\ 0 & \textit{if } b_i \mathbf{1}_{\{b_i \leq R_{\mathrm{II}}\}} < \max\limits_{j\neq i}\{b_j \mathbf{1}_{\{b_j \leq R_{\mathrm{II}}\}}, 0\} \end{cases}$$

in a general second-price auction with a rejection price R_{II}, where $\hat{u}_i(\cdot)$ is any strictly increasing function with $\hat{u}_i(0) = 0$.

Although these assumptions are much weaker than the original version, we can have the same conclusion in bidders' and the auctioneer's behavior.

Proposition 4. (General truth-telling for the auctioneer) *In a general second-price auction with a rejection price R_{II}, it is a symmetric equilibrium bidding strategy to bid according to $\beta_{\mathrm{II}}(v_i) = \min\{v_i, R_{\mathrm{II}}\}$, for any $i \in N$. And the optimal rejection price is $R^*_{\mathrm{II}} = v^*$.*[8]

Proof. First, we consider the equilibrium bidding strategy. Clearly, there is no incentive for bidder i to bid higher than v_i (may suffer a loss) or R_{II} (be rejected), for every $i \in N$. Suppose that bidder i bids an amount $b_i < \beta_{\mathrm{II}}(v_i)$. On the one hand, if bidding $\beta_{\mathrm{II}}(v_i)$ leads to lose, he loses by bidding b_i, too. One the other hand, note that $\hat{u}_i(\cdot)$ is a strictly increasing function, bidder i can gain his winning probability by bidding $\beta_{\mathrm{II}}(v_i)$ and his expected utility gains from the increment of winning probability is positive. Thus, bidding less than $\beta_{\mathrm{II}}(v_i)$ can never be an optimal strategy for bidder i, i.e., bidder i's bidding strategy is $\beta_{\mathrm{II}}(v_i) = \min\{v_i, R_{\mathrm{II}}\}$.

Therefore the auctioneer's expected utility can be computed as follows:

$$\begin{aligned} U_{\mathrm{II}}(R_{\mathrm{II}}) =& [1 - G(R_{\mathrm{II}}) - \sum_{i=1}^{n} (1 - F_i(R_{\mathrm{II}}))G_{-i}(R_{\mathrm{II}})]u(R_{\mathrm{II}}) \\ &+ \sum_{i=1}^{n} \int_{\underline{v}}^{R_{\mathrm{II}}} (1 - F_i(t))u(t)dG_{-i}(t), \end{aligned}$$

[8]Similarly to the original version, this proposition also holds even if $u(\cdot)$ is discontinuous on the interval $[\underline{v}, \bar{v}]$. This is because the only problem is the existence of $\sum_{i=1}^{n} \int_{\underline{v}}^{R_{\mathrm{II}}} (1 - F_i(t))u(t)dG_{-i}(t)$, which is solved by the monotonicity of $u(\cdot)$ and the integrability of $f_i(\cdot)$, where $i \in N$ and $G_{-i}(\cdot) = \prod_{j\neq i}^{n} F_j(\cdot)$.
The proof also applies to the case $v^* \geq \bar{v}$ and implies that a rejection price (or ceiling price) fails to increase the auctioneer's utility in a second-price sealed-bid auction with any assumption about bidders' and the auctioneer's attitude to risk.

where $G(\cdot) = \prod_{i=1}^{n} F_i(\cdot)$ and $G_{-i}(\cdot) = \prod_{j\neq i}^{n} F_j(\cdot)$. So

$$\begin{aligned}
U_{\mathrm{II}}(v^*) - U_{\mathrm{II}}(R_{\mathrm{II}}) =& [1 - G(v^*) - \sum_{i=1}^{n} (1 - F_i(v^*))G_{-i}(v^*)]u(v^*) \\
& - [1 - G(R_{\mathrm{II}}) - \sum_{i=1}^{n} (1 - F_i(R_{\mathrm{II}}))G_{-i}(R_{\mathrm{II}})]u(R_{\mathrm{II}}) \\
& + \sum_{i=1}^{n} \int_{R_{\mathrm{II}}}^{v^*} (1 - F_i(t))u(t)dG_{-i}(t).
\end{aligned}$$

Notice that $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, v^*]$. Then for any $R_{\mathrm{II}} \in [\underline{v}, v^*)$,

$$\begin{aligned}
U_{\mathrm{II}}(v^*) - U_{\mathrm{II}}(R_{\mathrm{II}}) > & [1 - G(v^*) - \sum_{i=1}^{n} (1 - F_i(v^*))G_{-i}(v^*)]u(v^*) \\
& - [1 - G(R_{\mathrm{II}}) - \sum_{i=1}^{n} (1 - F_i(R_{\mathrm{II}}))G_{-i}(R_{\mathrm{II}})]u(R_{\mathrm{II}}) \\
& + \sum_{i=1}^{n} \int_{R_{\mathrm{II}}}^{v^*} (1 - F_i(t))u(R_{\mathrm{II}})dG_{-i}(t) \\
= & [1 - G(v^*) - \sum_{i=1}^{n} (1 - F_i(v^*))G_{-i}(v^*)][u(v^*) - u(R_{\mathrm{II}})] \\
> & 0.
\end{aligned}$$

And since $u(\cdot)$ is strictly decreasing on the interval $(v^*, \bar{v}]$, for any $R_{\mathrm{II}} \in (v^*, \bar{v}]$,

$$\begin{aligned}
U_{\mathrm{II}}(v^*) - U_{\mathrm{II}}(R_{\mathrm{II}}) > & [1 - G(v^*) - \sum_{i=1}^{n} (1 - F_i(v^*))G_{-i}(v^*)]u(v^*) \\
& - [1 - G(R_{\mathrm{II}}) - \sum_{i=1}^{n} (1 - F_i(R_{\mathrm{II}}))G_{-i}(R_{\mathrm{II}})]u(R_{\mathrm{II}}) \\
& - \sum_{i=1}^{n} \int_{v^*}^{R_{\mathrm{II}}} (1 - F_i(t))u(v^*)dG_{-i}(t) \\
= & [1 - G(R_{\mathrm{II}}) - \sum_{i=1}^{n} (1 - F_i(R_{\mathrm{II}}))G_{-i}(R_{\mathrm{II}})][u(v^*) - u(R_{\mathrm{II}})] \\
> & 0.
\end{aligned}$$

Thus the optimal rejection price is $R_{\mathrm{II}}^* = v^*$. $\square$

4 First-price sealed-bid auction with rejection prices

We analyze a first-price sealed-bid auction with a rejection price by the same steps as the second-price sealed-bid auction with a rejection price R_{I}. Let $\beta(v)$ denote the equilibrium bidding strategy which is bidden by a bidder with value $v \in [\underline{v}, \bar{v}]$ in a standard first-price sealed-bid auction, i.e.,

$$\beta(v) = (\int_{\underline{v}}^{v} t dF^{n-1}(t))/F^{n-1}(v).$$

Based on Proposition 1, we may conjecture an equilibrium bidding strategy such that some bidders bid the rejection price and the others follow $\beta(v)$. Therefore, for a given rejection price R_{I}, there exists some value v_J that bidders whose values are higher than v_J bid the rejection price and the others (whose value is lower than v_J) follow $\beta(v)$. And the bidder with value v_J must be indifferent between bidding the rejection price R_{I} and the standard bid $\beta(v_J)$. The following lemma shows that in such an equilibrium bidding strategy, the value v_J uniquely exists if $R_{\mathrm{I}} \in (\underline{v}, \beta(\bar{v}))$ and it is strictly increasing in R_{I}.

Lemma 1. *For any given $R_{\mathrm{I}} \in (\underline{v}, \beta(\bar{v}))$, there uniquely exists $v_J \in (\underline{v}, \bar{v})$ which satisfies*

$$F^{n-1}(v_J)(v_J - \beta(v_J)) = \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_J - R_{\mathrm{I}}).$$

And $\frac{dv_J}{dR_{\mathrm{I}}} > 0$ when $R_{\mathrm{I}} \in (\underline{v}, \beta(\bar{v}))$.

Proof. For a given $R_{\mathrm{I}} \in (\underline{v}, \beta(\bar{v}))$, let

$$H(v) = \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v)(v - R_{\mathrm{I}}) - F^{n-1}(v)(v - \beta(v)),$$

where $v \in (\underline{v}, \bar{v})$. We show that $H(v)$ has only one zero point. We take the derivative with respect to v,

$$H'(v) = [\frac{f(v)(v - R_{\mathrm{I}})}{1 - F(v)} + 1][\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v) - F^{n-1}(v)].$$

Notice that $\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v) > F^{n-1}(v)$ for any $v \in (\underline{v}, \bar{v})$, so $H'(v) > 0$ for any $v \in [R_{\mathrm{I}}, \bar{v})$, i.e., $H(v)$ is strictly increasing on the interval $[R_{\mathrm{I}}, \bar{v})$. Further, since $\beta(v) < v$ for any $v \in (\underline{v}, \bar{v})$ and $\beta(v)$ is strictly increasing in v,[9] we can get

$$H(\beta^{-1}(R_{\mathrm{I}})) = [\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(\beta^{-1}(R_{\mathrm{I}})) - F^{n-1}(\beta^{-1}(R_{\mathrm{I}}))](\beta^{-1}(R_{\mathrm{I}}) - R_{\mathrm{I}}) > 0$$

[9] See Krishna (2009).

and
$$H(R_{\mathrm{I}}) = -F^{n-1}(R_{\mathrm{I}})(R_{\mathrm{I}} - \beta(R_{\mathrm{I}})) < 0.$$
So there uniquely exists $v \in (R_{\mathrm{I}}, \bar{v})$ such that $H(v) = 0$. Notice that $\beta(v) < v$ for any $v \in (\underline{v}, \bar{v})$, then $H(v) < 0$ for any $v \in (\underline{v}, R_{\mathrm{I}}]$. Therefore, $H(v)$ has only one zero point, i.e., $v_J \in (\underline{v}, \bar{v})$ uniquely exists.

Now, we show that $\frac{dv_J}{dR_{\mathrm{I}}} > 0$. Since
$$F^{n-1}(v_J)(v_J - \beta(v_J)) = \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_J - R_{\mathrm{I}}),$$
take the derivative of both sides with respect to v_J, we can get
$$\begin{aligned}\sum_{i=1}^{n} F^{n-i}(v_J)\frac{dR_{\mathrm{I}}}{dv_J} =& \sum_{i=1}^{n-1}(n-i)F^{n-i-1}(v_J)f(v_J)(v_J - R_{\mathrm{I}}) \\ &+ \sum_{i=1}^{n} F^{n-i}(v_J) - nF^{n-1}(v_J).\end{aligned}$$
Due to $v_J > R_{\mathrm{I}}$ and $\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v) > F^{n-1}(v)$, we can get $\frac{dR_{\mathrm{I}}}{dv_J} > 0$. Hence $\frac{dv_J}{dR_{\mathrm{I}}} > 0$.

□

Based on Lemma 1, we can have the next proposition that in some cases the conjecture that for a given rejection price R_{I} the bidders whose values are higher than v_J bid the rejection price and the others follow $\beta(v)$ is right.

Proposition 5. *Equilibrium strategies in a first-price sealed-bid auction with a rejection price R_{I} are given by,*
(i) if $R_{\mathrm{I}} < \beta(\bar{v})$,
$$\beta_{\mathrm{I}}(v_i) = \begin{cases} \beta(v_i) & \text{if } v_i < v_J \\ R_{\mathrm{I}} & \text{if } v_i \geq v_J \end{cases}$$
where v_J is a jump point which satisfies that
$$F^{n-1}(v_J)(v_J - \beta(v_J)) = \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_J - R_{\mathrm{I}});$$
(ii) if $R_{\mathrm{I}} \geq \beta(\bar{v})$, for every $v_i \in [\underline{v}, \bar{v}]$
$$\beta_{\mathrm{I}}(v_i) = \beta(v_i).$$

Proof. Obviously, if $R_{\mathrm{I}} \geq \beta(\bar{v})$, the rejection price can not work and bidders bid following the standard model, i.e., $\beta_{\mathrm{I}}(v_i) = \beta(v_i)$. Thus, we only need to prove the case $R_{\mathrm{I}} < \beta(\bar{v})$.

Suppose that all but bidder i follow the strategy given in the statement and let $u_i(v_i, (b_i, (\beta_{\mathrm{I}})_{-i}))$ denote bidder i's ex post expected utility when his bid is b_i. Then the ex post expected utility of bidder i can be computed:

$$u_i(v_i, (b_i, (\beta_{\mathrm{I}})_{-i})) = \begin{cases} 0 & \textit{if } b_i < \underline{v} \\ F^{n-1}(\beta^{-1}(b_i))(v_i - b_i) & \textit{if } \beta(v_J) \geq b_i \geq \underline{v} \\ F^{n-1}(v_J)(v_i - b_i) & \textit{if } R_{\mathrm{I}} > b_i > \beta(v_J) \\ \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_i - R_{\mathrm{I}}) & \textit{if } b_i = R_{\mathrm{I}} \\ 0 & \textit{if } b_i > R_{\mathrm{I}}. \end{cases}$$

Clearly, it is not optimal for bidder i to bid higher than R_{I} or lower than $\underline{v}$. On the one hand, if bidder i's value is $v_i < v_J$, note that the utility with $b_i \leq \beta(v_J)$ is same as the standard model, so we have $u_i(v_i, (\beta(v_i), (\beta_{\mathrm{I}})_{-i})) \geq u_i(v_i, (b_i, (\beta_{\mathrm{I}})_{-i}))$ for any $b_i \leq \beta(v_J)$. We also have, for any $R_{\mathrm{I}} > b_i > \beta(v_J)$,

$$u_i(v_i, (\beta(v_i), (\beta_{\mathrm{I}})_{-i})) \geq F^{n-1}(v_J)(v_i - \beta(v_J)) > F^{n-1}(v_J)(v_i - b_i)$$

and

$$\begin{aligned} u_i(v_i, (\beta(v_i), (\beta_{\mathrm{I}})_{-i})) &\geq F^{n-1}(v_J)(v_i - \beta(v_J)) \\ &> F^{n-1}(v_J)(v_J - \beta(v_J)) + \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_i - v_J) \\ &= \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_i - R_{\mathrm{I}}). \end{aligned}$$

Hence, it is optimal for bidder i with value $v_i < v_J$ to bid $\beta(v_i)$.

On the other hand, if bidder i's value is $v_i \geq v_J$, for any $R_{\mathrm{I}} > b_i > \beta(v_J)$

$$\begin{aligned} u_i(v_i, (R_{\mathrm{I}}, (\beta_{\mathrm{I}})_{-i})) &= \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_J - R_{\mathrm{I}}) + \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_i - v_J) \\ &= F^{n-1}(v_J)(v_J - \beta(v_J)) + \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v_i - v_J) \\ &\geq F^{n-1}(v_J)(v_J - \beta(v_J)) + F^{n-1}(v_J)(v_i - v_J) \\ &> F^{n-1}(v_J)(v_i - b_i) \end{aligned}$$

and for any $b_i \leq \beta(v_J)$

$$\begin{aligned} u_i(v_i, (R_{\mathrm{I}}, (\beta_{\mathrm{I}})_{-i})) &\geq F^{n-1}(v_J)(v_i - \beta(v_J)) \\ &= F^{n-1}(v_J)(v_J - \beta(v_J)) + F^{n-1}(v_J)(v_i - v_J) \\ &\geq F^{n-1}(\beta^{-1}(b_i))(v_J - b_i) + F^{n-1}(v_J)(v_i - v_J) \\ &\geq F^{n-1}(\beta^{-1}(b_i))(v_i - b_i). \end{aligned}$$

Hence, it is optimal for bidder i with value $v_i \geq v_J$ to bid R_{I}. $\square$

Proposition 5 implies that there exists a jump point if the rejection price works ($R_{\mathrm{I}} < \beta(\bar{v})$), i.e., such a rejection price induces some bidders bid higher than they bid in the standard model. One may imagine that the bidders might bid $\hat{\beta}_{\mathrm{I}}(v_i) = \min\{\beta(v_i), R_{\mathrm{I}}\}$ like the bidding strategy $\beta_{\mathrm{II}}(v_i)$. In this case, a bidder whose value is lower than $\beta^{-1}(R_{\mathrm{I}})$ can win the object only when all the other bidders' values are lower than his. However a bidder with value $\beta^{-1}(R_{\mathrm{I}})$ not only can win the object when all the other bidders' values are lower than his, but also may win the object when there exists some bidder whose value is higher than his. Notice that his profit is positive when he wins since $R_{\mathrm{I}} < \beta^{-1}(R_{\mathrm{I}})$. So the ex post expected utility of a bidder is discontinuous at $v = \beta^{-1}(R_{\mathrm{I}})$ in this case. Then by the same reason, a bidder whose value is lower than $\beta^{-1}(R_{\mathrm{I}})$ and sufficiently closes to it prefers bidding the rejection price to bidding the bid in the standard model. Therefore, in an equilibrium, the ex post expected utility of a bidder should be continuous at the lowest value among all values with which a bidder bids the rejection price.

In this section, we only focus on the equilibrium bidding strategy in Proposition 5. We now examine what effect such a rejection price has on the auctioneer's expected utility. The following proposition states that it is profitable for an auctioneer to set the optimal rejection price not higher than the maximizer, if the maximizer is lower than the maximum equilibrium bid in a standard model. We could let $v_J(R_{\mathrm{I}})$ denote the jump point with a rejection price R_{I} by Lemma 1 and the implicit function theorem. For notational simplicity, we drop R_{I} in $v_J(R_{\mathrm{I}})$ when there is no ambiguity.

Proposition 6. *In a first-price sealed-bid auction with a rejection price R_{I}, if $v^* < \beta(\bar{v})$, the optimal rejection price is $R_{\mathrm{I}}^* \in (\underline{v}, v^*]$. Moreover, if $u(\cdot)$ is differentiable at v^*, then the optimal rejection price is $R_{\mathrm{I}}^* \in (\underline{v}, v^*)$.*

Proof. First, we show the former statement. The auctioneer's expected utility with a rejection price R_{I} can be calculated as follows:

$$U_{\mathrm{I}}(R_{\mathrm{I}}) = \begin{cases} [1 - F^n(v_J(R_{\mathrm{I}}))]u(R_{\mathrm{I}}) + \int_{\underline{v}}^{v_J(R_{\mathrm{I}})} u(\beta(t))dF^n(t) & \text{if } R_{\mathrm{I}} < \beta(\bar{v}) \\ \int_{\underline{v}}^{\bar{v}} u(\beta(t))dF^n(t) & \text{if } R_{\mathrm{I}} \geq \beta(\bar{v}). \end{cases}$$

Due to Lemma 1, we have $v_J(R_{\mathrm{I}}) > v_J(v^*)$, for any $R_{\mathrm{I}} \in (v^*, \beta(\bar{v})]$. Note that $u(v^*) > u(v)$ for any $v \in (v^*, \bar{v}]$, so for any $R_{\mathrm{I}} \in (v^*, \beta(\bar{v})]$

$$\begin{aligned} U_{\mathrm{I}}(v^*) - U_{\mathrm{I}}(R_{\mathrm{I}}) &= [1 - F^n(v_J(v^*))]u(v^*) - [1 - F^n(v_J(R_{\mathrm{I}}))]u(R_{\mathrm{I}}) \\ &\quad - \int_{v_J(v^*)}^{v_J(R_{\mathrm{I}})} u(\beta(t))dF^n(t) \\ &> [F^n(v_J(R_{\mathrm{I}})) - F^n(v_J(v^*))]u(v^*) - \int_{v_J(v^*)}^{v_J(R_{\mathrm{I}})} u(\beta(t))dF^n(t) \\ &= \int_{v_J(v^*)}^{v_J(R_{\mathrm{I}})} [u(v^*) - u(\beta(t))]dF^n(t) \\ &> 0. \end{aligned}$$

For any $R_{\mathrm{I}} \in (\beta(\bar{v}), \bar{v}]$, it is easy to know that $U_{\mathrm{I}}(R_{\mathrm{I}}) = U_{\mathrm{I}}(\beta(\bar{v})) < U_{\mathrm{I}}(v^*)$. And since $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, v^*]$, then $U(v^*) > u(\underline{v}) = U(\underline{v})$. Since $U_{\mathrm{I}}(R_{\mathrm{I}})$ is continuous on the interval $[\underline{v}, \beta(\bar{v})]$, therefore, there exists $R_{\mathrm{I}}^* \in (\underline{v}, v^*]$ such that $U_{\mathrm{I}}(R_{\mathrm{I}}^*) = \max\limits_{R_{\mathrm{I}} \in [\underline{v}, \bar{v}]} U_{\mathrm{I}}(R_{\mathrm{I}})$.

Now, we show the latter statement. Since $u(\cdot)$ is differentiable at v^* and v^* is the maximizer, so we have $u'(v^*) = 0$ and $U_{\mathrm{I}}(\cdot)$ is also differentiable at v^*. Lemma 1 implies $\beta(v_J(v^*)) < v^* < v_J(v^*)$ and $\frac{dv_J(v^*)}{dR_{\mathrm{I}}} > 0$.[10] Then

$$\begin{aligned}
\frac{dU_{\mathrm{I}}(v^*)}{dR_{\mathrm{I}}} &= nF^{n-1}(v_J(v^*))f(v_J(v^*))[u(\beta(v_J(v^*))) - u(v^*)]\frac{dv_J(v^*)}{dR_{\mathrm{I}}} \\
&\quad + [1 - F^n(v_J(v^*))]u'(v^*) \\
&= nF^{n-1}(v_J(v^*))f(v_J(v^*))[u(\beta(v_J(v^*))) - u(v^*)]\frac{dv_J(v^*)}{dR_{\mathrm{I}}} \\
&< 0.
\end{aligned}$$

This implies that there exists $R_{\mathrm{I}} \in (\underline{v}, v^*)$ such that $U_{\mathrm{I}}(R_{\mathrm{I}}) > U_{\mathrm{I}}(v^*)$. Further, by the former statement, there exists $R_{\mathrm{I}}^* \in (\underline{v}, v^*)$ such that $U_{\mathrm{I}}(R_{\mathrm{I}}^*) = \max\limits_{R_{\mathrm{I}} \in [\underline{v}, \bar{v}]} U_{\mathrm{I}}(R_{\mathrm{I}})$. □

Remark 2. *If $v^* < \beta(\bar{v})$ and $u(\cdot)$ is linear on the interval $[\underline{v}, v^*]$, the optimal rejection price in a first-price sealed-bid auction is $R_{\mathrm{I}}^* = v^*$. The reason is as follows. It suffices to show that $R_{\mathrm{I}}^* \geq v^*$ by the former statement of Proposition 6. By the definition of $v_J(R_{\mathrm{I}})$, we have*

$$R_{\mathrm{I}} = v_J(R_{\mathrm{I}}) - \frac{n[1 - F(v_J(R_{\mathrm{I}}))]F^{n-1}(v_J(R_{\mathrm{I}}))[v_J(R_{\mathrm{I}}) - \beta(v_J(R_{\mathrm{I}}))]}{1 - F^n(v_J(R_{\mathrm{I}}))}.$$

So by the linearity, the auctioneer's expected utility with a rejection price R_{I} is

$$\begin{aligned}
U_{\mathrm{I}}(R_{\mathrm{I}}) &= u\{[1 - F^n(v_J(R_{\mathrm{I}}))]R_{\mathrm{I}} + \int_{\underline{v}}^{v_J(R_{\mathrm{I}})} \beta(t)dF^n(t)\} \\
&= u\{[1 - F^n(v_J(R_{\mathrm{I}}))]v_J(R_{\mathrm{I}}) + \int_{\underline{v}}^{v_J(R_{\mathrm{I}})} \beta(t)dF^n(t) \\
&\quad - n[1 - F(v_J(R_{\mathrm{I}}))]F^{n-1}(v_J(R_{\mathrm{I}}))[v_J(R_{\mathrm{I}}) - \beta(v_J(R_{\mathrm{I}}))]\}
\end{aligned}$$

for any $R_{\mathrm{I}} \in [\underline{v}, v^*]$. Therefore for any $R_{\mathrm{I}} \in (\underline{v}, v^*)$,

$$\frac{dU_{\mathrm{I}}(R_{\mathrm{I}})}{dR_{\mathrm{I}}} = u'(R_{\mathrm{I}})[1-F(v_J(R_{\mathrm{I}}))][\sum_{i=1}^{n} F^{n-i}(v_J(R_{\mathrm{I}})) - nF^{n-1}(v_J(R_{\mathrm{I}}))]\frac{dv_J(R_{\mathrm{I}})}{dR_{\mathrm{I}}} > 0.$$

And due to the continuity of $U_{\mathrm{I}}(R_{\mathrm{I}})$, we have $U_{\mathrm{I}}(v^*) > U_{\mathrm{I}}(R_{\mathrm{I}})$ for any $R_{\mathrm{I}} \in [\underline{v}, v^*)$. Hence, $R_{\mathrm{I}}^* \geq v^*$.

[10] See the proof of Lemma 1.

Proposition 6 implies that the optimal rejection price is not higher than the maximizer. Before we give an explanation, let $F_{\mathrm{I}}(v;R_{\mathrm{I}})$ denote the distribution function of the auctioneer's revenue v in a first-price sealed-bid auction with a rejection price R_{I}. It is easy to see that if $R_{\mathrm{I}} \in [\underline{v}, \beta(\bar{v}))$,

$$F_{\mathrm{I}}(v;R_{\mathrm{I}}) = \begin{cases} F^n(\beta^{-1}(v)) & \text{if } \beta(v_J(R_{\mathrm{I}})) > v \geq \underline{v} \\ F^n(v_J(R_{\mathrm{I}})) & \text{if } R_{\mathrm{I}} > v \geq \beta(v_J(R_{\mathrm{I}})) \\ 1 & \text{if } v = R_{\mathrm{I}} \end{cases}$$

and if $R_{\mathrm{I}} \in [\beta(\bar{v}), \bar{v}]$,

$$F_{\mathrm{I}}(v;R_{\mathrm{I}}) = F^n(\beta^{-1}(v))$$

for any $v \in [\underline{v}, \beta(\bar{v})]$. For a rejection price $R_{\mathrm{I}} > v^*$, notice that $F_{\mathrm{I}}(v;v^*) = F_{\mathrm{I}}(v;R_{\mathrm{I}})$ for any $v \in [\underline{v}, \beta(v_J(v^*))$ and $u(v^*) > u(v)$ for any $v \in (v^*, \bar{v}]$. Therefore, the auctioneer will get higher expected utility with the rejection price v^* than with the rejection price R_{I}, i.e., the optimal rejection price R_{I}^* cannot be higher than v^*.

For the case where $u'(v^*) = 0$, notice that the jump point is increasing in the rejection price, so that a rejection price which is lower than v^* makes bidders more likely to bid it, i.e., the auctioneer has a greater probability of receiving the rejection price. The increment of the probability brings a positive gain to the expected utility, but there is also a loss by receiving the lower rejection price. Because $u'(v^*) = 0$, if the lower rejection price is sufficiently close to v^*, then the loss can be negligible and there is only a gain to the expected utility. Hence the optimal rejection price is lower than the maximizer in this case. By the intuition, if the maximizer is larger than $\beta(\bar{v})$ but sufficiently close to it, there also exists an optimal rejection price $R_{\mathrm{I}}^* \in (\underline{v}, \beta(\bar{v}))$.[11]

Similarly to the second-price auction with a rejection price, we also want to know whether this mechanism could make a Pareto improvement to a standard first-price sealed-bid auction or not. Similarly, we let $u_{\mathrm{I}}(v;R_{\mathrm{I}})$ denote the ex post expected utility by a bidder with value $v \in [\underline{v}, \bar{v}]$ in a first-price sealed-bid auction with a rejection price R_{I}. With Proposition 5, we can calculate that if $R_{\mathrm{I}} < \beta(\bar{v})$,

$$u_{\mathrm{I}}(v;R_{\mathrm{I}}) = \begin{cases} \int_{\underline{v}}^{v} F^{n-1}(t)dt & \text{if } v_J > v \geq \underline{v} \\ \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J)(v - R_{\mathrm{I}}) & \text{if } \bar{v} \geq v \geq v_J. \end{cases}$$

It is easy to see that $u_{\mathrm{I}}(v;\bar{v}) = \int_{\underline{v}}^{v} F^{n-1}(t)dt$ denotes the ex post expected utility by a bidder with value v in a standard first-price auction.

Proposition 7. *Using a rejection price $R_{\mathrm{I}} < \beta(\bar{v})$ makes a Pareto improvement for bidders to a standard first-price sealed-bid auction if and only if $u_{\mathrm{I}}(\bar{v};R_{\mathrm{I}}) \geq$*

[11] Here is a simple example. There are 2 bidders whose values are i.i.d on $[0,1]$ according to a uniform distribution. Let the auctioneer's utility be $u(v) = 1.01v - v^2$, then it is easy to see that $v^* = 0.505 > 0.5 = \beta(\bar{v})$ and the optimal rejection price is $R_{\mathrm{I}}^* \approx 0.455 < 0.5$.

$u_{\rm I}(\bar{v};\bar{v})$*, namely*

$$\int_{v_J}^{\bar{v}} (\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J) - F^{n-1}(t))dt \geq 0.$$

Proof. Let $\Delta_{\rm I}(v) = u_{\rm I}(v;R_{\rm I}) - u_{\rm I}(v;\bar{v})$, then

$$\Delta_{\rm I}(v) = \begin{cases} 0 & \textit{if } v_J > v \geq \underline{v} \\ \int_{v_J}^{v} (\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J) - F^{n-1}(t))dt & \textit{if } \bar{v} \geq v \geq v_J. \end{cases}$$

It is suffices to show that $\Delta_{\rm I}(v) \geq 0$ for any $v \in [v_J, \bar{v}]$ and $\Delta_{\rm I}(v) > 0$ for some $v \in [v_J, \bar{v}]$ if and only if $u_{\rm I}(\bar{v};R_{\rm I}) \geq u_{\rm I}(\bar{v};\bar{v})$. We take the derivative with respect to v, then for any $v \in [v_J, \bar{v}]$

$$\Delta'_{\rm I}(v) = \frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J) - F^{n-1}(v)$$

and

$$\Delta''_{\rm I}(v) = -(n-1)F^{n-2}(v)f(v) < 0,$$

i.e., $\Delta_{\rm I}(v)$ is strictly concave on the interval $[v_J, \bar{v}]$. Thus $\Delta_{\rm I}(v) > \min\{\Delta_{\rm I}(v_J), \Delta_{\rm I}(\bar{v})\}$ for any $v \in (v_J, \bar{v})$. Notice that $\Delta_{\rm I}(v_J) = 0$, thus $\Delta_{\rm I}(v) > 0$ for any $v \in (v_J, \bar{v})$ if and only if

$$\int_{v_J}^{\bar{v}} (\frac{1}{n}\sum_{i=1}^{n} F^{n-i}(v_J) - F^{n-1}(t))dt \geq 0.$$

□

If $R_{\rm I} < \beta(\bar{v})$, similarly to $u_{\rm II}(\bar{v};R_{\rm II})$, $u_{\rm I}(\bar{v};R_{\rm I})$ can be written as

$$u_{\rm I}(\bar{v};R_{\rm I}) = \int_{\underline{v}}^{v_J(R_{\rm I})} F^{n-1}(t)dt + \int_{v_J(R_{\rm I})}^{\bar{v}} [(1-F(v_J(R_{\rm I})))\frac{t - v_J(R_{\rm I})}{\bar{v} - v_J(R_{\rm I})} + F(v_J(R_{\rm I}))]^{n-1}dt.$$

Compared to $u_{\rm I}(\bar{v};\bar{v}) = \int_{\underline{v}}^{\bar{v}} F^{n-1}(t)dt$, if for any $v \in [v_J(R_{\rm I}), \bar{v}]$,

$$F(v) = (1 - F(v_J(R_{\rm I})))\frac{v - v_J(R_{\rm I})}{\bar{v} - v_J(R_{\rm I})} + F(v_J(R_{\rm I})),$$

i.e., value $v \in [v_J(R_{\rm I}), \bar{v}]$ is uniformly distributed, then the bidder with the largest value gets the same ex post expected utility in a standard first-price sealed-bid auction and in a first-price sealed-bid auction with a rejection price $R_{\rm I}$. Then the following corollary is straightforward.

Corollary 3. *Using a rejection price* $R_{\rm I} < \beta(\bar{v})$ *makes a Pareto improvement for bidders to a standard first-price sealed-bid auction if for any* $v \in [v_J(R_{\rm I}), \bar{v}]$

$$F(v) \leq (1 - F(v_J(R_{\rm I})))\frac{v - v_J(R_{\rm I})}{\bar{v} - v_J(R_{\rm I})} + F(v_J(R_{\rm I})).$$

Proof. By Proposition 7, it suffices to show that $u_{\mathrm{I}}(\bar{v}; R_{\mathrm{I}}) \geq u_{\mathrm{I}}(\bar{v}; \bar{v})$. By the condition,

$$u_{\mathrm{I}}(\bar{v}; R_{\mathrm{I}}) - u_{\mathrm{I}}(\bar{v}; \bar{v}) = \int_{v_J(R_{\mathrm{I}})}^{\bar{v}} \{[(1 - F(v_J(R_{\mathrm{I}}))) \frac{t - v_J(R_{\mathrm{I}})}{\bar{v} - v_J(R_{\mathrm{I}})} + F(v_J(R_{\mathrm{I}}))]^{n-1} - F^{n-1}(t)\} dt \geq 0.$$

□

Based on Propositions 6 and 7, we can also show that using the optimal rejection price can make a Pareto improvement to a standard first-price sealed-bid auction by the following theorem.

Theorem 2. *If $v^* < \beta(\bar{v})$, using the optimal rejection price makes a Pareto improvement to a standard first-price sealed-bid auction if and only if*

$$\int_{v_J^*}^{\bar{v}} (\frac{1}{n} \sum_{i=1}^{n} F^{n-i}(v_J^*) - F^{n-1}(t)) dt \geq 0,$$

where $v_J^* = v_J(R_{\mathrm{I}}^*)$.

Proof. Clearly, due to Proposition 7, the ex post expected utilities of bidders are improved by the optimal rejection price. And the expected utility of auctioneer is also improved, since we have $U_{\mathrm{I}}(R_{\mathrm{I}}^*) > U_{\mathrm{I}}(\bar{v})$ in Proposition 6, where $U_{\mathrm{I}}(\bar{v})$ denotes the expected utility of the auctioneer in the standard first-price sealed-bid auction. □

Corollary 4. *If $v^* < \beta(\bar{v})$, using the optimal rejection price makes a Pareto improvement to a standard first-price sealed-bid auction if for any $v \in [v_J^*, \bar{v}]$*

$$F(v) \leq (1 - F(v_J^*)) \frac{v - v_J^*}{\bar{v} - v_J^*} + F(v_J^*),$$

where $v_J^* = v_J(R_{\mathrm{I}}^*)$.

Proof. This corollary holds by Corollary 3 and Theorem 2. □

5 Comparison

In this section, we investigate the preferences of an auctioneer with non-monotonic utility over the two auctions. In standard auction theory, Matthews (1979) and Waehrer et al. (1998) show that a risk-averse auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction under the standard equilibria. In our model, we draw a similar conclusion to theirs. Before we state that conclusion, we show their result by the following lemma.

Lemma 2. *An auctioneer with a utility function $w(\cdot)$ which is concave on the interval $[\underline{v}, \bar{v}]$ prefers a standard first-price sealed-bid auction to a standard second-price sealed-bid auction, that is,*

$$n \int_{\underline{v}}^{\bar{v}} (1 - F(t)) w(t) dF^{n-1}(t) \leq \int_{\underline{v}}^{\bar{v}} w(\beta(t)) dF^n(t).$$

Moreover, the inequality becomes strict if $w(\cdot)$ is not linear.

Proof. Let $\Delta(v) = n\int_{\underline{v}}^{v}(F(v)-F(t))w(t)dF^{n-1}(t) - \int_{\underline{v}}^{v} w(\beta(t))dF^{n}(t)$. Then taking the derivative with respect to v, we have

$$\Delta'(v) = nf(v)[\int_{\underline{v}}^{v} w(t)dF^{n-1}(t) - F^{n-1}(v)w(\beta(v))].$$

Since $w(\cdot)$ is concave on the interval $[\underline{v}, \bar{v}]$, by Jensen's inequality, for any $v \in [\underline{v}, \bar{v}]$

$$\int_{\underline{v}}^{v} \frac{w(t)dF^{n-1}(t)}{F^{n-1}(v)} \le w(\int_{\underline{v}}^{v} \frac{tdF^{n-1}(t)}{F^{n-1}(v)}) = w(\beta(v)).$$

This implies that $\Delta'(v) \le 0$ for any $v \in [\underline{v}, \bar{v}]$. Hence, $\Delta(\bar{v}) \le \Delta(\underline{v}) = 0$, i.e.,

$$n\int_{\underline{v}}^{\bar{v}} (1-F(t))w(t)dF^{n-1}(t) \le \int_{\underline{v}}^{\bar{v}} w(\beta(t))dF^{n}(t).$$

Moveover, if $w(\cdot)$ is not linear, then

$$\int_{\underline{v}}^{v'} \frac{w(t)dF^{n-1}(t)}{F^{n-1}(v')} < w(\int_{\underline{v}}^{v'} \frac{tdF^{n-1}(t)}{F^{n-1}(v')}) = w(\beta(v'))$$

for some $v' \in (\underline{v}, \bar{v})$. This implies that $\Delta'(v') < 0$ for some $v' \in (\underline{v}, \bar{v})$. Hence $\Delta(\bar{v}) < \Delta(\underline{v}) = 0$, i.e.,

$$n\int_{\underline{v}}^{\bar{v}} (1-F(t))w(t)dF^{n-1}(t) < \int_{\underline{v}}^{\bar{v}} w(\beta(t))dF^{n}(t).$$

□

This lemma is intuitive. Consider the event that a bidder with value $v \in [\underline{v}, \bar{v}]$ wins the object in a standard k^{th}-price sealed-bid auction, where $k \in \{\mathrm{I}, \mathrm{II}\}$. The bidder's payment is $\beta(v)$ in a standard first-price sealed-bid auction and due to the payment rule his payment is deterministic. His expected payment is also $\beta(v)$ in a standard second-price sealed-bid auction, however it is the expectation of the second-highest bid that is random. So a risk-averse auctioneer will get higher expected utility from this event in a standard first-price sealed-bid auction. And notice that this event happens with same probability in both auctions, therefore the auctioneer prefers a standard first-price sealed-bid auction to a standard second-price sealed-bid auction. We apply the same intuition to our model.

Proposition 8. *For a given rejection price $R \le v^*$, if $u(\cdot)$ is concave on the interval $[\underline{v}, R]$, then an auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction, that is, $U_{\mathrm{I}}(R) > U_{\mathrm{II}}(R)$.*

Proof. Let

$$w(v;R) = \begin{cases} u(v) & \textit{if } R \geq v \geq \underline{v} \\ u(R) & \textit{if } \bar{v} \geq v > R. \end{cases}$$

Obviously, $w(v;R)$ is concave but not linear on the interval $[\underline{v}, \bar{v}]$ since $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, R]$. Then we have

$$U_{\mathrm{II}}(R) = n \int_{\underline{v}}^{\bar{v}} (1 - F(t)) w(t;R) dF^{n-1}(t)$$

and if $R \in [\beta(\bar{v}), \bar{v}]$

$$U_{\mathrm{I}}(R) = \int_{\underline{v}}^{\bar{v}} w(\beta(t);R) dF^n(t).$$

Notice that $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, R]$ and $R > \beta(v_J(R))$. Thus if $R \in [\underline{v}, \beta(\bar{v}))$

$$\begin{aligned} U_{\mathrm{I}}(R) &> [1 - F^n(\beta^{-1}(R))] u(R) + \int_{\underline{v}}^{\beta^{-1}(R)} u(\beta(t)) dF^n(t) \\ &= \int_{\underline{v}}^{\bar{v}} w(\beta(t);R) dF^n(t). \end{aligned}$$

Therefore, by Lemma 2, we have for any $R \leq v^*$,

$$U_{\mathrm{I}}(R) \geq \int_{\underline{v}}^{\bar{v}} w(\beta(t);R) dF^n(t) > n \int_{\underline{v}}^{\bar{v}} (1 - F(t)) w(t;R) dF^{n-1}(t) = U_{\mathrm{II}}(R).$$

□

In a first-price sealed-bid auction with a rejection price $R \in [\underline{v}, \bar{v}]$, if bidders bid following the strategy $\hat{\beta}_{\mathrm{I}}(v) = min\{\beta(v), R\}$, then the auctioneer's expected utility is equal to $\int_{\underline{v}}^{\bar{v}} w(\beta(t);R) dF^n(t)$. By applying Lemma 2 to two standard auctions with a concave but non-linear utility function $w(v;R)$, we see that the expected utility is greater than the auctioneer's equilibrium expected utility under the standard second-price sealed-bid auction, which is equal to $U_{\mathrm{II}}(R)$ in our setting. In fact, in a first-price sealed-bid auction, some bidders bid higher than they bid following $\hat{\beta}_{\mathrm{I}}(v) = min\{\beta(v), R\}$. Since $R \leq v^*$ and $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, v^*]$, the auctioneer will get higher expected utility than that case. Hence, the auctioneer prefers a first-price sealed-bid auction. Notice that in Proposition 8, R could be equal to v^* which is the optimal rejection price in a second-price sealed-bid auction, then the following proposition is straightforward.

Proposition 9. *If $u(\cdot)$ is concave on the interval $[\underline{v}, v^*]$, then with the optimal rejection price*[12] *an auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction, that is, $\max_{R \in [\underline{v}, \bar{v}]} U_{\mathrm{I}}(R) > U_{\mathrm{II}}(v^*)$.*

[12] In a first-price sealed-bid auction, if $v^* \geq \beta(\bar{v})$ the auctioneer's expected utility may be maximized on the interval $[\beta(\bar{v}), \bar{v}]$, i.e., it may not be optimal for the auctioneer to use a rejection price. In this case, we regard any $R \in [\beta(\bar{v}), \bar{v}]$ as his optimal rejection price.

Proof. By Proposition 8, $\max_{R\in[\underline{v},\bar{v}]} U_{\mathrm{I}}(R) \geq U_{\mathrm{I}}(v^*) > U_{\mathrm{II}}(v^*)$. $\square$

We reach the above conclusions based on the auctioneer's risk-attitude. Next, we analyze the preferences of the auctioneer over the two auctions in a different way. From the auctioneer's point of view, a k^{th}-price sealed-bid auction with a rejection price can be regarded as a lottery since the revenue is random, where $k \in \{\mathrm{I}, \mathrm{II}\}$. Recall that we let $F_k(v; R_k)$ denote the distribution function of the auctioneer's revenue v in a k^{th}-price sealed-bid auction with a rejection price R_k. The following proposition shows that for a given rejection price $R \leq v^*$, if $F_{\mathrm{I}}(v; R)$ is first-order stochastic dominant over $F_{\mathrm{II}}(v; R)$ on the range where the auctioneer's revenue is lower than $\beta(v_J(R))$, the auctioneer prefers a first-price sealed-bid auction to a second-price sealed-bid auction.

Proposition 10. *For a given rejection price $R < \beta(\bar{v})$, if $R \leq v^*$ and*

$$nF^{n-1}(v) - (n-1)F^n(v) \geq F^n(\beta^{-1}(v))$$

for any $v \in [\underline{v}, \beta(v_J(R))]$,[13] *then an auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction, that is, $U_{\mathrm{I}}(R) > U_{\mathrm{II}}(R)$.*

Proof. Since $R < \beta(\bar{v})$ and for any $v \in [\underline{v}, \beta(v_J(R))]$

$$nF^{n-1}(v) - (n-1)F^n(v) \geq F^n(\beta^{-1}(v)),$$

then we have

$$F_{\mathrm{II}}(v; R) \geq F_{\mathrm{I}}(v; R)$$

for any $v \in [\underline{v}, \beta(v_J(R))]$ and

$$F_{\mathrm{II}}(v; R) > F_{\mathrm{I}}(v; R)$$

for any $v \in (\beta(v_J(R)), R)$. Since $u(\cdot)$ is strictly increasing on the interval $[\underline{v}, R]$, then

$$U_{\mathrm{I}}(R) - U_{\mathrm{II}}(R) = \int_{\underline{v}}^{R} u(t)dF_{\mathrm{I}}(t; R) - \int_{\underline{v}}^{R} u(t)dF_{\mathrm{II}}(t; R) > 0.$$

$\square$

By Propositions 2 and 6, we have $R^*_{\mathrm{I}} \leq v^* = R^*_{\mathrm{II}}$ if $v^* < \beta(\bar{v})$. Then, the following proposition can be shown immediately.

Proposition 11. *If $v^* < \beta(\bar{v})$ and*

$$nF^{n-1}(v) - (n-1)F^n(v) \geq F^n(\beta^{-1}(v))$$

for any $v \in [\underline{v}, \beta(v_J(v^*))]$, then with the optimal rejection price an auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction, that is, $U_{\mathrm{I}}(R^*_{\mathrm{I}}) \geq U_{\mathrm{II}}(v^*)$.

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[13] These conditions can be satisfied if $n = 2$ and $F(\cdot)$ is a uniform distribution function on $[0, 1]$ and $R \leq 4/9$.

Proof. By Proposition 10, $U_{\mathrm{I}}(R^*_{\mathrm{I}}) \geq U_{\mathrm{I}}(v^*) > U_{\mathrm{II}}(v^*)$. □

The intuition of the conditions of Propositions 10 and 11 is that the auctioneer will be more likely to acquire a low revenue in a second-price sealed-bid auction than in a first-price sealed-bid auction. This is because even if the winner's private value is high, the auctioneer may acquire a low revenue in a second-price sealed-bid auction.

The above four propositions show the same preference that the auctioneer strictly prefers a first-price sealed-bid auction. Notice that Propositions 8 and 9 only depend on the auctioneer's utility function $u(\cdot)$, and Propositions 10 and 11 only depend on the distribution function $F(\cdot)$ if $v^* < \beta(\bar{v})$. Therefore, in the case where $v^* < \beta(\bar{v})$, it is difficult to find some cases that an auctioneer prefers a second-price sealed-bid auction.[14]

Nonetheless, in the case where $v^* > \beta(\bar{v})$, for any rejection price the upper bound of the auctioneer's expected utility in a first-price sealed-bid auction is $u(\beta(\bar{v}))$. Notice that the lower bound of the auctioneer's expected utility in a second-price sealed-bid auction with the optimal rejection price is $[1 - F^n(v^*) - n(1 - F(v^*))F^{n-1}(v^*)]u(v^*)$, hence with the optimal rejection price the auctioneer strictly prefers a second-price sealed-bid auction to a first-price auction if $u(v^*) \geq u(\beta(\bar{v}))/[1 - F^n(v^*) - n(1 - F(v^*))F^{n-1}(v^*)]$.

6 Discussion

So far, we have assumed that the auctioneer's utility function is strictly increasing below the unique maximizer and strictly decreasing above the unique maximizer. In this section, we consider a more general non-monotonic utility function that $u(v^*) > u(v)$ for any $v \in [0, \infty)$, where $v^* \in (\underline{v}, \bar{v})$. For tractability, we also assume that $u(\cdot)$ is continuous on the interval $[0, \infty)$.

Notice that the equilibrium bidding strategies $\beta_{\mathrm{I}}(\cdot)$ and $\beta_{\mathrm{II}}(\cdot)$ are independent of the utility function $u(\cdot)$, therefore, in this section, we focus on the optimal rejection prices in first-price and second-price sealed-bid auctions. Before starting our discussion, we additionally assume that $\max_{v \in [0, \underline{v}]} u(v) < \min_{v \in (\underline{v}, v^*]} u(v)$ in order to exclude the case that the optimal rejection price is $R^*_k \in [0, \underline{v}]$, where $k \in \{\mathrm{I}, \mathrm{II}\}$. This is because, in that case, the auctioneer may want to sell the object at a price which is lower than $\underline{v}$.

At first, we consider a second-price sealed-bid auction with a rejection price. Recall that the function $F_{\mathrm{II}}(v; R_{\mathrm{II}})$ denotes the distribution function of the auctioneer's revenue v in a second-price sealed-bid auction with a rejection price R_{II}. For a rejection price $R_{\mathrm{II}} > v^*$, notice that $F_{\mathrm{II}}(v; v^*) = F_{\mathrm{II}}(v; R_{\mathrm{II}})$ for any $v \in [\underline{v}, v^*)$ and $u(v^*) > u(v)$ for any $v \in [0, \infty)$. Therefore, the auctioneer will get higher expected utility with the rejection price v^* than with the rejection price R_{II}, i.e., the optimal rejection price R^*_{II} cannot be higher than v^*. For a rejection price $R_{\mathrm{II}} < v^*$, notice that $F_{\mathrm{II}}(v; v^*)$ is first-order stochastic dominant

[14]In fact, we do not know of an example where an auctioneer prefers a second-price sealed-bid auction when $v^* < \beta(\bar{v})$.

over $F_{\mathrm{II}}(v; R_{\mathrm{II}})$. So if $u(\cdot)$ is weakly increasing on the interval $[\underline{v}, v^*]$, then the auctioneer strictly prefers the rejection price v^*. This is because $u(\cdot)$ is not constant on the interval $[\underline{v}, v^*]$ by the assumption that $u(v^*) > u(v)$ for any $v \in [0, \infty)$.

Remark 3. *In a second-price sealed-bid auction with a rejection price R_{II}, the optimal rejection price is $R^*_{\mathrm{II}} \in (\underline{v}, v^*]$. Moreover, if $u(\cdot)$ is weakly increasing on the interval $[\underline{v}, v^*]$, then the optimal rejection price is $R^*_{\mathrm{II}} = v^*$.*

For a first-price sealed-bid auction with a rejection price, notice that the proof of Proposition 6 can be supported by the assumption that $u(v^*) > u(v)$ for any $v \in [0, \infty)$, i.e., we can prove the same results as Proposition 6.

Remark 4. *In a first-price sealed-bid auction with a rejection price R_{I}, if $v^* < \beta(\bar{v})$, the optimal rejection price is $R^*_{\mathrm{I}} \in (\underline{v}, v^*]$. Moreover, if $u(\cdot)$ is differentiable at v^*, then the optimal rejection price is $R^*_{\mathrm{I}} \in (\underline{v}, v^*)$.*

7 Conclusion

We have studied an auctioneer whose utility is non-monotonic and has a unique maximizer. This kind of auctioneer is not uncommon in the real world, such as the local government in China. When such an auctioneer sells objects in an auction, he is willing to use a rejection price to maximize his utility.

First, we analyzed a second-price sealed-bid auction with a rejection price. The equilibrium bidding strategy in this auction is straightforward. That is, a bidder bids the lower one between his value and the rejection price. Therefore, the auctioneer announces truthfully before auction starts, i.e., chooses the optimal rejection price which equals the unique maximizer. And whatever bidders' attitudes to risk are, the conclusion is the same. We have compared such a mechanism to a standard second-price sealed-bid auction, and found that if the ex post expected utility of the bidder who has the maximum value can be improved by the rejection price, then all bidders' ex post expected utilities can be improved. Further, using the optimal rejection price makes a Pareto improvement to the standard model.

Second, we studied the behavior of bidders and the auctioneer in a first-price sealed-bid auction with a rejection price. We found that the rejection price works only if it is lower than the the maximum equilibrium bid in a standard first-price sealed-bid auction. We focus attention on the case where it works and have shown that there exists a jump point in the equilibrium bidding strategy. In this case, the bid of bidder whose value is lower than the jump point, is the same as the standard model, and the bidder whose value is higher than it bids the rejection price. If the maximizer is also lower than maximum equilibrium bid in a standard model, the optimal rejection price for the auctioneer is not higher than the maximizer. Moreover, if his utility function is smooth at the maximizer, due to the existence of the jump point, the optimal rejection price will be lower than the maximizer. And we also have proved that if the ex post expected utility of the bidder with the maximum value can be improved, such a mechanism makes a Pareto improvement to the standard model.

Finally, we analyzed the auctioneer's preferences over the two auctions. We found that the auctioneer strictly prefers a first-price sealed-bid auction to a second-price sealed-bid auction if the increasing part of his utility function is concave. The same conclusion can be reached if the distribution of revenues in a first-price sealed-bid auction is first-order stochastic dominant over it in a second-price sealed-bid auction. And we also found some cases that the auctioneer strictly prefers a second-price sealed-bid auction to a first-price sealed-bid auction.

The natural extension of this work is to consider a more realistic rejecting strategy that the auctioneer rejects bids with probabilities. One strategy that can be taken into account is that the auctioneer accepts the bids which are lower than some amount and rejects the bids which are higher than it with probability, i.e. the auctioneer can use a mixed strategy. This is because the auctioneer may not directly announce that amount, i.e., the bidders may acquire the information about it with uncertainty. Note that, in this paper, we assume that bidders are able to bid higher than the rejection price, though get nothing. The rejection price becomes a pure rejecting strategy in a general framework. Another more realistic rejecting strategy is that the higher bid is more likely to be rejected, i.e. the rejecting probability is increasing in the bid. We leave the challenging work for future research.

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